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ARTICLE INFO

Article history:

Received 5 July 2008

Available online 28 July 2009

Submitted by M. Passare

Keywords:

Complex q -difference equation

Meromorphic solution

Growth

ABSTRACT

We investigate the growth of transcendental meromorphic solutions of some complex q -difference equations and find lower bounds for Nevanlinna lower order for meromorphic solutions of such equations. We also obtain a q -difference version of Tumura–Clunie theorem.

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1. Introduction and results

Throughout this paper, we use standard notations in the Nevanlinna theory (see e.g. [11,15,17]). Let $f(z)$ be a meromorphic function. Here and in the following the word “meromorphic” means meromorphic in the whole complex plane. Let $m(r, f)$, $N(r, f)$ and $T(r, f)$ denote the proximity function, the counting function and the characteristic function of $f(z)$ respectively. We also use notations $\rho(f)$, $\lambda(f)$, $\bar{\lambda}(f)$ and $\lambda(\frac{1}{f})$, $\bar{\lambda}(\frac{1}{f})$ for the order, the exponent of convergence of zeros (distinct zeros) and the exponent of convergence of poles (distinct poles) of $f(z)$ respectively.

Recently, a number of papers (see e.g. [2,4,5,10]) focus on the complex difference and q -difference. There are also papers (see e.g. [1,3,8,9,13,14,16]) focusing on the existence and the growth of meromorphic solutions of difference equations and q -difference equations.

In [13], Heittokangas et al. considered the essential growth problem for transcendental meromorphic solutions of complex difference equations, which is to find lower bounds for their characteristic functions, and obtained the following results.

Theorem A. Let $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$ and let $m \geq 2$. Suppose that y is a transcendental meromorphic solution of difference equation

$$\sum_{i=1}^n a_i(z)y(z+c_i) = \sum_{i=0}^m b_i(z)y(z)^i \quad (1.1)$$

with rational coefficients $a_i(z)$, $b_i(z)$. Denote $C = \max\{|c_1|, \dots, |c_n|\}$.

- (1) If y is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that $\log M(r, y) \geq Km^{r/C}$ holds for all $r \geq r_0$.
- (2) If y has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that $n(r, y) \geq Km^{r/C}$ holds for all $r \geq r_0$.

[☆] This project was supported by the National Natural Science Foundation of China (No. 10871076).

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(3) Thus, all transcendental meromorphic solutions of (1.1) have infinite lower order.

Theorem B. Let $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$ and y be a transcendental meromorphic solution of difference equation

$$\sum_{i=1}^n d_i(z)y(z+c_i) = \frac{a_0(z) + a_1(z)y(z) + \dots + a_p(z)y(z)^p}{b_0(z) + b_1(z)y(z) + \dots + b_t(z)y(z)^t}, \quad (1.2)$$

where all coefficients in (1.2) are of growth $o(T(r, y))$ without an exceptional set as $r \rightarrow \infty$, and d_i 's are non-vanishing. If $d = \max\{p, t\} > n$, then for any ε ($0 < \varepsilon < \frac{d-n}{d+n}$), there exists an $r_0 > 0$ such that $T(r, y) \geq K(\frac{d}{n}(\frac{1-\varepsilon}{1+\varepsilon}))^{r/C}$ for all $r \geq r_0$, where $C = \max\{|c_1|, \dots, |c_n|\}$ and $K > 0$ is a constant.

Theorem C. Suppose that all coefficients in (1.2) are of growth $S(r, y)$ and that all other assumptions of Theorem B hold. Then $\mu(y) = \infty$.

In this paper, we consider a similar growth problem for transcendental meromorphic solutions of complex q -difference equations instead of difference equations, where the usual shift $f(z+c)$ will be replaced by the q -shift $f(qz)$, and obtain the following results.

Theorem 1. Suppose that f is a transcendental meromorphic solution of equation

$$\sum_{j=1}^n a_j(z)f(q^j z) = \sum_{i=0}^d b_i(z)f(z)^i, \quad (1.3)$$

where $q \in \mathbb{C}$, $|q| > 1$, $d \geq 2$ and the coefficients $a_j(z)$, $b_i(z)$ are rational functions.

- (1) If f is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that for all $r \geq r_0$, $\log M(r, f) \geq Kd^{\frac{\log r}{n \log |q|}}$.
- (2) If f has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that for all $r \geq r_0$, $n(r, f) \geq Kd^{\frac{\log r}{n \log |q|}}$.
- (3) Thus, the lower order of f satisfies $\mu(f) \geq \frac{\log d}{n \log |q|}$.

Example 1. A function $f(z) = \frac{e^z}{z}$ satisfies $\sum_{j=1}^n \frac{2^j}{z^{2^j-1}} f(2^j z) = \sum_{i=1}^n f(z)^{2^i}$, where $|q| = |2| > 1$. Since $n < 2^n$ for all $n \in \mathbb{N}$, we have $\log M(r, f) = r - \log r \geq Kn^{\frac{\log r}{n \log 2}}$ ($r \rightarrow \infty$) and $\mu(f) = \sigma(f) = 1 > \frac{\log n}{n \log 2}$. This shows that the strict inequality in the result " $\mu(f) \geq \frac{\log d}{n \log |q|}$ " of Theorem 1 may hold.

Theorem 2. Suppose that f is a transcendental meromorphic solution of equation

$$\sum_{j=1}^n a_j(z)f(q^j z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \quad (1.4)$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$ are rational functions and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geq 2$. If f has infinitely many poles, then for sufficiently large r , $n(r, f) \geq Kd^{\frac{\log r}{n \log |q|}}$ holds for some constant $K > 0$. Thus, the lower order of f , which has infinitely many poles, satisfies $\mu(f) \geq \frac{\log d}{n \log |q|}$.

Theorem 3. Suppose that f is a transcendental meromorphic solution of equation

$$\frac{\sum_{\lambda \in I} d_\lambda(z)f(qz)^{i_{\lambda,1}} f(q^2 z)^{i_{\lambda,2}} \dots f(q^n z)^{i_{\lambda,n}}}{\sum_{\mu \in J} e_\mu(z)f(qz)^{j_{\mu,1}} f(q^2 z)^{j_{\mu,2}} \dots f(q^n z)^{j_{\mu,n}}} = \frac{a_0(z) + a_1(z)f(z) + \dots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \dots + b_t(z)f(z)^t}, \quad (1.5)$$

where $I = \{(i_{\lambda,1}, i_{\lambda,2}, \dots, i_{\lambda,n})\}$, $J = \{(j_{\mu,1}, j_{\mu,2}, \dots, j_{\mu,n})\}$ are two finite index sets,

$$\max_{\lambda, \mu} \{i_{\lambda,1} + i_{\lambda,2} + \dots + i_{\lambda,n}, j_{\mu,1} + j_{\mu,2} + \dots + j_{\mu,n}\} = \sigma,$$

$q \in \mathbb{C}$, $|q| > 1$ and all coefficients of (1.5) are of growth $S(r, f)$. If $d = \max\{p, t\} > 2n\sigma$, then for sufficiently large r , $T(r, f) \geq K(\frac{d}{2n\sigma})^{\frac{\log r}{n \log |q|}}$, where $K (> 0)$ is a constant. Thus, the lower order of f satisfies $\mu(f) \geq \frac{\log d - \log 2n\sigma}{n \log |q|}$.

Recently, meromorphic solutions of complex difference equations have gained increasing interest, due to the apparent role of the existence of such solutions of finite order for the integrability of difference equations. For example, in [9], Halburd and Korhonen showed that the existence of sufficiently many meromorphic solutions of finite order is enough to single out the second difference Painlevé equation

$$f(z+1) + f(z-1) = \frac{(\lambda z + \mu)f(z) + \nu}{1 - f(z)^2},$$

where λ, μ, ν are complex constants, from a more general class of equations

$$f(z+1) + f(z-1) = R(z, f(z)),$$

where R is rational in both arguments, see [9, Theorem 4.1]. A key tool in the reasoning of [9] is the following result.

Theorem D. Let f be a non-rational meromorphic solution of equation

$$f(z+1) + f(z-1) = R(z, f(z)) = \frac{c_2(z)f(z)^2 + c_1(z)f(z) + c_0(z)}{f(z)^2 + a(z)f(z) + b(z)},$$

where $R(z, f)$ is irreducible in f , and a, b, c_0, c_1, c_2 are rational functions such that $a^2 - 4b$ does not vanish identically. If there exist $r_0 > 0$ and $\alpha < 2$ such that

$$\bar{N}(r, f(z+1) + f(z-1)) \leq \alpha \bar{N}(r+3, f(z))$$

for all $r > r_0$, then f has to be of infinite order.

Laine, Rieppo and Silvennoinen extended Theorem D into more general type in [16], which is omitted here. In this paper, we proceed to prove a q -difference counterpart of the above results.

Theorem 4. Suppose that f is a transcendental meromorphic solution of equation

$$\sum_{j=1}^n a_j(z) f(q^j z) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \quad (1.6)$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$ are non-vanishing small functions relative to f and P, Q are relatively prime polynomials in f over the field of small functions relative to f . Moreover, we assume that $t = \deg_f Q > 0$, $n = \max\{p, t\} = \max\{\deg_f P, \deg_f Q\}$, and that, without restricting generality, Q is a monic polynomial. If there exists $\alpha \in [0, n)$ such that for all sufficiently large r ,

$$\bar{N}\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) \leq \alpha \bar{N}(|q|^n r, f(z)) + S(r, f), \quad (1.7)$$

then either the order $\rho(f) > 0$, or $Q(z, f(z)) \equiv (f(z) + h(z))^t$, where $h(z)$ is a small meromorphic function.

2. Lemmas for proofs of theorems

Lemma 1 (Valiron–Mohon'ko). (See [15].) Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f ,

$$R(z, f(z)) = \frac{\sum_{i=0}^m a_i(z) f(z)^i}{\sum_{j=0}^n b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j} \{T(r, a_i), T(r, b_j)\}$.

The inequality

$$T\left(r, \sum_{J \subseteq I} \alpha_J \left(\prod_{j \in J} f_j\right)\right) \leq \sum_{i \in I} T(r, f_i), \quad \alpha_J \text{ are constants}$$

can be found in [18, p. 1004], which is an important tool while considering complex functional equations. A difference version can also be found in [7, Theorem B.16]. Next, we prove the following lemma, which is more general than the above two results.

Lemma 2. Let f_1, f_2, \dots, f_n be meromorphic functions. Then

$$T\left(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}\right) \leq \sigma \sum_{i=1}^n T(r, f_i) + \log s,$$

where $I = \{(i_{\lambda,1}, i_{\lambda,2}, \dots, i_{\lambda,n})\}$ is an index set consisting of s elements, and $\sigma = \max_{\lambda \in I} \{i_{\lambda,1} + i_{\lambda,2} + \dots + i_{\lambda,n}\}$.

Proof. We prove first that

$$N\left(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}\right) \leq \sigma \sum_{i=1}^n N(r, f_i). \quad (2.1)$$

If $\sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}$ has a pole at z_0 of multiplicity K , then there exists at least one index $\lambda \in I$ such that $f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}$ has a pole at z_0 of multiplicity $K_1 (\geq K)$. Thus there exists a subset $\{s_1, s_2, \dots, s_\nu\}$ of $\{1, 2, \dots, n\}$ such that each one of $f_{s_1}, f_{s_2}, \dots, f_{s_\nu}$ has a pole at z_0 . Suppose $m_{s_j} (\geq 1)$ ($j = 1, 2, \dots, \nu$) are the multiplicities of f_{s_j} at z_0 respectively. Then we have

$$\begin{aligned} K &\leq K_1 = m_{s_1} i_{\lambda, s_1} + m_{s_2} i_{\lambda, s_2} + \dots + m_{s_\nu} i_{\lambda, s_\nu} \\ &\leq (m_{s_1} + m_{s_2} + \dots + m_{s_\nu})(i_{\lambda, s_1} + i_{\lambda, s_2} + \dots + i_{\lambda, s_\nu}) \leq \sigma(m_{s_1} + m_{s_2} + \dots + m_{s_\nu}). \end{aligned}$$

So, $n(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}) \leq \sigma \sum_{i=1}^n n(r, f_i)$ holds, by which (2.1) follows.

We next prove that

$$m\left(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}\right) \leq \sigma \sum_{i=1}^n m(r, f_i) + \log s. \quad (2.2)$$

For $i = 1, 2, \dots, n$, we define $f_i^*(z) = f_i(z)$ when $|f_i(z)| > 1$ and $f_i^*(z) = 1$ when $|f_i(z)| \leq 1$. Thus,

$$\begin{aligned} \left| \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}} \right| &\leq \sum_{\lambda \in I} |f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}| \leq \sum_{\lambda \in I} |f_1^{*i_{\lambda,1}} f_2^{*i_{\lambda,2}} \dots f_n^{*i_{\lambda,n}}| \\ &\leq \sum_{\lambda \in I} |f_1^* f_2^* \dots f_n^*|^{i_{\lambda,1} + i_{\lambda,2} + \dots + i_{\lambda,n}} \leq \sum_{\lambda \in I} |f_1^* f_2^* \dots f_n^*|^\sigma = s |f_1^* f_2^* \dots f_n^*|^\sigma. \end{aligned} \quad (2.3)$$

By the definition of f_i^* , $i = 1, 2, \dots, n$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_i(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f_i^*(re^{i\theta})| d\theta. \quad (2.4)$$

Thus, by (2.3) and (2.4) we have

$$m\left(r, \sum_{\lambda \in I} f_1^{i_{\lambda,1}} f_2^{i_{\lambda,2}} \dots f_n^{i_{\lambda,n}}\right) \leq \sigma \sum_{i=1}^n m(r, f_i^*) + \log s \leq \sigma \sum_{i=1}^n m(r, f_i) + \log s,$$

that is (2.2). Combining (2.1) and (2.2), we get the result immediately. \square

Lemma 3. (See [15].) Let $g : (0, +\infty) \rightarrow \mathbb{R}$, $h : (0, +\infty) \rightarrow \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

The following Lemma 4 is a variant of the famous Tumura–Clunie theorem (see [6,19]).

Lemma 4. (See [20].) Let f be a meromorphic function, and let ϕ be given by $\phi = f^n + a_{n-1}f^{n-1} + \dots + a_0$, where a_0, a_1, \dots, a_{n-1} are small meromorphic functions relative to f . Then either

$$\phi = \left(f + \frac{a_{n-1}}{n}\right)^n$$

or

$$T(r, f) \leq \bar{N}\left(r, \frac{1}{\phi}\right) + \bar{N}(r, f) + S(r, f).$$

Lemma 5. (See [16].) Let f be a non-constant meromorphic function and let $P(z, f)$, $Q(z, f)$ be two polynomials in f with meromorphic coefficients small relative to f . If P and Q have no common factors of positive degree in f over the field of small functions relative to f , then

$$\bar{N}\left(r, \frac{1}{Q(z, f)}\right) \leq \bar{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + S(r, f).$$

The following Lemma 6 is a special case of [12, Lemma 4].

Lemma 6. If $T: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing function such that $\overline{\lim}_{r \rightarrow \infty} \frac{\log T(r)}{\log r} = 0$, then the set $E = \{r: T(C_1 r) \geq C_2 T(r)\}$ has logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

The following result follows immediately by Lemma 6.

Lemma 7. Let f be a non-constant meromorphic function, $\beta > 1$, $\alpha < 1$ are given constants, and let $F \subset \mathbb{R}^+$ be the set of all r such that $\bar{N}(r, f) \leq \alpha \bar{N}(\beta r, f)$. If the logarithmic density of F is non-zero, that is $\log \text{dens } F > 0$, then the exponent of convergence of distinct poles $\lambda(\frac{1}{f})$ is non-zero. Thus, $\rho(f)$ is non-zero.

Remark. (See [3].) We shall also use the observation that

$$M(r, f(qz)) = M(|q|r, f), \quad N(r, f(qz)) = N(|q|r, f) + O(1)$$

and

$$T(r, f(qz)) = T(|q|r, f) + O(1)$$

hold for any meromorphic function f and any non-zero constant q .

3. Proof of Theorem 1

We multiply out the denominators of the coefficients $a_j(z)$, $b_i(z)$ in (1.3) to obtain

$$\sum_{j=1}^n A_j(z) f(q^j z) = \sum_{i=0}^d B_i(z) f(z)^i, \quad (3.1)$$

where the coefficients $A_j(z)$, $B_i(z)$ are polynomials.

(1) Suppose first that f , the solution of (1.3) (or (3.1)), is transcendental entire. Set $p_j = \deg A_j$ ($j = 1, 2, \dots, n$), $q_i = \deg B_i$ ($i = 0, 1, \dots, d$). Taking $m = \max\{p_1, \dots, p_n\} + 1$, and noting that $|q| > 1$, $M(r, f(q^j z)) = M(|q|^j r, f)$, we have that

$$M\left(r, \sum_{i=0}^d B_i(z) f(z)^i\right) = M\left(r, \sum_{j=1}^n A_j(z) f(q^j z)\right) \leq nr^m M(|q|^n r, f), \quad (3.2)$$

when r is sufficiently large. Since f is a transcendental entire function and B_i ($i = 0, 1, \dots, d$) are polynomials, we have $M(r, \sum_{i=0}^{d-1} B_i(z) f(z)^i) = o(M(r, f(z)^d))$. Thus

$$M\left(r, \sum_{i=0}^d B_i(z) f(z)^i\right) \geq \frac{1}{2} M(r, B_d(z) f(z)^d), \quad (3.3)$$

when r is sufficiently large. We have by (3.2) and (3.3) that

$$\log M(|q|^n r, f) \geq d \log M(r, f) + g(r), \quad (3.4)$$

where $|g(r)| < K \log r$ for some $K > 0$ and sufficiently large r . Iterating (3.4), we have

$$\log M(|q|^{nk} r, f) \geq d^k \log M(r, f) + E_k(r) \quad (k \in \mathbb{N}), \quad (3.5)$$

where

$$\begin{aligned} |E_k(r)| &= |d^{k-1} g(r) + d^{k-2} g(|q|^n r) + \dots + g(|q|^{n(k-1)} r)| \\ &\leq K d^{k-1} \sum_{i=0}^{k-1} \frac{\log(|q|^{ni} r)}{d^i} \leq K d^{k-1} \sum_{i=0}^{\infty} \frac{\log(|q|^{ni} r)}{d^i}. \end{aligned} \quad (3.6)$$

We assert that the series $\sum_{i=0}^{\infty} \frac{\log(|q|^{ni}r)}{d^i}$ is convergent when $d \geq 2$. In fact, since $|q| > 1$, we have $\log(|q|^{ni}r) = \log |q|^{ni} + \log r \leq \log |q|^{ni} \cdot \log r = in \log |q| \cdot \log r$. So, we have

$$\sum_{i=0}^{\infty} \frac{\log(|q|^{ni}r)}{d^i} \leq n \log |q| \cdot \log r \sum_{i=0}^{\infty} \frac{i}{d^i}. \quad (3.7)$$

By (3.7) and the fact that $\sum_{i=0}^{\infty} \frac{i}{d^i} < \infty$ when $d \geq 2$, we see that the series $\sum_{i=0}^{\infty} \frac{\log(|q|^{ni}r)}{d^i}$ is convergent. Therefore, by (3.6) and (3.7), we have

$$|E_k(r)| \leq K' d^k \log r, \quad (3.8)$$

where $K' (> 0)$ is some constant. Since f is a transcendental entire function, we have

$$\log M(r, f) \geq 2K' \log r \quad (3.9)$$

for sufficiently large r . By (3.5), (3.8) and (3.9), we see that there exists $r_0 \geq e$ such that for $r \geq r_0$,

$$\log M(|q|^{nk}r, f) \geq K' d^k \log r. \quad (3.10)$$

Thus, for each sufficiently large s , there exists a $k \in \mathbb{N}$ such that

$$s \in [|q|^{nk}r_0, |q|^{n(k+1)}r_0), \quad \text{i.e. } k > \frac{\log s - \log(|q|^{nk}r_0)}{n \log |q|}. \quad (3.11)$$

We have by (3.10) and (3.11) that

$$\log M(s, f) \geq \log M(|q|^{nk}r_0, f) \geq K' d^k \log r_0 \geq K' d^k \geq K'' d^{\frac{\log s}{n \log |q|}},$$

where $K'' = K' d^{-\frac{\log(|q|^{nk}r_0)}{n \log |q|}}$. Hence, we have proved the assertion when f is entire.

Suppose now that f , the solution of (1.3) (or (3.1)), is meromorphic with finitely many poles. Then there exists a polynomial $P(z)$ such that $g(z) = P(z)f(z)$ is entire. Substituting $f(z) = \frac{g(z)}{P(z)}$ into (3.1) and again multiplying away the denominators, we will obtain a equation similar to (3.1). Applying the same reasoning above to $g(z)$, we obtain that for sufficiently large r ,

$$\log M(r, f) = \log M(r, g) + O(1) \geq (K'' - \varepsilon) d^{\frac{\log r}{n \log |q|}} = K''' d^{\frac{\log r}{n \log |q|}},$$

where $K''' (> 0)$ is some constant. Thus, part (1) is proved.

(2) Since part (2) is a particular case of Theorem 2, we omit the proof here.

(3) By $K d^{\frac{\log r}{n \log |q|}} \leq \log M(r, f) \leq 3T(2r, f)$ or $K d^{\frac{\log r}{n \log |q|}} \leq n(r, f) \leq \frac{1}{\log 2} N(2r, f) \leq \frac{1}{\log 2} T(2r, f)$ respectively, we immediately obtain $\mu(f) \geq \frac{\log d}{n \log |q|}$.

4. Proof of Theorem 2

Since the coefficients of $R(z, f)$ are rational functions and f has infinitely many poles, we can choose a sufficiently large constant $R (> 0)$ such that the coefficients of $R(z, f)$ have no zeros or poles in $\{z \in \mathbb{C}: |z| > R\}$ and that we can choose a pole z_0 of f of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Then the right side of (1.4) has a pole of multiplicity $d\tau$ at z_0 . Hence, there exists at least one index $j_1 \in \{1, 2, \dots, n\}$ such that $q^{j_1}z_0$ is a pole of f of multiplicity $\tau_1 \geq d\tau$. Replacing z by $q^{j_1}z$ in (1.4), we have

$$\sum_{j=1}^n a_j(q^{j_1}z_0) f(q^{j+j_1}z_0) = R(q^{j_1}z_0, f(q^{j_1}z_0)). \quad (4.1)$$

Since $|q^{j_1}z_0| > |z_0|$, the coefficients of $R(z, f)$ cannot have a zero or a pole at $q^{j_1}z_0$, thus the right side of (4.1) has a pole of multiplicity $d\tau_1$ at $q^{j_1}z_0$. Hence, there exists at least one index $j_2 \in \{1, 2, \dots, n\}$ such that $q^{j_1+j_2}z_0$ is a pole of f of multiplicity $\tau_2 \geq d\tau_1 \geq d^2\tau$.

We proceed to follow the step above. Since the coefficients of $R(z, f)$ have no zeros or poles in $\{z \in \mathbb{C}: |z| > R\}$ and f has infinitely many poles again, we may construct poles $\zeta_k = q^{j_1+j_2+\dots+j_k}z_0$ ($j_i \in \{1, 2, \dots, n\}$, $i = 1, 2, \dots, k$) of f of multiplicity τ_k for all $k \in \mathbb{N}$, satisfying $\tau_k \geq d^k\tau \rightarrow \infty$ ($k \rightarrow \infty$). Clearly, $|\zeta_k| \rightarrow \infty$ ($k \rightarrow \infty$). So for sufficiently large k , say $k \geq k_0$,

$$\tau d^k \leq \tau(1 + d + \dots + d^k) \leq n(|\zeta_k|, f) = n(|q|^{j_1+j_2+\dots+j_k}|z_0|, f) \leq n((|q|^n)^k|z_0|, f). \quad (4.2)$$

Thus, for each sufficiently large r , there exists a $k \in \mathbb{N}$ such that $r \in [|q|^{nk}|z_0|, |q|^{n(k+1)}|z_0|)$. Using the same method as in the proof of Theorem 1(1), we obtain by (4.2) that

$$n(r, f) \geq \tau d^k \geq \tau d^{\frac{\log r - \log(|q|^n |z_0|)}{n \log |q|}} \geq K d^{\frac{\log r}{n \log |q|}},$$

where $K = \tau d^{-\frac{\log(|q|^n |z_0|)}{n \log |q|}}$.

Finally, since $K d^{\frac{\log r}{n \log |q|}} \leq n(r, f) \leq \frac{1}{\log 2} N(2r, f) \leq \frac{1}{\log 2} T(2r, f)$ for all $r \geq r_0$, we obtain $\mu(f) \geq \frac{\log d}{n \log |q|}$.

5. Proof of Theorem 3

By Remark, we have $T(r, f(q^j z)) = T(|q|^j r, f) + O(1)$. Noting that $|q| > 1$, by (1.5), Lemmas 1 and 2, we have that for any given ε ($0 < \varepsilon < \frac{d-2n\sigma}{d+2n\sigma}$),

$$\begin{aligned} d(1-\varepsilon)T(r, f) &\leq dT(r, f) + S(r, f) \leq 2\sigma \sum_{j=1}^n T(|q|^j r, f) + S(r, f) \\ &\leq 2n\sigma T(|q|^n r, f) + S(r, f) \leq 2n\sigma(1+\varepsilon)T(|q|^n r, f), \end{aligned} \quad (5.1)$$

outside of a possible exceptional set of finite linear measure. By Lemma 3 and (5.1), it follows that for any given $\alpha > 1$, there exists an $r_0 > 0$ such that

$$d(1-\varepsilon)T(r, f) \leq 2n\sigma(1+\varepsilon)T(\alpha|q|^n r, f)$$

holds for all $r \geq r_0$. Hence

$$T(\alpha|q|^n r, f) \geq \frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)} T(r, f), \quad r \geq r_0. \quad (5.2)$$

Inductively, for any $k \in \mathbb{N}$, we have by (5.1) and (5.2) that

$$T((\alpha|q|^n)^k r, f) \geq \left(\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)} \right)^k T(r, f), \quad r \geq r_0. \quad (5.3)$$

For sufficiently large s , using the same method as in the proof of Theorem 1, we obtain by (5.3) that

$$T(s, f) \geq \left(\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)} \right)^{\frac{\log s - \log r}{\log \alpha |q|^n}} T(r_0, f). \quad (5.4)$$

Letting $\varepsilon \rightarrow 0$ and $\alpha \rightarrow 1$, we have by (5.4) that

$$T(s, f) \geq \left(\frac{d}{2n\sigma} \right)^{\frac{\log s - \log |q|^n r_0}{\log |q|^n}} T(r_0, f) = K \left(\frac{d}{2n\sigma} \right)^{\frac{\log s}{\log |q|^n}},$$

where $K = \left(\frac{d}{2n\sigma} \right)^{\frac{-\log |q|^n r_0}{\log |q|^n}} T(r_0, f)$ (> 0) is a constant.

Thus, we get $\mu(f) \geq \frac{\log d - \log 2n\sigma}{n \log |q|}$.

6. Proof of Theorem 4

Suppose the second alternative of the assertion do not hold. Then by Lemmas 4 and 5, we get

$$T(r, f) \leq \bar{N}\left(r, \frac{1}{Q}\right) + \bar{N}(r, f) + S(r, f) \leq \bar{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + \bar{N}(r, f) + S(r, f). \quad (6.1)$$

By (1.6), (1.8) and (6.1), we obtain that

$$\begin{aligned} T(r, f) - \bar{N}(r, f) &\leq \bar{N}\left(r, \frac{P(z, f)}{Q(z, f)}\right) + S(r, f) \\ &= \bar{N}\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + S(r, f) \leq \alpha \bar{N}(|q|^n r, f) + S(r, f). \end{aligned} \quad (6.2)$$

Assume contrary to the assertion, that is $\rho(f) = 0$. Then Lemma 6 and Remark imply that for any constant $C > 1$,

$$T(r, f(q^j z)) = T(|q|^j r, f) + O(1) < CT(r, f), \quad j = 1, \dots, n \quad (6.3)$$

on a set of logarithmic density 1.

We see that if a set is of finite linear measure, then the set is of logarithmic density 0. Thus, combining (6.3), we obtain that for $j = 1, \dots, n$,

$$S(r, f(q^j z)) = o(T(r, f(z))) \quad (6.4)$$

on a set of logarithmic density 1. Now, (6.2) applies to $f(q^j z)$ ($j = 1, \dots, n$), then by (6.4) we have

$$T(r, f(q^j z)) - \bar{N}(r, f(q^j z)) \leq \alpha \bar{N}(|q|^n r, f(q^j z)) + o(T(r, f)) \quad (6.5)$$

on a set of logarithmic density 1. Applying Lemma 1 on both sides of (1.6), we conclude by (1.7) and Remark that

$$\begin{aligned} nT(r, f) &= T\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + o(T(r, f)) \\ &= T\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) - \bar{N}\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + \bar{N}\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + o(T(r, f)) \\ &\leq m\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + N_1\left(r, \sum_{j=1}^n a_j(z) f(q^j z)\right) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &\leq \sum_{j=1}^n (m(r, f(q^j z)) + N_1(r, f(q^j z))) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &= \sum_{j=1}^n (T(r, f(q^j z)) - \bar{N}(r, f(q^j z))) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \end{aligned} \quad (6.6)$$

on a set of logarithmic density 1, where $N_1(r, f) = N(r, f) - \bar{N}(r, f)$. We have by (6.5) and (6.6) that

$$\begin{aligned} nT(r, f) &\leq \sum_{j=1}^n \alpha \bar{N}(|q|^n r, f(q^j z)) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &\leq n\alpha \bar{N}(|q|^{2n} r, f) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &= (n+1)\alpha \bar{N}(|q|^{2n} r, f) + o(T(r, f)) \end{aligned} \quad (6.7)$$

on a set of logarithmic density 1. Therefore, we have obtained by (6.7) that

$$T(r, f) - \bar{N}(r, f) \leq \frac{n+1}{n} \alpha \bar{N}(|q|^{2n} r, f) - \bar{N}(r, f) + o(T(r, f)) \quad (6.8)$$

on a set of logarithmic density 1.

We now proceed, inductively, to prove that

$$T(r, f) - \bar{N}(r, f) \leq \frac{n+m}{n} \alpha \bar{N}(|q|^{2mn} r, f) - m\bar{N}(r, f) + o(T(r, f)) \quad (6.9)$$

on a set of logarithmic density 1. Having already proved the case $m = 1$ in (6.8), we continue to the inductive step. To this end, observe that the above reasoning also applies to the functions $f(q^j z)$, $j = 1, 2, \dots, n$ instead of $f(z)$. Therefore, we may apply the inductive assertion to obtain by (6.6) that

$$\begin{aligned} nT(r, f) &\leq \sum_{j=1}^n (T(r, f(q^j z)) - \bar{N}(r, f(q^j z))) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &\leq \sum_{j=1}^n \left(\frac{n+m}{n} \alpha \bar{N}(|q|^{2mn} r, f(q^j z)) - m\bar{N}(r, f(q^j z)) \right) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &\leq \sum_{j=1}^n \left(\frac{n+m}{n} \alpha \bar{N}(|q|^{(2m+1)n} r, f(z)) - m\bar{N}(r, f) \right) + \alpha \bar{N}(|q|^n r, f) + o(T(r, f)) \\ &\leq (n+m+1)\alpha \bar{N}(|q|^{(2m+1)n} r, f) - mn\bar{N}(r, f) + o(T(r, f)) \\ &\leq (n+m+1)\alpha \bar{N}(|q|^{2(m+1)n} r, f) - mn\bar{N}(r, f) + o(T(r, f)) \end{aligned}$$

on a set of logarithmic density 1. Therefore, we conclude that

$$T(r, f) - \bar{N}(r, f) \leq \frac{n+m+1}{n} \alpha \bar{N}(|q|^{2(m+1)n} r, f) - (m+1) \bar{N}(r, f) + o(T(r, f))$$

on a set of logarithmic density 1, completing the induction (6.9).

Thus, noting that $T(r, f) - o(T(r, f)) \geq 0$, we immediately see by (6.9) that

$$\bar{N}(r, f) \leq \frac{n+m}{n(m-1)} \alpha \bar{N}(|q|^{2mn} r, f)$$

on a set of logarithmic density 1. Setting $\alpha' = \frac{n+m}{n(m-1)} \alpha$, we obtain

$$\bar{N}(r, f) \leq \alpha' \bar{N}(|q|^{2mn} r, f) \quad (6.10)$$

on a set of logarithmic density 1. Since $\alpha \in [0, n)$, we see that for sufficiently large m ,

$$\alpha' = \frac{n+m}{n(m-1)} \alpha = \left(\frac{1}{m-1} + \frac{1}{n} \frac{m}{m-1} \right) \alpha < 1. \quad (6.11)$$

So by Lemma 7, (6.10), (6.11) and $|q|^{2mn} > 1$, we get $\rho(f) > 0$, a contradiction.

Acknowledgments

The authors are grateful to the referees for a number of helpful suggestions to improve the readability of the paper.

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