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Heat equation with memory: Lack of controllability to rest ☆

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ABSTRACT

We prove that the one-dimensional heat equation with memory cannot be controlled to rest for large classes of memory kernels and controls. The approach is based on the application of the theory of interpolation in Paley–Wiener spaces.

1. Introduction

First we give an informal description of the subject of this paper. The second part of the present introduction contains the technical details.

Several "controllability" results have been recently proved for the heat equation with memory, see for example [2,12,16–18,20]. These papers study control problems for equations of the following form (here $t > 0$ and $x \in (0, \pi)$), but also the case $x \in \mathbb{R}^n$ has been considered):

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s, \qquad \theta(0) = \xi \tag{1}$$

and θ is furthermore acted upon by a control so that $\theta(t) = \theta(t; u)$.

Exact controllability has been studied in the references above, i.e. the question whether an arbitrary final target $\eta(\cdot)$ can be hit by using a suitable control. In particular, the target can be 0. In this case we have a kind of null controllability which however need not be controllability to rest, since even if the trajectory hits 0, the solution may leave 0 in the future (see the comments below). Using a term which was once popular among specialists of systems with delays, this is a kind of "relative controllability", see [8]. In contrast with this, we say that the system is *controllable to rest* when for every initial condition ξ we can find a control u with compact support and such that the corresponding solution $\theta(t; u)$ has compact support too.

Controllability to rest is not impossible, see Section 1.1, but we shall prove that it is an exceptional property in the presence of memory. This, may be, will not be a surprise but our point in this paper is that the obstacle to controllability

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is not so much the infinite memory of the equation but the joint facts that the equation has infinite memory and that $\theta(t)$ takes values in an infinite-dimensional Hilbert space. We shall see (in Section 1.1) that if $\theta(t)$ takes values in $\mathbb{R}^n$ (and the term θ_{xx} is replaced by $A\theta$, A a matrix), then controllability may well be possible also if the kernel $N(t)$ has the properties required by the negative results presented in Section 3. For simplicity, the examples are given in the scalar case, $n = 1$.

Our negative results cover in particular the cases that the kernel $N(t)$ is a linear combination of exponentials or it is a kernel of Abel type, as studied for example in [5,6]. The proofs use the Fourier methods and the Laplace transform in order to reduce controllability to a moment/interpolation problem in a Paley–Wiener space, see [1,10]. This is done in Section 2 where all the relevant definitions are given.

Now we make precise the previous informal considerations. We study the following three control problems:

Problem (i). Boundary control.

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s, \quad x \in (0,\pi),\ t > 0,$$
$$\theta(t,0) = u(t) \in L^2_{\mathrm{loc}}[0,\infty), \qquad \theta(t,\pi) = 0, \qquad \theta(0,x) = \xi(x) \in L^2(0,\pi). \tag{2}$$

Problem (ii). Distributed control with a given profile.

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s + b(x)u(t), \quad x \in (0,\pi),\ t > 0,$$
$$\theta(t,0) = \theta(t,\pi) = 0, \qquad \theta(0,x) = \xi(x) \in L^2(0,\pi). \tag{3}$$

Here $b \in L^2(0,\pi)$ is a given function and $u \in L^2_{\mathrm{loc}}(0,+\infty)$.

Problem (iii). Control distributed on a subdomain.

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s + u(x,t), \quad x \in (0,\pi),\ t > 0,$$
$$\theta(t,0) = \theta(t,\pi) = 0, \qquad \theta(0,x) = \xi(x) \in L^2(0,\pi). \tag{4}$$

Here $u \in L^2_{\mathrm{loc}}([0,+\infty) \times [0,\pi])$ is a control supported (in x) on an interval $[\beta,\gamma]$ properly contained in $[0,\pi]$.

Now we give the following definition which apply to each one of the cases above. Solutions will be rigorously defined later on.

Definition 1. The initial vector $\xi \in L^2(0,\pi)$ is *controllable to rest* if we can find a time $T > 0$ and a control $u(t)$ such that $u(t) = 0$ and $\theta(t;u) = 0$ for $t > T$.

The system is *controllable to rest* if for every initial vector $\xi \in L^2(0,\pi)$ we can find a time $T > 0$ and a control $u(t)$ such that $u(t) = 0$ for $t > T$ and the corresponding solution $\theta(t)$ satisfies $\theta(t) = 0$ for $t > T$; if the controllability time T can be chosen independent of ξ then we say that the system is *controllable to rest in time T*.

Controllability to rest *is not* a kind of "null controllability", in the sense used for systems with finite delays, see for example [13,14]. In fact, let us consider the following simple example of system with finite delay h:

$$\dot{\theta}(t) = A_0\theta(t) + A_1\theta(t-h) + Bu(t),$$

where now $\theta \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A_0, A_1, B are matrices of suitable dimensions. The initial condition is $\theta(s) = \xi(s)$, $s \in [-h,0]$. According to the general definition of state, see [9], the function $s \to \theta(t+s)$, $s \in [-h,0]$ can be chosen as the state of the system at time t. The system is "null controllable" at time T when there exists a control $u(t)$, $0 \leqslant t \leqslant T$ such that the *state* at time T is zero; i.e. $\theta(T+s) = 0$ for every $s \in [-h,0]$ (note that if we extend $u(t)$ with $u(t) = 0$ for $t > T$ then also the state is zero for $t > T$). In general, the state of Eq. (1) at time t is $\theta(s)$, $s \leqslant t$ (see [7]) and "null controllability" at time T would imply $\theta(t) = 0$ for every $t \geqslant 0$. This is impossible, unless the initial condition is $\xi = 0$. So, we study the weaker condition that $\theta(t)$ and the control $u(t)$ are zero for $t > T > 0$. To avoid any risk of confusion, we use the term "controllability to rest" for this property. It is clear that null controllability and controllability to rest are equivalent for systems with finite memory.

1.1. Preliminary examples

We first consider two control problems, with $\theta(t) \in \mathbb{R}$. Controllability to rest holds in these examples, in spite of the fact that the kernels $N(t)$ fulfill the assumptions of the negative results given in Section 3.

We introduce the notation

$$G(\lambda) = \alpha + \hat{N}(\lambda) = \alpha + \int_0^{+\infty} e^{-\lambda t} N(t)\, dt$$

(i.e. $\hat{}$ denotes Laplace transform). The negative results in Section 3 are expressed using this function.

We are interested in the first example since in this case $G(\lambda)$ admits zeros.

Example 1. The first example is

$$\dot{\theta} = 3\theta - 2\int_0^t \theta(s)\, ds + u(t), \qquad \theta(0) = \xi. \tag{5}$$

Here $\theta \in \mathbb{R}$. Note that $G(\lambda) = 3 - 2/\lambda$ so that $G(2/3) = 0$.

We introduce

$$y(t) = \int_0^t \theta(s)\, ds$$

(so that $y(0) = 0$). System (5) can be controlled to rest provided that we can find $T > 0$ and a control u which is zero for $t > T$ and such that

$$\theta(T) = 0, \qquad y(T) = 0. \tag{6}$$

In fact let $u(t) = 0$ for $t = T + \tau$. Then $\theta(T + \tau)$ solves

$$\frac{d}{dt}\theta(T + \tau) = 3\theta(T + \tau) - 2\int_0^\tau \theta(T + r)\, dr - 2y(T).$$

The solution $\tau \to \theta(T + \tau)$ is identically zero if it happens that $\theta(T) = 0$, $y(T) = 0$ and (and, in fact, solely in this case).

Now we observe that a state space realization of the map from u to θ is

$$\theta' = 3\theta - 2y + u, \qquad y' = \theta. \tag{7}$$

This system is controllable since

$$\operatorname{rank}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] = \operatorname{rank}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = 2.$$

The function $G(\lambda)$ in the second example has a double pole and no zero.

Example 2. Here again θ is scalar and

$$\theta' = \int_0^t (t - s)\theta(s)\, ds + u(t), \qquad \theta(0) = \xi,$$

i.e.

$$\alpha = 0, \qquad N(t) = t \quad \text{so that} \quad G(\lambda) = \hat{N}(\lambda) = \frac{1}{\lambda^2}.$$

It is easy to see that controllability to rest holds if we can find $T > 0$ and $u(t)$ such that hold:

$$\begin{cases} \theta(T) = 0, \\ y(T) = 0, \\ \zeta(T) = 0 \end{cases} \quad \text{where} \quad \begin{cases} y(t) = \displaystyle\int_0^t (t - s)\theta(s)\, ds, \\ \zeta(t) = \displaystyle\int_0^t \theta(s)\, ds. \end{cases}$$

So we have to study the controllability of the linear finite-dimensional system

$$\dot\theta = y + u, \qquad \dot y = \zeta, \qquad \dot\zeta = \theta.$$

We omit the simple verification that this system is controllable.

We consider now an example of a system of the form (1) which is controllable to rest. This is an integrated form of the wave equation.

Example 3. We consider

$$\theta_t = \int_0^t \theta_{xx}(s)\,\mathrm{d}s, \qquad \theta(0) = \xi, \qquad \theta(t,0) = u(t), \qquad \theta(t,\pi) = 0,$$

$\theta(t,\cdot) \in L^2(0,\pi)$. We introduce

$$y(t) = \int_0^t \theta_{xx}(s)\,\mathrm{d}s$$

so that we get the wave equation

$$\theta_t = y, \qquad y_t = \theta_{xx}, \qquad \theta(t,0) = u(t), \qquad \theta(t,\pi) = 0,$$

i.e. the wave equation, which is controllable, see [1,10].

Our point in this paper is that every system of the form (1) which is controllable to rest and which has a "rational kernel", i.e. a kernel with rational Laplace transform, is a wave equation in disguise.

2. Controllability, interpolation and moment problems

We first outline the definition of solutions for Problems (i) and (ii). See Section 3.2 for Problem (iii). We consider the formulation of Problem (i) but the arguments below are easily adapted to Problem (ii). See [4] for the use of similar ideas in the non-delayed case.

We note the following equality which holds for every function $\psi(x) \in H^2(0,\pi)$ such that $\psi(0) = \kappa$, $\psi(1) = 0$ and for every $\phi(x) \in H^2(0,\pi) \cap H_0^1(0,\pi)$ (here $\langle \phi, \psi \rangle$ is the inner product in $L^2(0,\pi)$):

$$\langle \psi_{xx}, \phi \rangle = \phi'(0)\kappa + \big\langle \psi(\cdot), \phi_{xx}(\cdot) \big\rangle = \phi'(0)\kappa + \langle \psi, A\phi \rangle, \tag{8}$$

where A is the operator

$$\operatorname{dom} A = H^2(0,\pi) \cap H_0^1(0,\pi) \subseteq L^2(0,\pi), \qquad (A\phi)(x) = \phi_{xx}(x) \in L^2(0,\pi). \tag{9}$$

This formula replaces the boundary condition with an additive term, and suggests the following definition: the function $\theta(t) = \theta(t,x) \in L^2_{\mathrm{loc}}[0,+\infty; L^2(0,\pi))$ solves Problem (i) when the following equality holds for every $\phi \in \operatorname{dom} A$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle \theta(t), \phi \big\rangle = \alpha\big[\big\langle \theta(t), A\phi \big\rangle + \phi'(0)u(t)\big] + \int_0^t N(t-s)\big[\big\langle \theta(s), A\phi \big\rangle + \phi'(0)u(s)\big]\,\mathrm{d}s, \qquad \theta(0) = \xi. \tag{10}$$

The normalized eigenvectors of A are the functions $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$ (the corresponding eigenvalue is $-n^2$) and this is an orthonormal basis in $L^2(0,\pi)$. Hence we can expand

$$\theta(t) = \theta(t,x) = \sum_{n=1}^{+\infty} \theta_n(t)\phi_n(x). \tag{11}$$

If in particular $\phi = \phi_n$ in (10) we see that the nth component $\theta_n(t) = \langle \theta(t), \phi_n \rangle$ satisfies

$$\theta_n'(t) = \alpha\big[-n^2\theta_n(t) + \phi_n'(0)u(t)\big] + \int_0^t N(t-s)\big[-n^2\theta_n(s) + \phi_n'(0)u(s)\big]\,\mathrm{d}s, \qquad \theta_n(0) = \xi_n = \langle \xi, \phi_n \rangle. \tag{12}$$

The solution of Problem (ii) can be defined in an analogous way and we see that if $\theta(t)$ solves Problem (ii) then $\theta_n(t) = \langle \theta(t), \phi_n \rangle$ solves

$$\theta_n'(t) = -\alpha n^2\theta_n(t) - n^2\int_0^t N(t-s)\theta_n(s)\,\mathrm{d}s + b_n u(t), \qquad \theta_n(0) = \xi_n = \langle \xi, \phi_n \rangle, \tag{13}$$

where

$$b_n = \langle b, \phi_n \rangle.$$

By definition, if the initial condition ξ can be controlled to rest, then we can find a control u with support in $[0,T]$ such that each function $\theta_n(t)$ which solves (12), when studying Problem (i), or (13) when studying Problem (ii), must have compact support too. We are going to prove that this is impossible for most of the kernels met in applications, relying on frequency domain techniques.

As in Section 1.1, let

$$G(\lambda) = \alpha + \hat{N}(\lambda) = \alpha + \int_0^{+\infty} e^{-\lambda t} N(t)\,\mathrm{d}t.$$

We denote $\mathcal{P}W_+$ the linear space of the Laplace transforms of square integrable functions on $(0,+\infty)$ with compact support. The characterization of this space is known, $\hat{\phi}(\lambda) \in \mathcal{P}W_+$ if it is an *entire function* such that

- there exist real numbers M and T (which depend on ϕ) such that $|\phi(\lambda)| \leqslant Me^{T|\lambda|}$;
- $\sup_{x \geqslant 0} \int_{-\infty}^{+\infty} |\phi(x+iy)|^2\,\mathrm{d}y < +\infty$.

An important example is $[1-\exp(-\lambda T)]/\lambda$. In terms of complex analysis, an element of $\mathcal{P}W_+$ is in a classical Paley–Wiener space and it is bounded in the right half plane.

We now compute the Laplace transform of $\theta_n(t)$ in (12) or (13). Using $\phi_n'(0) = \sqrt{\frac{2}{\pi}}n$, we get

$$\text{Problem (i)} \quad \hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2 G(\lambda)} \left(\xi_n + \frac{n\sqrt{2}G(\lambda)}{\sqrt{\pi}} \hat{u}(\lambda) \right), \tag{14}$$

$$\text{Problem (ii)} \quad \hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2 G(\lambda)} \left(b_n \hat{u}(\lambda) + \xi_n \right), \quad b_n = \langle b, \phi_n \rangle. \tag{15}$$

Hence, if controllability to rest is possible then for any given initial condition $\xi = \theta(0) \in L^2(0,\pi)$, we can find an input $\hat{u}(\lambda) \in \mathcal{P}W_+$ such that for every n the component $\hat{\theta}_n(\lambda)$ belong to $\mathcal{P}W_+$. In particular, the functions $\hat{\theta}_n(\lambda)$ cannot have singularities at the roots of the denominator $\lambda + n^2 G(\lambda)$ of the expressions in (14)–(15), since the elements of $\mathcal{P}W_+$ are entire functions. Hence the control u must satisfy the following equalities:

$$\text{Problem (i)} \quad \hat{u}(\lambda) = \sqrt{\frac{\pi}{2}} \frac{n}{\lambda} \xi_n \quad \text{when } \lambda + n^2 G(\lambda) = 0, \tag{16}$$

$$\text{Problem (ii)} \quad \hat{u}(\lambda) = -\frac{1}{b_n} \xi_n \quad \text{when } \lambda + n^2 G(\lambda) = 0. \tag{17}$$

(We may suppose $b_n \neq 0$ for every n since if $b_m = 0$ for some m we cannot control the mth component $\theta_m(t)$.)

These problems are *interpolation problems.*

We sum up: controllability to rest implies that the previous interpolation problems are solvable for every $\{\xi_n\} \in l^2$.

Now let the system be controllable to rest in time T. Using the definition of the Laplace transform, the interpolation problems (17) can be written as

$$\int_0^T e^{-\lambda t} u(t)\,\mathrm{d}t = -\frac{1}{b_n} \xi_n \quad \text{when } \lambda + n^2 G(\lambda) = 0. \tag{18}$$

(It is possible that for a fixed n the equation $\lambda + n^2 G(\lambda) = 0$ has more than one root. In this case condition (18) must be satisfied at each root.)

Problem (16) takes the form

$$\int_0^T e^{-\lambda t} u(t)\,\mathrm{d}t = \sqrt{\frac{\pi}{2}} \frac{n}{\lambda} \xi_n \quad \text{when } \lambda + n^2 G(\lambda) = 0. \tag{19}$$

The interpolation problems (16) and (17), when written in the equivalent form (18), respectively (19), are called *moment problems* (with respect to a family of exponentials), see [1,10,21].

Our negative results on the controllability to rest will be derived since, under appropriate conditions, we can see that there are obstructions to the solution of the interpolation/moment problem. We list two of such obstructions, which will be explicitly used. The first is trivial.

Obstruction 1. We see directly from the definition that if there is a sequence λ_n of zeros of $\lambda + n^2 G(\lambda)$ which converges to μ, then the interpolation problems (16) and (17) are not generally solvable. To show this, we choose $\xi_{2k+1} = 0$ and we note that $\hat{u}(\lambda)$ is an entire function so that the conditions $u(\lambda_{2k+1}) = 0$ and $\lambda_{2k+1} \to \mu$ imply that $\hat{u}(\lambda)$ has to be identically zero and this forces every component ξ_{2k} to be zero too: neither interpolation condition (16) nor (17) are possible unless $\xi = 0$.

An informal interpretation of this negative result which suggests where to search for more powerful obstructions, is as follows: a subfamily $\{e^{\lambda_{n_k} t}\}$ of the family of the exponentials in (18), respectively (19), is "so rich" that the corresponding equalities uniquely identify the input $u(t)$. In this case, it will not be possible to solve the remaining equalities. This informal argument suggest that results which concerns completeness of families of exponentials might be useful to prove negative results for controllability to rest. In fact, as seen in Lemma 1 below, a powerful obstruction to interpolation follows from the following completeness condition.

Obstruction 2. Let $\{\lambda_n\}$ be a sequence of complex numbers with the following property: there exists a number $\gamma > 1$ such that

$$\sum_{n=1}^{+\infty} \frac{1}{|\lambda_n|^\gamma} = +\infty. \tag{20}$$

If (20) holds then the sequence of exponentials $\{e^{\lambda_n t}\}$ is complete in $L^2(0, T)$ for every $T > 0$.

See [19, p. 105, "complement" to Remark 2] for an even more general formulation.

Note that the condition in Obstruction 1 implies condition (20).

We see in Lemma 1 below that when the condition in Obstruction 2 holds then our interpolation/moment problem cannot be solved if ξ has to be an arbitrary initial condition in $L^2(0, \pi)$. In fact, Lemma 1 shows a stronger consequence, that we shall use in Remark 4 and in Theorem 8:

Lemma 1. *Let condition* (20) *be fulfilled. Let r be nonnegative and let us consider the moment problems* (18) *or* (19) *with* $\xi_n = \langle \xi, \phi_n \rangle$, *and* $\xi \in \mathrm{dom}(-A)^r$. *These moment problems cannot be solved.*

Proof. We fix an index k and we consider the equalities required by the moment problems for every index $n \neq k$. Condition (20) is satisfied by the numbers λ_n, $n \neq k$ so that exponential family $\{e^{-\lambda_n t},\ n \neq k\}$ is complete.

We consider the special initial condition $\xi = c\phi_k$, where c is constant. Clearly, $\xi \in \mathrm{dom}(-A)^r$. The moment equalities for $n \neq k$ are

$$\int_0^T e^{-\lambda_n t} u(t)\, \mathrm{d}t = 0, \quad n \neq k. \tag{21}$$

Since the exponential family $\{e^{-\lambda_n t},\ n \neq k\}$ is complete, this implies $u = 0$. Then the value of $\int_0^T e^{-\lambda_k t} u(t)\, \mathrm{d}t$ is zero too and cannot be assigned at will. The moment problems (18) or (19) can only be solved if $c = 0$. $\square$

3. Lack of controllability to rest

The negative results are proved in this section. It is convenient to consider first the case of boundary controls, and the case of a distributed control with a profile, i.e. the cases in which the control takes values in $\mathbb{R}$, first.

3.1. Problems (i) and (ii)

Example 3 shows that system (1) can be controllable to rest. However, our negative results show that this is an exceptional case. The first negative result is as follows:

Theorem 2. *If there exists a zero* λ_0 *of* $G(\lambda) = \alpha + \hat{N}(\lambda)$ *then controllability to rest in the cases* (i), *i.e. Eq.* (2), *and* (ii), *Eq.* (3), *is impossible.*

Proof. We shall see the existence of a sequence $\{\lambda_n\}$ of zeros of

$$\lambda + n^2 G(\lambda),$$

one λ_n for each n, which is convergent. Obstruction 1 then shows that interpolation is impossible.

The sequence $\{\lambda_n\}$ exists, from Rouchè's Theorem: let λ_0 be a zero of $G(\lambda)$. We consider a disk centered at λ_0 in which $G(\lambda)$ has no singularity and such that on the boundary $|G(\lambda)| > \mu > 0$. Let $\nu = \max |\lambda|$ on the boundary of the disk. Clearly, for n large enough we have $n^2 \mu > \nu$ so that the function $n^2 G(\lambda) + \lambda$ has a zero λ_n in this disk, for every large n, as was to be proved. $\square$

In particular, if $\alpha > 0$ and $N(t) = e^{\gamma t}$, or if $\alpha = 0$ and $N(t) = e^{\gamma t} + e^{\sigma t}$, then controllability to rest is impossible. This is to be contrasted with Example 1.

An example of a function $G(\lambda)$ without zeros is $G(\lambda) = \hat{N}(\lambda) = 1/\lambda^2$ (hence $\alpha = 0$) as in Example 2 and this case is not covered by Theorem 2. So, we prove the next negative result.

Theorem 3. *Let*

$$G(\lambda) = \frac{1}{\lambda^\nu + g(\lambda)},$$

where $\nu \geqslant 2$ *and* $|g(\lambda)| < M|\lambda|^{\nu-1}$. *The estimate for* $g(\lambda)$ *is assumed in a sector* $\mathcal{S}$ *with the vertex at* 0 *and containing a ray* $\arg z = \frac{2\pi k+\pi}{\nu+1}$.

Under this assumption, controllability to rest in the cases (i), *i.e. Eq.* (2), *and* (ii), *Eq.* (3), *is impossible.*

Proof. The proof depends on Obstruction 2.

The equation $\lambda + n^2 G(\lambda) = 0$ gives

$$0 = \left(\lambda^{\nu+1} + n^2\right) + \lambda g(\lambda). \tag{22}$$

We compare $\lambda g(\lambda)$ and

$$f_n(\lambda) = \lambda^{\nu+1} + n^2.$$

The zeros of $f_n(\lambda)$ are $(-n^2)^{1/(\nu+1)}$ and lie on the lines identified by the $(\nu+1)$-roots of (-1), $e^{(i\pi+2ki\pi)/(\nu+1)}$, $0 \leqslant k \leqslant \nu$. By assumption, at least one of them belong to the sector $\mathcal{S}$. Let this root be $e^{i\pi(2k_0+1)/(\nu+1)}$.

Consider the sequence of the roots

$$\zeta_n = n^{2/(\nu+1)} e^{i\pi(2k_0+1)/(\nu+1)}.$$

These belong to a straight line in the sector $\mathcal{S}$. Moreover, there exists a number $\gamma > 1$ such that $\sum 1/|\zeta_n|^\gamma = +\infty$, since $2/(\nu+1) < 1$. Obstruction 2 implies that for the set of the zeros of the functions $f_n(\lambda)$ our interpolation problems cannot be solved for general data.

We prove that this negative property is inherited by the zeros of $\lambda + n^2 G(\lambda)$ as follows: using again Rouchè's Theorem, we prove that there exists a number $\sigma > 0$ and a number N_0 such that for $n > N_0$ the function $\lambda + n^2 G(\lambda)$ has a zero μ_n in a disk of radius σ centered at each zero of ζ_n. So, we have also $\sum 1/|\mu_n|^\gamma = +\infty$ and interpolation is impossible.

Let Γ_n be the circle

$$\Gamma_n\colon\ \lambda = \zeta_n + \sigma e^{i\omega}, \quad \omega \in [0, 2\pi].$$

The number σ is still to be determined. We compare $|\lambda|^\nu$ with $f_n(\lambda) = (\lambda^{\nu+1} + n^2)$ on this circle.

We obtain

$$|\lambda|^\nu = \left|\zeta_n + \sigma e^{i\omega}\right|^\nu \leqslant n^{2\nu/(\nu+1)}\left[1 + \frac{\sigma}{n^{2/(\nu+1)}}\right]^\nu = n^{2\nu/(\nu+1)}\left[1 + \sum_{k=1}^{\nu}\binom{\nu}{k}\frac{\sigma^k}{n^{2k/(\nu+1)}}\right].$$

Hence, there exists a number M such that the following estimate holds on Γ_n:

$$\left|\lambda g(\lambda)\right| \leqslant M|\lambda|^\nu \leqslant M n^{2\nu/(\nu+1)}\left[1 + \sum_{k=1}^{\nu}\binom{\nu}{k}\frac{\sigma^k}{n^{2k/(\nu+1)}}\right]. \tag{23}$$

Note that the sum converges to zero for $n \to +\infty$. We now consider $f_n(\lambda)$ on Γ_n. We recall $\zeta_n^{(\nu+1)} = -n^2$.

$$\begin{aligned}\left|f_n(\lambda)\right| &= \left|\left[\zeta_n + \sigma e^{i\omega}\right]^{\nu+1} + n^2\right| = n^2\left|1 - \left[1 + \frac{\sigma e^{i\omega}}{\zeta_n}\right]^{\nu+1}\right| = n^2\left|\sum_{k=1}^{\nu+1}\binom{\nu+1}{k}\frac{[\sigma e^{i\omega}]^k}{\zeta_n^k}\right| \\ &\geqslant n^{2\nu/(\nu+1)}\sigma(\nu+1)\left[1 - \left|\sum_{k=2}^{\nu+1}\binom{\nu+1}{k}\frac{[\sigma e^{i\omega}]^k}{\sigma(\nu+1)\zeta_n^{k-1}}\right|\right].\end{aligned} \tag{24}$$

The sum converges to 0 for $n \to +\infty$, uniformly for $\omega \in [0, 2\pi]$. We compare with (23). We choose the number σ so to have

$$\sigma > \frac{4M}{\nu+1}.$$

With this choice the following inequality holds for every n:

$$(\nu+1)\sigma n^{2\nu/(\nu+1)} > 4M n^{2\nu/(\nu+1)}.$$

The value of σ is now fixed.

We observe that the disks are contained in $\mathcal{S}$ for every n which is large enough and that the sum in (23) tends to zero so that there exists N' such that for every $n > N'$ and $\lambda \in \Gamma_n$ we have

$$(\nu+1)\sigma n^{2\nu/(\nu+1)} > 3Mn^{2\nu/(\nu+1)}\left\{1+\sum_{k=1}^{\nu}\binom{\nu}{k}\left[\frac{\sigma^k}{n^{2k/(\nu+1)}}\right]\right\} \geqslant 3\left|\lambda g(\lambda)\right|. \tag{25}$$

We consider (24). The fact that the sum converges to zero uniformly for $\omega \in [0, 2\pi]$ shows the existence of $N_0 > N'$ such that, for $n > N_0$ and $\lambda \in \Gamma_n$ we have

$$\left|f(\lambda)\right| \geqslant \frac{3}{2}\left|\lambda g(\lambda)\right|. \tag{26}$$

Rouchè's Theorem shows that $\lambda + n^2 G(\lambda)$ has a zero μ_n in every disk bounded by Γ_n provided that $n > N_0$ and this implies $\sum 1/|\mu_n|^\gamma = +\infty$. Hence, interpolation is impossible. □

Remark 4. We observe:

- The previous results are in sharp contrast with the picture we have when $\theta \in \mathbb{R}^n$. Note that, in spite of the fact that controllability to rest is impossible, control to rest of the projection on a finite-dimensional eigenspace of A might be achievable.
- The assumption on the sector containing one of the rays $\arg z = \frac{2\pi k+\pi}{\nu+1}$ is satisfied if it happens that the required inequality on the function $g(\lambda)$ holds in an angle larger than $2\pi/(\nu+1)$ radians, since $2\pi/(\nu+1)$ is the angle among the rays.
- Theorem 3 can be applied in particular if $\alpha = 0$ and $\hat{N}(\lambda) = 1/d(\lambda)$ where $d(\lambda)$ is a polynomial of degree at least 2, as in Example 2.
- When $G(\lambda)$ has zeros or multiple poles, we proved the existence of chain of roots at which interpolation is impossible. We do not exclude the existence of chain of roots at which interpolation is possible. Let us consider the following example: the kernel is $N(t) = 1+t$ and $\alpha = 0$. The equation to be solved in order to identify the interpolation points is

  $$G(\lambda) = h(\lambda) = n^2(\lambda+1) + \lambda^3 = 0.$$

  For every number n this equation has one solution in the interval $[-1, -1+1/n]$ since $G(-1) = 0$ while $G(-1+1/n^2) = \frac{3}{n^2} - \frac{3}{n^4} + \frac{1}{n^6} > 0$. So, there is a chain of roots $\{\lambda_n^{(1)}\}$ which accumulates to -1 and interpolation at these roots is impossible, see Obstruction 1.

  In fact, we have two more chains of roots:

  $$\lambda_n^{(2)} = -in + \chi_1^{(n)}, \qquad \left|\chi_1^{(n)}\right| < 1, \qquad \lambda_n^{(3)} = in + \chi_2^{(n)}, \qquad \left|\chi_2^{(n)}\right| < 2.$$

  This is easily seen using Rouchè's Theorem. So, controllability to rest is equivalent to the following interpolation problem:

  $$\hat{u}\left(\lambda_n^{(1)}\right) = -\frac{\sqrt{\pi}}{n}\frac{[\lambda_n^{(1)}]^2}{1+\lambda_n^{(1)}}\xi_n, \qquad \hat{u}\left(\lambda_n^{(2)}\right) = -\frac{\sqrt{\pi}}{n}\frac{[\lambda_n^{(2)}]^2}{1+\lambda_n^{(2)}}\xi_n, \qquad \hat{u}\left(\lambda_n^{(3)}\right) = -\frac{\sqrt{\pi}}{n}\frac{[\lambda_n^{(3)}]^2}{1+\lambda_n^{(3)}}\xi_n.$$

  The interpolation problem for the chains $\{\lambda_n^{(2)}\}$ and $\{\lambda_n^{(3)}\}$ is solvable. The obstruction to controllability is due solely to the chain $\{\lambda_n^{(1)}\}$.
- Different "obstructions" to interpolation or moment problems can be used to prove the negative results above. For example we mention Blaschke condition which, when written with respect to the right-hand plane, is

  $$\sum_{n=1}^{\infty}\frac{\Re e\,\mu_n}{1+|\mu_n|^2} < +\infty.$$

  This is necessary and sufficient in order that for every n the interpolation problem

  $$\hat{f}(\mu_k) = \begin{cases} 1, & k = n, \\ 0, & \text{else}, \end{cases}$$

  has a solution $\hat{f}$ which is the Laplace transform of a square integrable function.

Abel kernels are often met in applications and are widely studied (see for example [5,6]). An Abel kernel has the form

$$N(t) = \frac{1}{\Gamma(1-\gamma)}t^{-\gamma} \quad \text{i.e.} \quad \hat{N}(\lambda) = \frac{1}{\lambda^{1-\gamma}},$$

where $0 < \gamma < 1$ and $\Gamma(\lambda)$ is the Euler Γ-function.

We have:

Theorem 5. *If $\hat{N}(\lambda) = 1/\lambda^{1-\gamma}$, $\gamma \in (0,1)$, then controllability to rest is impossible.*

Proof. In fact, let $u \in \mathcal{P}W_+$ and let $\theta(t)$ be the corresponding solution to Problem (i). Then we have

$$\hat{\theta}_n(\lambda) = \frac{1}{\lambda + n^2[\alpha + 1/\lambda^\sigma]}\left\{\xi_n + \frac{n}{\sqrt{\pi/2}}\left(\alpha + \frac{1}{\lambda^\sigma}\right)\hat{u}(\lambda)\right\},$$

where $\sigma = 1-\gamma$. We prove that if γ, i.e. σ is not an integer, and $\hat{\theta}_n(\lambda)$ is analytical at the origin, then $\hat{u}$ is singular there. For this, we replace $\lambda = \rho e^{i\omega}$ (ρ "small"). If $\hat{\theta}(\lambda)$ is regular at $\lambda = 0$ then we have $\lim_{\omega\to 0}\hat{\theta}(\rho e^{i\omega}) = \lim_{\omega\to 2\pi}\hat{\theta}(\rho e^{i\omega})$ and the same for $\hat{u}(\rho e^{i\omega})$. So, the following equality must hold:

$$\frac{1}{\rho + n^2[\alpha + 1/\rho^\sigma]}\left[\xi_n + \frac{n\sqrt{2}}{\sqrt{\pi}}\left(\alpha + \frac{1}{\rho^\sigma}\right)\hat{u}(\rho)\right] = \frac{1}{\rho + n^2[\alpha + 1/(\rho^\sigma e^{2\pi\sigma i})]}\left[\xi_n + \frac{n\sqrt{2}}{\sqrt{\pi}}\left(\alpha + \frac{1}{\rho^\sigma e^{2\pi\sigma i}}\right)\hat{u}(\rho)\right].$$

We reduce to the same denominator and we find the equality

$$n\left(1 - e^{-2\pi\sigma i}\right)\frac{\xi_n}{\rho} = \sqrt{\frac{2}{\pi}}\left(1 - e^{-2\pi\sigma i}\right)\hat{u}(\rho).$$

This equality must hold for every index n. If γ, i.e. σ, is not an integer then $(1 - e^{2\pi\sigma i}) \neq 0$ and we see that $\hat{u}(\rho)$ is unbounded for $\rho \to 0+$ if $\xi_n \neq 0$ for one index n. This contradiction shows that controllability to rest can be achieved only if the initial condition ξ is $\xi = 0$.

The case of Problem (ii) can be treated similarly. □

Remark 6. We note:

- The fact that θ evolves in an infinite-dimensional space has not been used in the proof of Theorem 5, which holds also if $\theta = x \in \mathbb{R}^n$ (and the control entering in the equation, of course). For example, controllability to rest is impossible even for the system

$$\dot{x}(t) = \int_0^t \frac{1}{(t-s)^\gamma} x(s)\,\mathrm{d}s + u(t), \quad x \in \mathbb{R}.$$

- The results in this section have been stated for boundary control systems with Dirichlet control. If the control is in the Neumann condition than we get similar interpolation problems and completely analogous negative results. The difference is that for Neumann controls we obtain, instead of (14),

$$\hat{\theta}_n(\lambda) = -\frac{1}{\lambda + n^2 G(\lambda)}\left(\xi_n + \phi_n(0)G(\lambda)\hat{u}(\lambda)\right).$$

  Here $\phi_n(t)$ are the eigenfunctions of the Neumann problem, so that $\phi_n(0) \neq 0$.

3.2. Case (iii): Control distributed on a subdomain

In this section the system is acted upon by a distributed control which is supported in an interval $[\beta,\gamma] \subsetneq [0,\pi]$. Hence, there is an interval $[a,a_1] \subseteq [0,\pi]$ which does not intersect $[\beta,\gamma]$. Clearly, we can choose either $a = 0$ or $a_1 = \pi$. We consider the case $a_1 = \pi$. The case $a = 0$ is treated analogously. We consider the restriction of the solution $\theta(t,\cdot)$ of Eq. (4) to the interval $(a,a_1) = (a,\pi)$. This function solves the equation

$$\theta_t = \alpha\theta_{xx} + \int_0^t N(t-s)\theta_{xx}(s)\,\mathrm{d}s, \quad x \in (a,\pi),\ t > 0, \tag{27}$$

with initial condition $\theta(0,x) = \xi(x)$ for $x \in (a,b)$ and boundary conditions

$$\theta(t,\pi) = 0, \qquad \theta(t,a+) = \theta(t,a-),$$

provided that we can give a meaning to $\theta(t,a-)$.

In this case, $\theta(t,a-) = v(t)$ acts as a boundary control and, needless to say, the negative results we already proved for the interval $(0,\pi)$ hold also if the system is considered on the interval $[a,\pi]$. Hence, if the kernel has the properties of the previous section then controllability to rest of the restriction of $\theta(t,\cdot)$ to (a,π) is impossible. So, it is also impossible to control the system to rest on the interval $[0,\pi]$.

We are going to give conditions under which the $v(t) = \theta(t, a-)$ can be computed.

We consider the following equation in $X = L^2(0, \pi)$:

$$\theta_t = \alpha A\theta + \int_0^t N(t-s)A\theta(s)\,\mathrm{d}s + Bu(t), \qquad \theta(0) = \xi, \tag{28}$$

where A is the operator in (9) and $B \in \mathcal{L}(\mathbb{R}, X)$ is given by $Bu = b(x)u$.

Lemma 7. *Let $\theta(t)$ solve* (28) *with $u \in L^2_{\mathrm{loc}}([0, +\infty); X)$ and let $\xi \in H^1_0(0, \pi) = \mathrm{dom}(-A)^{1/2}$. We have that $\theta(t, \cdot) \in H^1_0(0, \pi)$ for every $t \geqslant 0$, $\alpha > 0$ and $N(t) \in H^1(0, T)$ for every $T > 0$.*

Proof. We fix any $T > 0$. We consider case (a) first. In this case $\alpha > 0$ and we can assume $\alpha = 1$ without restriction. Then (see [3, Section 2]), $\theta(t) = \theta(t, x)$ solves

$$\theta(t) = e^{At}\xi - \int_0^t N(t-r)\theta(r)\,\mathrm{d}r + \int_0^t e^{A(t-s)}\left[N(0)\theta(s) + \int_0^s N'(s-r)\theta(r)\,\mathrm{d}r\right]\mathrm{d}s + \int_0^t e^{A(t-s)}Bu(s)\,\mathrm{d}s. \tag{29}$$

Here e^{At} is the holomorphic semigroup generated by the operator A in (9).

We consider now the following Volterra integral equation:

$$y(t) = e^{At}(-A)^{1/2}\xi + \int_0^t (-A)^{1/2}e^{A(t-s)}Bu(s)\,\mathrm{d}s - \int_0^t N(t-r)y(r)\,\mathrm{d}r + \int_0^t e^{A(t-s)}\left[y(s) + \int_0^s N'(s-r)y(r)\,\mathrm{d}r\right]\mathrm{d}s.$$

It is known that the transformation

$$u \to \int_0^t (-A)^{1/2}e^{A(t-s)}Bu(s)\,\mathrm{d}s$$

is continuous from $L^2(0, T; L^2(0, \pi))$ to itself (see [11] for an even more general case). So, both this Volterra integral equation and Eq. (29) are solvable and have a unique solution. Hence it must be $\theta(t) = (-A)^{-1/2}y(t) \in \mathrm{dom}(-A)^{1/2} = H^1_0(0, \pi)$. This shows that $v(t) = \theta(t, a-)$ can be computed, as we wanted to prove. $\square$

In conclusion:

Theorem 8. *Let the assumptions in Lemma* 7 *hold and let the kernel $N(t)$ satisfies the assumptions of one of the theorems in Section* 3.1. *Then, control system* (4)*, with $b(x)$ supported in $[\alpha, \beta] \subsetneq [0, \pi]$, is not controllable to rest.*

Note that the initial conditions we considered in the proof of Theorem 8 belong to $H^1_0(0, \pi)$ but the obstructions to interpolation applies to this set of initial conditions as well, see Lemma 1.

4. Conclusion

In this paper we presented negative results on the controllability to rest for the heat equation with memory, which show that the cases in which controllability to rest is achievable must be very particular: cases in which the Laplace transform of the kernel does not have neither branch points nor zeros or multiple poles. Even more, we proved that if $N(t)$ has a rational Laplace transform then $\hat{N}(\lambda)$ must have the form $a/(\lambda + b)$. I.e., in this case the heat equation with memory must be an “integrated” form of a wave type equation (the wave equation when $b = 0$ as in Example 3).

See [15] for a different kind of negative results due to the effect of infinite memory.

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