Value-sharing of meromorphic functions on a Riemann surface

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ARTICLE INFO	ABSTRACT
	We present some results on two meromorphic functions from S to $\widehat{\mathbb{C}}$ sharing a number of values where S is a Riemann surface of one of the following types: compact, compact minus finitely many points, the unit disk, a torus, the complex plane.

0. Introduction

We write $\widehat{\mathbb{C}}$ (or sometimes $\mathbb{P}^1$) for the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Let R be a Riemann surface. Two meromorphic functions $f_1, f_2 : R \to \widehat{\mathbb{C}}$ are said to share the value $a \in \widehat{\mathbb{C}}$ if for every $u \in R$ we have $f_1(u) = a \Leftrightarrow f_2(u) = a$. If moreover f_1 takes the value a at each u with the same multiplicity as f_2, we say that f_1 and f_2 share the value a CM (counting multiplicities). If we do not know the multiplicities or do not care, we say that a is shared IM (ignoring multiplicities).

In [12] Sauer proved among other results that if S is a compact Riemann surface of genus $g > 0$ then two different non-constant meromorphic functions on S cannot share more than $2 + 2\sqrt{g}$ values. If moreover one of the shared values is shared CM, the bound can be strengthened to $\frac{1}{4}(9 + \sqrt{32g + 17})$. In [13] these bounds have been slightly improved, and bounds in terms of other invariants of S have also been given.

In this paper we show that there do indeed exist compact Riemann surfaces that can carry two meromorphic functions with many shared values and that one can even prescribe the shared values (Theorem 1). This question had been left open in [12] and [13]. The best examples one can find in the literature have 4 shared values. We also investigate how many values two functions that are meromorphic on a punctured compact Riemann surface can share (Section 3). In the last section we use the concept of weighted sharing to refine some known results.

1. Basic facts

Recall that the gonality d of a compact Riemann surface S is defined to be the smallest integer m such that S can be realized as an m-sheeted branched covering of the Riemann sphere. Equivalently, d is the smallest possible degree of a non-constant meromorphic function on S.

Theorem A. *(See [13].) Let S be a compact Riemann surface of genus $g > 0$ and gonality d. Let $f_1, f_2 : S \to \widehat{\mathbb{C}}$ be two different non-constant meromorphic functions sharing n values.*

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(a) *Then*

$$n \leqslant \min\left\{2 + \sqrt{2g+2}, 2d + 1, 4 + \frac{2(g-1)}{d}\right\}.$$

(b) *If moreover one of these n values is shared CM, then we even have*

$$n \leqslant \min\left\{\frac{1}{2}(5 + \sqrt{4g+5-2d}), 2d + 1, 3 + \frac{2(g-1)}{d}\right\}.$$

(c) *If $g = 0$, i.e. $S = \mathbb{P}^1$, the optimal bound in both cases is known to be* 3.

Actually, [3, Theorem 1] claims an inequality that is stronger than the key lemmas in [12] and [13] and that would consequently imply stronger results than Theorem A. But trying to follow the proof in [3], I have not been able to obtain the claimed inequality.

Besides the three articles already mentioned, the only other instance in the literature dealing with value-sharing of meromorphic functions on a compact Riemann surface (other than the Riemann sphere) seems to be [2]. It investigates functions sharing sets of values and is formulated in terms of functions on algebraic curves, which allows working over any algebraically closed field of characteristic 0, not just over $\mathbb{C}$. We point out that by the nature of their proofs the results in [12,13] and some of the results we will obtain below (notably Theorem 1) also hold in such a general setting.

The most famous result on value-sharing is of course Rolf Nevanlinna's theorem, that two meromorphic functions in the complex plane that share 5 values must be equal. A true generalization of this result to Riemann surfaces would, for example, be a statement about meromorphic functions on a punctured compact Riemann surface. I thank Jun-Muk Hwang who pointed this out to me and thus triggered my interest in this problem. Obviously, this is more difficult than working on a compact Riemann surface, as there are many more meromorphic functions, and algebraic arguments cannot suffice. Luckily, the local result one would like to have has already been proved by R. Nevanlinna.

Theorem B. *(See [9].) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions in a neighbourhood of the point $z = \infty$, where they have an essential singularity. If for FIVE different values (finite or not) of w the equalities*

$$f_1(z) = w, \qquad f_2(z) = w,$$

outside some circle $|z| = r_0$ are satisfied for exactly the same values of z, then f_1 and f_2 are identical.

Of course one can transform this into a statement about meromorphic functions with an essential singularity in a punctured disk.

Together with Theorem A(c), which is folklore, Theorem B implies Nevanlinna's Five Value Theorem in the complex plane in the same way as the Big Picard Theorem implies the Little Picard Theorem.

2. Compact Riemann surfaces

No example seems to be known of a compact Riemann surface S and two different non-constant meromorphic functions $f_1, f_2 : S \to \widehat{\mathbb{C}}$ that share more than 4 values. Moreover, [13] shows that "most" compact Riemann surfaces of a given genus do not allow more than 7 shared values.

So it is legitimate to wonder whether in this case there is an absolute bound on the number of shared values, valid for all compact Riemann surfaces. The final remarks of [13] advocate this point of view, suggesting that a possible approach could be to prove the existence of a bound for the number of shared values of two different meromorphic functions on the open unit disk. The logical connection is immediate by pulling back the functions from the Riemann surface to its universal covering, which for $g \geqslant 2$ is the open unit disk.

I am grateful to Jörg Winkelmann who showed to me that one can construct different meromorphic functions on the open unit disk with any finite number of shared values. Although his examples do not come from compact Riemann surfaces, they convinced me that there probably is no uniform bound for all compact Riemann surfaces and that one should rather try to find examples of compact Riemann surfaces that allow many shared values. By an algebraic argument we will now construct such examples. In fact, one can even prescribe the shared values.

Theorem 1. *Let $a_1, a_2, \ldots, a_n \in \widehat{\mathbb{C}}$ be n different values with $n \geqslant 2$.*

(a) *There exist a compact Riemann surface S of genus $g \leqslant n^2$ and two different non-constant meromorphic functions f_1 and f_2 from S to $\widehat{\mathbb{C}}$ that share the values $a_1, \ldots, a_n$.*

(b) *There exist a compact Riemann surface S of genus $g \leqslant 2n^2 - 5n + 3$ and two different non-constant meromorphic functions f_1 and f_2 from S to $\widehat{\mathbb{C}}$ that share the values $a_1, \ldots, a_{n-1}$ IM and the value a_n CM.*

Proof. (a) After a Möbius transformation we can assume that all the values a_i are finite and non-zero. We consider the polynomial

$$F(X, Y) = (X - Y)^{n+1} + Y \prod_{i=1}^{n}(Y - a_i) \prod_{i=1}^{n}(X - a_i).$$

As a polynomial in X the highest coefficient is 1, all other coefficients are divisible by Y, and the absolute term is not divisible by Y^2. (Here we are using that the a_i are non-zero.) So it is an Eisenstein polynomial in X with respect to Y. By the Eisenstein criterion (see for example [15, Proposition III.1.14]) it is therefore irreducible in X. Since the highest term is X^{n+1}, we also cannot factor out a polynomial that depends only on Y. So $F(X, Y)$ is irreducible.

Let S be the compact Riemann surface of the algebraic equation

$$F(X, Y) = 0.$$

Then the field of meromorphic functions on S is $\mathbb{C}(X, Y)$ where X and Y are related by $F(X, Y) = 0$. In particular, X and Y are functions of degree $n + 1$ from S to $\widehat{\mathbb{C}}$. If X takes the value a_i at some point of S, we see from the equation that Y cannot have a pole at that point and that it must take the same value. And vice versa. Thus X and Y share the value a_i.

To estimate the genus of S we apply the Castelnuovo inequality. See [15, Theorem III.10.3] for an algebraic proof or [1, Theorem 3.5] for a proof that is more in the spirit of Riemann surfaces. The condition that the two maps do not factor over another Riemann surface corresponds to the condition that X and Y generate the function field of S, which holds by construction. From the Castelnuovo inequality we get

$$g \leqslant \big(\deg(X) - 1\big)\big(\deg(Y) - 1\big) = n^2.$$

(b) We can suppose that $a_n = \infty$ and that $a_1, \ldots, a_{n-1}$ are non-zero. This time we look at the Riemann surface corresponding to $F(X, Y) = 0$ where

$$F(X, Y) = (X - Y)^{2n-1} + Y \prod_{i=1}^{n-1}(Y - a_i) \prod_{i=1}^{n-1}(X - a_i).$$

As above we see that $F(X, Y)$ is irreducible and that the functions X and Y share the values $a_1, \ldots, a_{n-1}$. From the equation we also see that if X or Y has a pole at some point, then the other function must also have a pole at that point with the same multiplicity.

Finally, $F(X, Y) = 0$ is a plane curve of degree $d = 2n - 1$. Using the formula for the genus of a plane curve ([1, p. 5] or [15, Proposition III.10.5]), we get

$$g \leqslant \frac{(d - 1)(d - 2)}{2} = (n - 1)(2n - 3). \qquad \square$$

Remark 2. Together with the results from [12] (or Theorem A) this answers a question asked in [12]. In both cases (all values shared IM, or one of the values shared CM) the order of magnitude of the optimal bound in terms of the genus is $\sqrt{g}$.

There is still some room for further fine-tuning since the upper bounds from Theorem A grow asymptotically like $\sqrt{2g}$ resp. $\sqrt{g}$ whereas the examples from Theorem 1 grow asymptotically like $\sqrt{g}$ resp. $\sqrt{g/2}$.

Corollary 3. *Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \widehat{\mathbb{C}}$. Then for every k with $1 \leqslant k \leqslant n$ there exist two different meromorphic functions h_1 and h_2 from the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ to $\widehat{\mathbb{C}}$ that*

- *omit the values $a_1, \ldots, a_{k-1}$;*
- *take every other value in $\widehat{\mathbb{C}}$ infinitely often;*
- *share the values $a_k, \ldots, a_n$;*
- *even share the value a_n CM.*

Proof. Take a compact Riemann surface S of genus $g \geqslant 2$ and two different functions $f_i : S \to \widehat{\mathbb{C}}$ sharing $a_1, \ldots, a_n$ as in Theorem 1(b). Remove the inverse images of $a_1, \ldots, a_{k-1}$ from S, and restrict f_i to functions $\widetilde{f_i}$ on the punctured Riemann surface R. Now we simply have to take $h_i = \widetilde{f_i} \circ \pi$ where $\pi : \mathbb{D} \to R$ is the universal covering. $\square$

For a recent detailed study of value-sharing of meromorphic functions in the unit disk see [16].

3. Punctured Riemann surfaces

Let P_1 be a point of $\mathbb{P}^1$. By Nevanlinna's Five Value Theorem, the maximum number of values that two different non-constant functions that are meromorphic on the punctured Riemann surface $\mathbb{P}^1 - \{P_1\}$ can share is 4, realized for example by e^z and e^{-z} on $\mathbb{C}$.

The same bound holds for $\mathbb{P}^1$ minus two points. After a Möbius transformation we can assume that f_1 and f_2 are meromorphic on $\mathbb{C} - \{0\}$. Then $f_1(e^z)$ and $f_2(e^z)$ are meromorphic on $\mathbb{C}$ and share the same values; so their number is bounded by 4.

Examples 4.

(a) Let $\zeta_r = e^{\frac{2\pi i}{r}}$ be a primitive r-th root of unity with $r \geqslant 2$. Then the functions

$$f_1(z) = z \quad \text{and} \quad f_2(z) = \zeta_r z$$

are obviously meromorphic on

$$\mathbb{P}^1 - \{\zeta_r^k \colon k = 0, \ldots, r-1\}$$

and share the $r + 2$ values

$$\infty, \ 0 \ \text{and} \ \zeta_r^k \quad (k = 0, \ldots, r-1).$$

(b) For any fixed choice of three points P_1, P_2, P_3 on $\mathbb{P}^1$ we can always construct two meromorphic functions on $\mathbb{P}^1 - \{P_1, P_2, P_3\}$ that share 5 values. Let $T(z)$ be the Möbius transformation that maps P_1, P_2, P_3 to $1, \zeta_3, \zeta_3^2$. Then $T(z)$ and $\zeta_3 T(z)$ share $\infty, 0, 1, \zeta_3, \zeta_3^2$, actually all CM.

(c) On any fixed Riemann surface $\mathbb{P}^1 - \{P_1, \ldots, P_r\}$ with $r \geqslant 4$ we can realize at least 6 shared values. Let $T_1(z)$ be the Möbius transformation that maps P_1, P_2, P_3 to $\infty, 0, 1$ and let $c = T_1(P_4)$. Fix a square-root $\sqrt{c}$ and set $T_2(z) = \frac{z + \sqrt{c}}{z - \sqrt{c}}$. Then T_2 maps $\infty, 0, 1, c$ to $1, -1, a, -a$. So the functions $T_2(T_1(z))$ and $-T_2(T_1(z))$ from $\mathbb{P}^1 - \{P_1, \ldots, P_r\}$ to $\widehat{\mathbb{C}}$ share $\infty, 0, 1, -1, a, -a$, actually all CM.

(d) If $g \geqslant 2$ and $R_1, \ldots, R_{2g+2} \in \mathbb{C}$ are different, the compact Riemann surface S corresponding to

$$Y^2 = \prod_{i=1}^{2g+2} (X - R_i)$$

is hyperelliptic of genus g. Moreover, the hyperelliptic map $\kappa : S \to \mathbb{P}^1$, corresponding to $(X, Y) \mapsto X$, is ramified exactly above the points R_i. If $\{R_1, \ldots, R_{2g+2}\}$ contains all r-th roots of unity and $P_i = \kappa^{-1}(\zeta_r^i)$ for $i = 1, 2, \ldots, r$, then κ and $\zeta_r \kappa$ share the $r + 2$ values $\infty, 0, 1, \zeta_r, \ldots, \zeta_r^{r-1}$ on $S - \{P_1, \ldots, P_r\}$.

In Examples 4(a), (b) and (d) we actually have constructed r different functions that all share the same $r + 2$ values CM. Simple as these constructions may be, under certain conditions they will turn out to be essentially the only ones that realize the maximal possible number of shared values.

Now we combine Theorem B and algebraic arguments to derive upper bounds. Recall that if S is a compact Riemann surface and $P_1, \ldots, P_r$ are r different points on S with $0 \leqslant r < \infty$, then the Euler characteristic of the Riemann surface $R = S - \{P_1, \ldots, P_r\}$ is defined to be

$$\chi(R) = 2 - 2g(S) - r.$$

If $\chi(R) < 0$ then R is called hyperbolic. For $r > 0$ this only excludes the two cases $g = 0$, $r \leqslant 2$, which we have discussed at the beginning of this section.

Theorem 5. *Let S be a compact Riemann surface of genus g and gonality d. Let $R = S - \{P_1, \ldots, P_r\}$ where $P_1, \ldots, P_r$ are r different points on S.*

If R is hyperbolic, then two different non-constant meromorphic functions on R can share at most

$$4 + \frac{2g - 2 + r}{d} = 4 - \frac{\chi(R)}{d}$$

values.

Moreover, if $4 - \frac{\chi(R)}{d}$ is an integer and if f_1 and f_2 realize this bound, then f_1 and f_2 must both be meromorphic on the compact Riemann surface S and $\deg(f_1) = \deg(f_2) = d$.

Proof. First let us assume that one of the functions, say f_1 has an essential singularity at one of the points P_i. By the Big Picard Theorem, in every neighbourhood of P_i the function f_1 can omit at most two of the shared values. Every other shared value is a limit of $f_2(z)$ when z approaches P_i along the corresponding inverse images. So if there are 4 or more shared values, then f_2 also has an essential singularity at P_i. By Theorem B in this case f_1 and f_2 cannot share more than 4 values.

Now suppose that none of the points P_i is an essential singularity. Then f_1 and f_2 extend to meromorphic functions on the compact Riemann surface S. Let $d_i = \deg(f_i)$. Without loss of generality we can assume $d_1 \leqslant d_2$. After a Möbius transformation we can also assume that all n shared values $a_1, \ldots, a_n$ lie in $\mathbb{C}$. Let M consist of all $u \in S$ with $f_2(u) \in \{a_1, \ldots, a_n\}$. Then

$$r_2(M) := \sum_{u \in M} \left(\text{mult}_{f_2}(u) - 1 \right)$$

measures the ramification of the covering $f_2 : S \to \widehat{\mathbb{C}}$ above these values. Applying the Hurwitz formula we get

$$r_2(M) \leqslant r_2(S) = 2g(S) - 2 - d_2\left(2g(\widehat{\mathbb{C}}) - 2\right) = 2g - 2 + 2d_2.$$

Also, every element of $M \cap R$ is a zero of $f_1 - f_2$, which is a function of degree $\leqslant 2d_2$. Together we obtain

$$nd_2 = |M \cap R| + \left| M \cap \{P_1, \ldots, P_r\} \right| + r_2(M) \leqslant 2d_2 + r + 2g - 2 + 2d_2,$$

and after division

$$n \leqslant 4 + \frac{2g - 2 + r}{d_2} = 4 - \frac{\chi(R)}{d_2}.$$

Since (by definition) $d \leqslant d_1 \leqslant d_2$, this establishes the bound, and it also shows that reaching the bound is only possible if $d_2 = d$. $\quad\square$

Corollary 6. *Let $P_1, \ldots, P_r$ be r different points on $\mathbb{P}^1$ with $r \geqslant 3$. Then two different non-constant functions that are meromorphic on $\mathbb{P}^1 - \{P_1, \ldots, P_r\}$ cannot share more than $r + 2$ values.*

Moreover, this bound is optimal (at least for a suitable choice of $P_1, \ldots, P_r$). If f_1 and f_2 attain this bound, they both must be functions of degree 1 on $\mathbb{P}^1$, that is, they must be fractional linear transformations.

Proof. Specialize Theorem 5. The bound is sharp by Example 4(a). $\quad\square$

Note that if $r > 4$ we do not claim that the bound in Corollary 6 is sharp for every choice of $P_1, \ldots, P_r$.

Example 7. If the 5 points $P_1, \ldots, P_5$ from $\mathbb{P}^1$ are not all lying on one circle or on one straight line, then two different non-constant meromorphic functions on $R = \mathbb{P}^1 - \{P_1, \ldots, P_5\}$ cannot share more than 6 values.

Indeed, if f_1 and f_2 share 7 values, then by Corollary 6 they must be functions of degree 1 on $\mathbb{P}^1$. Moreover, two of the shared values must obviously be taken at points of R. Applying Möbius transformations to the values and to the arguments we can assume that these two shared values are ∞ (taken at the point ∞) and 0 (taken at the point 0). Hence $f_i(z) = c_i z$ with $c_i \in \mathbb{C}^*$. Without loss of generality we can assume $f_1(z) = z$. Then f_2 must permute the 5 punctures. This is only possible if f_2 is a rotation around 0 and if the 5 punctures are lying on a circle around 0. Since Möbius transformations respect circles on $\mathbb{P}^1$, the 5 original points $P_1, \ldots, P_5$ must lie on such a circle.

For $r = 0$ Theorem 5 gives one of the bounds from Theorem A. In view of the other bounds in Theorem A one might think that in the case where d is small and g is large it should be possible to get stronger bounds than Theorem 5. But even under this condition the bound in Theorem 5 is often sharp.

Proposition 8. *For every compact Riemann surface S there are infinitely many $r \in \mathbb{N}$ for which the bound in Theorem 5 is sharp (provided the points $P_1, \ldots, P_r$ are suitably chosen), and the values are even shared CM.*

Proof. Fix a covering $\pi : S \to \mathbb{P}^1$ of degree d. We can assume that all ramified values $R_1, \ldots, R_m$ of π are in $\mathbb{C}^*$.

If $Q_1, \ldots, Q_s$ are s different points in $\mathbb{C}^*$, containing all R_i, then by the Hurwitz formula there are exactly $r = ds - (2d + 2g - 2)$ points P_i of S lying above $Q_1, \ldots, Q_s$.

If moreover the set $\{Q_1, \ldots, Q_s\}$ is closed under $Q_i \mapsto -Q_i$, then the functions π and $-\pi$ from $S - \{P_1, \ldots, P_r\}$ to $\widehat{\mathbb{C}}$ share the $2 + s$ values $\infty, 0, Q_1, \ldots, Q_s$ CM and $2 + s = 4 + \frac{2g - 2 + r}{d}$. $\quad\square$

However, for r in a certain range one can indeed improve on Theorem 5.

Theorem 9. *Let S be a compact Riemann surface of genus g and gonality d. Let $R = S - \{P_1, \ldots, P_r\}$ where $P_1, \ldots, P_r$ are r different points on S.*

If $r \geqslant 2d$, then two different non-constant meromorphic functions on R can share at most $r + 2$ values.

Moreover, if f_1 and f_2 realize this bound and $r > 2d$, then f_1 and f_2 must both be meromorphic on S with $\deg(f_1) = \deg(f_2) = d$, and the coverings $f_i : S \to \mathbb{P}^1$ are totally ramified at all points $P_1, \ldots, P_r$.

Proof. As explained at the beginning of the proof of Theorem 5, we can assume that f_1 and f_2 are meromorphic on S. Without loss of generality $\deg(f_1) = d_1 \leqslant d_2 = \deg(f_2)$.

If $(d-1)(d_2-1) \geqslant g$, i.e. $d_2 \geqslant \frac{g-1+d}{d-1}$, from the last inequality in the proof of Theorem 5 we get

$$n \leqslant 4 + \frac{2g-2+r}{d_2} \leqslant 4 + \frac{(2g-2+r)(d-1)}{g-1+d} = 4 + 2(d-1) + \frac{(r-2d)(d-1)}{g-1+d},$$

which is smaller than $2 + 2d + r - 2d = r + 2$ if $r > 2d$ and equal to $r + 2$ if $r = 2d$.

If $(d-1)(d_2-1) < g$, we proceed by induction on d, using Corollary 6 as induction basis.

If $d > 1$ let F be the function field of S. Fix a rational subfield R of index d in F. Let M be the compositum of R and $\mathbb{C}(f_2)$ and let c be the index of M in F. By Castelnuovo's inequality we have

$$g(M) \leqslant \left(\frac{d}{c} - 1\right)\left(\frac{d_2}{c} - 1\right).$$

Now $c = 1$ would be equivalent to $M = F$ and hence contradict the Castelnuovo inequality. So M is a proper subfield of F.

If the compositum of M and $\mathbb{C}(f_1)$ were F, then again by Castelnuovo's inequality we would obtain $g \leqslant cg(M) + (c-1)(d_1-1)$, and hence

$$(d-1)(d_2-1) < g \leqslant c\left(\frac{d}{c} - 1\right)\left(\frac{d_2}{c} - 1\right) + (c-1)(d_2-1).$$

Subtracting $(c-1)(d_2-1)$ we get the contradiction $(d-c)(d_2-1) < (d-c)(\frac{d_2}{c} - 1)$. We conclude that R, $\mathbb{C}(f_1)$ and $\mathbb{C}(f_2)$ are contained in a proper subfield L of F. Let $\delta = [F : L]$. Then

$$f_i = \widetilde{f_i} \circ \kappa$$

where $\widetilde{f_1}$ and $\widetilde{f_2}$ are meromorphic of degrees $\frac{d_1}{\delta}$ resp. $\frac{d_2}{\delta}$ on the compact Riemann surface $\widetilde{S}$ corresponding to L and κ is the covering map from S to $\widetilde{S}$.

Now the image of $\{P_1, \ldots, P_r\}$ under κ is a subset of $\widetilde{S}$ of cardinality $\widetilde{r}$ where obviously

$$\frac{r}{\delta} \leqslant \widetilde{r} \leqslant r.$$

Moreover, $\widetilde{r} \geqslant 2\frac{d}{\delta}$ and $\frac{d}{\delta}$ is the gonality of $\widetilde{S}$. By induction $\widetilde{f_1}$ and $\widetilde{f_2}$ can share at most $\widetilde{r} + 2$ values, and if they share $\widetilde{r} + 2$ values then $\deg(\widetilde{f_i}) = \frac{d}{\delta}$ and $\widetilde{f_1}$ and $\widetilde{f_2}$ are totally ramified at all $\kappa(P_i)$.

This proves the first statement of the theorem. It also shows that having $r + 2$ shared values is only possible if $\widetilde{r} = r$, i.e. if κ is totally ramified at $P_1, \ldots, P_r$. $\quad\square$

Remarks 10.

(a) In the range $2d \leqslant r < \frac{2g}{d-1} + 2$ the bound in Theorem 9 is better than Theorem 5 if $d > 1$. Of course, this range might be empty if d is sufficiently big with respect to g.

(b) By Example 4(d), for each pair (g, r) with $4 \leqslant r \leqslant 2g + 2$ there exists a hyperelliptic Riemann surface S, points $P_1, \ldots, P_r$ on S, and meromorphic f_1, f_2 on $S - \{P_1, \ldots, P_r\}$ sharing $r + 2$ values.

On the other hand, if S is the hyperelliptic surface corresponding to

$$Y^2 = (X-1)(X-2)(X-3)(X-1-i)(X-2-i)(X-3-i),$$

then by Theorem 9 and Example 7 for every choice of five points $P_1, \ldots, P_5$ on S we cannot get more than 6 shared values.

(c) More generally, if $d \geqslant 3$ and $g = \frac{1}{2}(d-1)(md-2)$ with $m \geqslant 2$, then for every r with $2d \leqslant r \leqslant md = \frac{2g}{d-1} + 2$ there exist a compact Riemann surface S of genus g and gonality d, points $P_1, \ldots, P_r$ on S, and meromorphic f_1 and f_2 on $S - \{P_1, \ldots, P_r\}$ that share $r + 2$ values.

Explicitly, let S be the Riemann surface of the function field $F = \mathbb{C}(X, Y)$ with $Y^d = f(X)$ where $f(X) \in \mathbb{C}[X]$ is square-free, of degree md and divisible by $X^r - 1$. Using the Hurwitz formula one can calculate the genus of every intermediate field between $\mathbb{C}(X)$ and F. Then the Castelnuovo inequality shows that F indeed has gonality d. Let π be the covering map from S to $\mathbb{P}^1$ corresponding to the extension $F/\mathbb{C}(X)$. Let $P_i = \pi^{-1}(\zeta_r^i)$ for $i = 1, 2, \ldots, r$. Then π and $\zeta_r\pi$ share the values $\infty, 0, 1, \zeta_r, \ldots, \zeta_r^{r-1}$ on $S - \{P_1, \ldots, P_r\}$.

We finish this section with another example of which one can easily construct many explicit instances.

Example 11. Let F be the compositum of two quadratic extensions of $\mathbb{C}(z)$. Then F is a Galois extension of $\mathbb{C}(z)$ with Galois group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Assume that F has genus $g \geqslant 10$ and that each of the three intermediate quadratic fields has genus at least 2. Then the Castelnuovo inequality implies that the gonality of F is 4 and that $\mathbb{C}(z)$ is the only rational subfield over which F has degree 4. Moreover, no place is totally ramified in $F/\mathbb{C}(z)$.

Let S be the compact Riemann surface of F. Then Theorem 9 implies that for any choice of $r > 8$ different points $P_1, \ldots, P_r$ on S two meromorphic functions on $S - \{P_1, \ldots, P_r\}$ cannot share more than $r + 1$ values.

4. Weighted sharing

In order to refine the results we recall the concept of weighted sharing as introduced by Lahiri in [7]. A shared value a is shared with weight $m \in \mathbb{N}_0 \cup \{\infty\}$ if for all inverse images u of a we have

$$\mathrm{mult}_{f_1}(u) = \mu \leqslant m \quad \Leftrightarrow \quad \mathrm{mult}_{f_2}(u) = \mu \leqslant m$$

and

$$\mathrm{mult}_{f_1}(u) > m \quad \Leftrightarrow \quad \mathrm{mult}_{f_2}(u) > m.$$

In particular, f_1 and f_2 sharing the value a with weight one means that the simple a-points of f_1 are exactly the simple a-points of f_2 and the multiple a-points of f_1 are exactly the multiple a-points of f_2, where in the latter case the multiplicities are not necessarily the same.

Obviously, sharing with weight 0 simply means sharing IM, and sharing with weight ∞ is the same as sharing CM.

In the sequel we write $(m_1, \ldots, m_n)$ to indicate that the value a_i is shared with weight m_i.

Since a meromorphic function on a compact Riemann surface S is determined up to a multiplicative constant by its divisor, we cannot have two different non-constant meromorphic functions on S sharing 3 values with weights $(\infty, \infty, 0)$. (Apply a Möbius transformation to move the two CM-shared values to 0 and ∞.) But for every $m \in \mathbb{N}$ sharing with weights $(\infty, m, 0)$ is possible on every S.

Example 12. From [10] we take the example of the functions

$$f_1(z) = \frac{-4z^3}{(z-1)^3(z+1)} \quad \text{and} \quad f_2(z) = \frac{-4z}{(z-1)(z+1)^3}$$

from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. They share the value 1 CM (taken with multiplicity 1 at the zeroes of $(z^2+1)(z^2+2z-1)$), and the values 0 and ∞, both IM. Since the value 0 is taken exactly at the points 0 and ∞, the functions $h_i(z) = f_i(z^{m+1})$ take this value at these two points with multiplicity at least $m + 1$. So, somewhat trivially, they share the value 0 with weight m.

Finally, if S is any compact Riemann surface, there is a covering $\pi: S \to \mathbb{P}^1$. Then $h_1 \circ \pi$ and $h_2 \circ \pi$ are two meromorphic functions on S that share the values 1, 0 and ∞ with respective weights $(\infty, m, 0)$. Applying a suitable Möbius transformation, we can get any 3 values shared with these weights.

Remark 13. If $a_1, a_2, \ldots, a_n \in \widehat{\mathbb{C}}$ and $m \in \mathbb{N}$, with the same trick we can even construct a compact Riemann surface R and two functions h_1, h_2 from R to $\widehat{\mathbb{C}}$ that share the values $a_1, \ldots, a_n$ with respective weights $(\infty, m, m, \ldots, m)$. We only have to take S and f_1, f_2 as in Theorem 1(b), take a finite covering $\pi: R \to S$ of compact Riemann surfaces that is totally ramified above all inverse images of the shared values with ramification index at least $m + 1$, and set $h_i = f_i \circ \pi$.

The same weighted sharing can then of course also be obtained in Corollary 3.

Admittedly, in the preceding examples the weight m of the sharing does not really tell much, as we simply have artificially increased the multiplicities. However, if one restricts the nature of the underlying Riemann surface, one can get non-trivial information, as we will show now by improving a result from [13].

The following lemma can be easily obtained as a special case of known results on two meromorphic functions in the complex plane that share 4 values (cf. [4,5,8,17]). We prefer to give a direct algebraic proof. Besides being conceptually much simpler, it has the advantage to be valid for rational functions on an elliptic curve over any algebraically closed field of characteristic 0.

Lemma 14. *Let S be a compact Riemann surface of genus 1 and f_1, f_2 two non-constant meromorphic functions from S to $\widehat{\mathbb{C}}$ that share the 4 values $a_1, a_2, a_3, a_4 \in \widehat{\mathbb{C}}$. Let $M = f_i^{-1}\{a_1, a_2, a_3, a_4\} \subseteq S$. Then*

(a) $\deg(f_1) = \deg(f_2)$;
(b) $|M| = 2\deg(f_i)$;
(c) *the map $f_i: S \to \widehat{\mathbb{C}}$ is unramified outside M;*

(d) $f_1(u) \neq f_2(u)$ for every $u \notin M$;
(e) at each point $u \in M$ at least one of the two functions takes the shared value $a_j = f_i(u)$ with multiplicity 1.

Proof. Let $d_i = \deg(f_i)$. Without loss of generality we can assume $d_1 \leqslant d_2$. Let

$$r_2(M) := \sum_{u \in M} \big(\text{mult}_{f_2}(u) - 1\big).$$

Applying the Hurwitz formula to the covering $f_2 : S \to \widehat{\mathbb{C}}$ we get

$$r_2(M) \leqslant r_2(S) = 2g(S) - 2 - d_2\big(2g(\widehat{\mathbb{C}}) - 2\big) = 2d_2.$$

After a Möbius transformation we can assume $a_1, \ldots, a_4 \in \mathbb{C}$. Since every $u \in M$ is a zero of $f_1 - f_2$, we have

$$d_1 + d_2 \geqslant \deg(f_1 - f_2) \geqslant |M| = 4d_2 - r_2(M) \geqslant 2d_2.$$

This shows $d_1 = d_2$ and $|M| = 2d_2$ and also $r_2(M) = 2d_2$, which means that f_2 is unramified outside M. Since $d_1 = d_2$ we can interchange f_1 and f_2, so f_1 is also unramified outside M.

Moreover, we see $\deg(f_1 - f_2) = 2d_2$. This shows that f_1 and f_2 have no common poles. Finally, since $f_1 - f_2$ vanishes at the $2d_2$ different points in M, it cannot vanish outside M (so claim (d) holds) and it cannot have a multiple zero, which implies statement (e). $\square$

Corollary 15. *Let S be a compact Riemann surface of genus 1 and f_1, f_2 two non-constant meromorphic functions from S to $\widehat{\mathbb{C}}$ that share 4 values with respective weights $(1, 0, 0, 0)$. Then $f_1 = f_2$.*

Proof. By Lemma 14(e) the value that is shared with weight one is actually shared CM. By part (b) of Theorem A this implies $f_1 = f_2$. $\square$

Corollary 16. *Let f_1 and f_2 be two non-constant elliptic functions on the complex plane (not necessarily with commensurable period lattices). If f_1 and f_2 share 4 values, of which one is shared with weight one, then the functions f_1 and f_2 are equal.*

Proof. Let Λ_i be the period lattice of f_i. Translating the variable z we can assume that $f_1(0)$ is one of the shared values. Then f_1 (and also f_2) takes this value at all points of the $\mathbb{Z}$-module $\Lambda_1 + \Lambda_2$. Since f_1 is not constant, $\Lambda_1 + \Lambda_2$ must be discrete and hence a rank 2 lattice. Thus Λ_1 and Λ_2 are commensurable. Let Λ be the rank 2 lattice $\Lambda_1 \cap \Lambda_2$. Then we can consider f_1 and f_2 as meromorphic functions on the torus $\mathbb{C}/\Lambda$ and apply the previous corollary. $\square$

This result, presumably well known to specialists, has inspired the following modification of a famous problem. Let f_1 and f_2 be two non-constant meromorphic functions in the complex plane sharing 4 values. Gundersen [4] has shown that if 2 of these 4 values are shared CM then all 4 values must be shared CM. He also asked whether one CM-shared value would already be enough for the same conclusion. Although there are positive answers under different additional conditions (see for example [8,5], [17, Chapter 4], [6]), this is still an open problem.

But the notion of weighted sharing opens up infinitely many more possibilities between weights $(\infty, 0, 0, 0)$ and $(\infty, \infty, 0, 0)$. So one might ask: Does sharing with weights $(\infty, m, 0, 0)$ for some $m \in \mathbb{N}$ imply weights $(\infty, \infty, \infty, \infty)$? It turns out that a much weaker condition, namely that the weights of sharing are $(1, 1, 0, 0)$, already suffices.

Theorem 17. *Let f_1 and f_2 be two non-constant meromorphic functions in the complex plane sharing 4 values. If 2 of these values are shared with weight one, then all 4 values are shared CM.*

Proof. If f_1 and f_2 share 4 values, then by [8, Lemma 1] for each shared value a_j the counting function of the points that are multiple a_j-points for both functions is $S(r, f_i)$. Thus a value that is shared with weight one is shared "CM" in the sense of [8], that is, the counting function of the points where the value is taken with different multiplicities is $S(r, f_i)$. But by [5, Theorem C*], two "CM"-shared values imply that all four values are shared CM. $\square$

Modifying the condition that one value is shared CM in the other direction by relaxing it, we obtain the following problem.

Question. Given $m \in \mathbb{N}$. If two non-constant meromorphic functions in the complex plane share 4 values with respective weights $(m, 0, 0, 0)$, does this imply that all four values are shared CM?

This is presumably a very difficult question. A positive answer would settle the famous problem on 1 CM plus 3 IM shared values. And a counterexample would be a new example of two meromorphic functions in the complex plane sharing 4 values not all of which are shared CM. There are essentially only 3 known examples of such functions, namely the ones described in [4,14,11].

Acknowledgments

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