J. Math. Anal. Appl. 366 (2010) 128–136

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Formal first integrals for periodic systems ☆

Jia Jiao [a,b,*], Shaoyun Shi [a,c], Zhiguo Xu [a]

[a] Institute of Mathematics, Jilin University, Changchun 130012, PR China
[b] College of Science, Dalian Nationalities University, Dalian 116600, PR China
[c] Key Laboratory of Symbolic Computation and Knowledge Engineering of Ministry of Education, Jilin University, Changchun 130012, PR China

ARTICLE INFO

*Article history:*
Received 8 January 2009
Available online 4 January 2010
Submitted by J. Mawhin

*Keywords:*
Formal first integral
Periodic systems
Floquet's theory

ABSTRACT

In the present paper, we give an elementary proof for the result of Li et al. (2003) [6] about nonexistence of formal first integrals for periodic systems in a neighborhood of a constant solution. Moreover, we present a criterion about partial existence of formal first integrals for the periodic system, by using the Floquet's theory.



## 1. Introduction

Early in the 19th century, Poincaré [10] presented a simple criterion on nonexistence of analytic first integral for autonomous differential equations

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{C}^n, \tag{1}$$

where $f(x)$ is a vector-valued analytic function satisfying $f(0) = 0$. He showed that if the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Jacobi matrix $A = Df(0)$ are non-resonant, i.e., they do not satisfy any condition of the form

$$\sum_{j=1}^{n} k_j \lambda_j = 0, \quad k_j \in \mathbb{Z}^+, \qquad \sum_{j=1}^{n} k_j \neq 0,$$

then system (1) does not have any nontrivial analytic first integral in a neighborhood of $x = 0$.

Along Poincaré's idea, many works have been done in this field. In 1983, based on the monodromy properties around particular solutions, Ziglin [16] studied the nonexistence of first integrals for Hamiltonian systems. Yoshida [13] derived some necessary conditions for quasihomogeneous systems to have first integrals. In 1996, Furta [3] provided an elementary proof of Poincaré's result, and furthermore, he studied the nonexistence and partial existence of analytic first integrals for semi-quasihomogeneous systems. Recently, the authors in [5] extended the Furta's result. Li et al. [6] generalized Poincaré's result to the case that an eigenvalue of the matrix $A$ is zero and the other eigenvalues are non-resonant. In [4], Goriely investigated the partial integrability, i.e., the existence of a certain number of first integrals less than the number required for the complete integration. More concrete relation between the number of first integrals of a given differential system and

☆ This work is supported by NSFC grant (10771083), National 973 project of China (2006CD805903), SRFDP grant (20060183017), Program for New Century Excellent Talents in University and 985 project of Jilin University.

* Corresponding author at: College of Science, Dalian Nationalities University, Dalian 116600, PR China.
*E-mail addresses:* jiaolinrui@sina.com (J. Jiao), shisy@jlu.edu.cn (S. Shi), zhiguo@email.jlu.edu.cn (Z. Xu).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2009.12.049

its corresponding linear system was given by Zhang [15]. And in [2], the relation between resonance and the number of first integrals for quasi-periodic differential systems was also considered. More related works can be found in [7–9,11,12,14].

In this paper, we will present some criteria about nonexistence and partial existence of first integrals for the periodic differential system

$$\dot{x} = f(t, x), \tag{2}$$

where $(t, x) \in \mathbf{S}^1 \times \mathbb{C}^n$ with $\mathbf{S}^1 = \mathbb{R}/(\mathbb{N}T)$, $f(t, x)$ is $C^r$ in $\mathbf{S}^1 \times \mathbb{C}^n$ with $r \geqslant 1$ and $f(t+T, x) = f(t, x)$. Here, a non-constant function $\Phi(t, x)$ defined on $\mathbf{S}^1 \times \mathbf{U}$, with $\mathbf{U}$ an open subset of $\mathbb{C}^n$, is called a *first integral* of system (2) if it is $T$-periodic with respect to $t$ and constant along every flow defined in $\mathbf{U}$. If $\Phi(t, x)$ is differentiable, then this definition can be written as the condition

$$\frac{\partial \Phi(t, x)}{\partial t} + \left\langle \frac{\partial \Phi(t, x)}{\partial x}, f(t, x) \right\rangle \equiv 0. \tag{3}$$

If $\Phi(t, x)$ is a formal series in $x$ and satisfies equality (3), then $\Phi(t, x)$ is called a formal first integral of system (2).

Suppose that $x = 0$ is a constant solution of system (2), i.e., $f(t, 0) \equiv 0$, then system (2) can be rewritten as

$$\dot{x} = A(t)x + g(t, x) \tag{4}$$

near some neighborhood of $x = 0$, where $A(t) = \frac{\partial f}{\partial x}(t, 0)$, $A(t+T, x) = A(t, x)$, and $g(t, x) = O(\|x\|^2)$ is $T$-periodic in $t$.

By Floquet's theory [1], there exists a $T$-periodic function $Q(t)$ such that under the transformation

$$x = Q(t)y, \tag{5}$$

system (4) is transformed to

$$\dot{y} = By + h(t, y), \tag{6}$$

where $B$ is a constant matrix, and $h(t, y) = O(\|y\|^2)$ is $T$-periodic in $t$. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of $B$ are called the characteristic exponents (or Floquet exponents) of system

$$\dot{x} = A(t)x \tag{7}$$

and the eigenvalues of $e^{TB}$, i.e., $\mu_1 = \exp(\lambda_1 T), \ldots, \mu_n = \exp(\lambda_n T)$, are called the characteristic multipliers of the system (7).

In 2003, by using the theory of linear operators and normal forms, Li et al. [6] obtained the following result.

**Theorem 1.** *Assume that $x = 0$ is a constant solution of system* (4)*. If the characteristic multipliers of system* (7) *do not satisfy any resonant equality of the type*

$$\prod_{i=1}^{n} \mu_i^{k_i} = 1, \quad k_i \in \mathbb{Z}^+, \qquad \sum_{i=1}^{n} k_i \geqslant 1, \tag{8}$$

*then system* (4) *does not have any nontrivial formal first integral in a neighborhood of $x = 0$.*

In this paper, firstly, we give an elementary proof for Theorem 1. Secondly, we present a criterion about partial existence of formal integrals for periodic differential system. Finally, we give some examples to illustrate our results.

## 2. An elementary proof of Theorem 1

Suppose that system (4) has a nontrivial formal first integral $\Phi(t, x)$ in a neighborhood of $x = 0$. Evidently, we know that $\Psi(t, y) = \Phi(t, Q(t)y)$ is a formal first integral of system (6) and $\Psi(t+T, y) = \Psi(t, y)$, from the definition of first integral, it requires that

$$\frac{\partial \Psi(t, y)}{\partial t} + \left\langle \frac{\partial \Psi(t, y)}{\partial y}, By + h(t, y) \right\rangle \equiv 0. \tag{9}$$

Expanding the considered function $\Psi(t, y)$ in the formal power series with respect to $y$, we get

$$\Psi(t, y) = \Psi_0(t) + \Psi_l(t, y) + \Psi_{l+1}(t, y) + \cdots, \tag{10}$$

where $l \in \mathbb{N}$ and $\Psi_i(t, y)$ is a homogeneous polynomial of degree $i$ in $y$ with $T$-periodic coefficients in $t$. Substitute (10) into (9) and equate all the terms in (9) of the same order with respect to $y$ zero. For $y^0$, we get

$$\frac{\partial \Psi_0(t)}{\partial t} \equiv 0,$$

that is to say, $\Psi_0(t) \equiv \Psi_0(0)$. Without any loss of generality, we set $\Psi_0(t) = 0$. Equating the terms of $y^l$ we obtain

$$\frac{\partial \Psi_l(t,y)}{\partial t} + \left\langle \frac{\partial \Psi_l(t,y)}{\partial y}, By \right\rangle \equiv 0, \tag{11}$$

this implies that $\Psi_l(t,y)$ is an integral of the linear system

$$\dot{y} = By. \tag{12}$$

Since matrix $B$ can be changed to a Jordan canonical form under a nonsingular linear transformation, for simplicity, we assume that $B$ has been a Jordan canonical form, i.e.,

$$B = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_{\bar{m}} \end{pmatrix}, \qquad J_r = \begin{pmatrix} \lambda_r & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_r \end{pmatrix},$$

where $J_r$ is a Jordan block with degree equal to $i_r$, $i_1 + \cdots + i_{\bar{m}} = n$.

Make the following transformation of variables

$$y = Cz, \tag{13}$$

where

$$C = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_m \end{pmatrix}, \qquad C_r = \begin{pmatrix} 1 & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon^{i_r-1} \end{pmatrix},$$

$C_r$ is a Jordan block with degree equal to $i_r$, $i_1 + \cdots + i_m = n$, and $\varepsilon > 0$ is a constant.

Under the transformation (13), system (12) can be rewritten as

$$\dot{z} = (D + \varepsilon \tilde{D})z, \tag{14}$$

where

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_m \end{pmatrix}, \qquad D_r = \begin{pmatrix} \lambda_r & & & \\ & \lambda_r & & \\ & & \ddots & \\ & & & \lambda_r \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} \tilde{D}_1 & & & \\ & \tilde{D}_2 & & \\ & & \ddots & \\ & & & \tilde{D}_m \end{pmatrix}, \qquad \tilde{D}_r = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Obviously, $\bar{\Psi}(t,z,\varepsilon) = \Psi_l(t,Cz)$ is an integral of the linear system (14). Thus

$$\frac{\partial \bar{\Psi}}{\partial t}(t,z,\varepsilon) + \left\langle \frac{\partial \bar{\Psi}}{\partial z}(t,z,\varepsilon), (D + \varepsilon \tilde{D})z \right\rangle \equiv 0. \tag{15}$$

Since

$$\Psi_l(t,y) = \sum_{k_1+\cdots+k_n=l} \Psi_{k_1\cdots k_n}(t) y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n},$$

$\bar{\Psi}(t,z,\varepsilon)$ has the form

$$\begin{aligned} \bar{\Psi}(t,z,\varepsilon) &= \Psi_l(t,Cz) \\ &= \sum_{k_1+\cdots+k_n=l} \Psi_{k_1\cdots k_n}(t)(z_1)^{k_1}(\varepsilon z_2)^{k_2} \cdots \left(\varepsilon^{i_1-1} z_{i_r}\right)^{k_{i_1}} (z_{i_1+1})^{k_{i_1+1}} (\varepsilon z_{i_1+2})^{k_{i_1+2}} \cdots \left(\varepsilon^{i_2-1} z_{i_1+i_2}\right)^{k_{i_1+i_2}} \cdots \\ &\quad (z_{i_1+\cdots+i_{m-1}+1})^{k_{i_1+\cdots+i_{m-1}+1}} (\varepsilon z_{i_1+\cdots+i_{m-1}+2})^{k_{i_1+\cdots+i_{m-1}+1}} \cdots \left(\varepsilon^{i_m-1} z_n\right)^{k_n} \\ &= \Psi_l^0(t,z) + \varepsilon \Psi_l^1(t,z) + \cdots + \varepsilon^{l^*} \Psi_l^{l^*}(t,z), \end{aligned} \tag{16}$$

where

$$\Psi_l^j(t,z) = \sum_{k_1+\cdots+k_n=l} \Psi_{k_1\cdots k_n}^j(t) z_1^{k_1}\cdots z_n^{k_n}, \quad 0 \leqslant j \leqslant l^*, \tag{17}$$

$l^* \in \mathbb{Z}^+$ is a certain constant and $\Psi_{k_1\cdots k_n}^j(t+T) = \Psi_{k_1\cdots k_n}^j(t)$.

Now substitute (16) into (15) and equate all the same order with respect to $\varepsilon$ zero. Suppose $\Psi_l^0(t,z) = \Psi_l^1(t,z) = \cdots = \Psi_l^{M-1}(t,z) \equiv 0$, $\Psi_l^M(t,z) \not\equiv 0$ $(0 \leqslant M \leqslant l^*)$. Then $\Psi_l^M(t,z)$ has to satisfy the following equation

$$\frac{\partial \Psi_l^M(t,z)}{\partial t} + \left\langle \frac{\partial \Psi_l^M(t,z)}{\partial z}, Dz \right\rangle \equiv 0. \tag{18}$$

Substituting (17) into (18), we obtain

$$\sum_{k_1+\cdots+k_n=l} \left[ \frac{d\Psi_{k_1\cdots k_n}^M(t)}{dt} + \langle \Lambda, \mathbf{k} \rangle \Psi_{k_1\cdots k_n}^M(t) \right] z_1^{k_1}\cdots z_n^{k_n} \equiv 0,$$

where $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mathbf{k} = (k_1, \ldots, k_n)$. For any monomial $z_1^{k_1}\cdots z_n^{k_n}$, we have

$$\frac{d\Psi_{k_1\cdots k_n}^M(t)}{dt} + \langle \Lambda, \mathbf{k} \rangle \Psi_{k_1\cdots k_n}^M(t) \equiv 0.$$

From the above equation,

$$\Psi_{k_1\cdots k_n}^M(t) = \Psi_{k_1\cdots k_n}^M(0) e^{-\langle \Lambda, \mathbf{k} \rangle t}. \tag{19}$$

Since the coefficient $\Psi_{k_1\cdots k_n}^M(t)$ is a $T$-periodic function in $t$, we get

$$e^{-\langle \Lambda, \mathbf{k} \rangle T} = 1.$$

Then

$$\prod_{i=1}^n \mu_i^{k_i} = \prod_{i=1}^n \left(e^{\lambda_i T}\right)^{k_i} = \prod_{i=1}^n e^{\lambda_i k_i T} = e^{\langle \Lambda, \mathbf{k} \rangle T} = 1.$$

Thus, a resonant condition of (8) type has to be fulfilled for any nonzero coefficient $\Psi_{k_1\cdots k_n}^M(t)$, which contradicts the conditions of Theorem 1. The theorem is proved.

By the above proof, we can obtain the following result.

**Corollary 1.** *If system* (2) *has a nontrivial formal first integral $\Phi(t,x)$ in a neighborhood of the constant solution $x = 0$, then $\Psi(t,y) = \Phi(t, Q(t)y)$ is a formal integral of* (6) *in the vicinity of $y = 0$. Moreover, there exists a homogeneous function $\Psi_l(t,y)$ of degree $l$ with respect to $y$ which is an integral of system* (12) *in a neighborhood of $y = 0$. In fact, it is the leading term of the formal power series of $\Psi(t,y)$ with respect to $y$.*

## 3. Partial existence of formal first integrals

In this section, we are concerned with the partial integrability of periodic system (2). According to the proof of Theorem 1, we know that if system (2) has a nontrivial formal first integral $\Phi(t,x)$, then there exists a nonzero vector $\mathbf{k} \in \mathbb{Z}_+^n$ such that

$$\Theta = \left\{ \mathbf{k} = (k_1, \ldots, k_n)\colon \prod_{i=1}^n \mu_i^{k_i} = 1,\ k_i \in \mathbb{Z}^+ \right\} \tag{20}$$

is a nonempty subset of $\mathbb{Z}_+^n$. We say $\kappa_1, \ldots, \kappa_s \in \Theta$ are the least generating elements of $\Theta$ if both of the following conditions hold:

(1) For any $\kappa \in \Theta$, there exist $a_1, \ldots, a_s \in \mathbb{Z}^+$ such that $\kappa = \sum_i^s a_i \kappa_i$.
(2) $s \in \mathbb{N}$ is the minimal number satisfying the first condition.

Suppose that system (2) admits $s$ nontrivial formal integrals $\Phi^1(t,y), \ldots, \Phi^s(t,y)$, then according to Corollary 1, linear system (12) has $s$ nontrivial homogeneous formal first integrals $\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)$ with respect to $y$. Here, we assume that $B$ is diagonalizable, just for simplicity, $B$ has already been a diagonal form.

Now, we can state our main result of this paper.

**Theorem 2.** *Assume system* (2) *has a constant solution $x = 0$ and admits $s$ ($1 \leqslant s < n$) nontrivial formal integrals $\Phi^1(t,x), \dots, \Phi^s(t,x)$. If $\Psi^1_{l_1}(t,y), \dots, \Psi^s_{l_s}(t,y)$ are functionally independent in $y$ and the number of the least generating elements of set $\Theta$ is $s$, then any other nontrivial formal integral $\Phi(t,x)$ of system* (2) *must be a function of $\Phi^1(t,x), \dots, \Phi^s(t,x)$, i.e.,*

$$\Phi(t,x) = \mathcal{D}\big(\Phi^1(t,x), \dots, \Phi^s(t,x)\big),$$

*where $\mathcal{D}$ is a formal series.*

We remark that Theorem 2 is, in some sense, a generalization of Theorem 3 of [6].
In order to prove Theorem 2, we will get the help of the following two lemmas.

**Lemma 1.** *Suppose that the number of the least generating elements of set $\Theta$ is $s$ ($1 \leqslant s < n$), and let $\kappa_1 = (\kappa_{11}, \dots, \kappa_{1n}), \dots, \kappa_s = (\kappa_{s1}, \dots, \kappa_{sn})$ be the least generating elements of set $\Theta$. Then the following statements hold*:

(a) *$\rho^i(t,y) = e^{-\langle \Lambda, \kappa_i\rangle t} y_1^{\kappa_{i1}} \cdots y_n^{\kappa_{in}}$ for $i = 1, \dots, s$ are $s$ first integrals of the linear system* (12).
(b) *Any nontrivial homogeneous formal first integral $\Psi_l(t,y)$ of* (12) *is a polynomial function of $\rho^1(t,y), \dots, \rho^s(t,y)$.*

**Proof.** (a) It follows easily by a direct calculation.
(b) If

$$\Psi_l(t,y) = \sum_{k_1+\cdots+k_n=l} \Psi_{k_1\cdots k_n}(t) y_1^{k_1} \cdots y_n^{k_n}$$

is a nontrivial homogeneous integral of linear system (12), from the definition of first integral, it requires that

$$\frac{\partial \Psi_l(t,y)}{\partial t} + \left\langle \frac{\partial \Psi_l(t,y)}{\partial y}, By \right\rangle = \sum_{k_1+\cdots+k_n=l} \left[ \frac{\partial \Psi_{k_1\cdots k_n}(t)}{\partial t} + \langle \Lambda, \mathbf{k}\rangle \Psi_{k_1\cdots k_n}(t) \right] y_1^{k_1} \cdots y_n^{k_n} \equiv 0. \tag{21}$$

Therefore, any nonzero coefficient $\Psi_{k_1\cdots k_n}(t)$ should satisfy

$$\frac{\partial \Psi_{k_1\cdots k_n}(t)}{\partial t} + \langle \Lambda, \mathbf{k}\rangle \Psi_{k_1\cdots k_n}(t) \equiv 0,$$

so

$$\Psi_{k_1\cdots k_n}(t) = \Psi_{k_1\cdots k_n}(0) e^{-\langle \Lambda, \mathbf{k}\rangle t}. \tag{22}$$

Since for any nonzero coefficient $\Psi_{k_1\cdots k_n}(t)$, geometric point $\mathbf{k} = (k_1, \dots, k_n) \in \Theta$. Therefore there exist $a_1, \dots, a_s \in \mathbb{Z}^+$ such that

$$\mathbf{k} = a_1\kappa_1 + \cdots + a_s\kappa_s,$$

or equivalently

$$k_i = a_1\kappa_{1i} + \cdots + a_s\kappa_{si}, \quad i = 1, 2, \dots, n.$$

Thus we have

$$\begin{aligned} e^{-\langle \Lambda, \mathbf{k}\rangle t} y_1^{k_1} \cdots y_n^{k_n} &= e^{-a_1\langle \Lambda, \kappa_1\rangle} \cdots e^{-a_s\langle \Lambda, \kappa_s\rangle} y_1^{a_1\kappa_{11}+\cdots+a_s\kappa_{s1}} \cdots y_n^{a_1\kappa_{1n}+\cdots+a_s\kappa_{sn}} \\ &= \big(e^{-\langle \Lambda, \kappa_1\rangle t} y_1^{\kappa_{11}} \cdots y_n^{\kappa_{1n}}\big)^{a_1} \cdots \big(e^{-\langle \Lambda, \kappa_s\rangle t} y_1^{\kappa_{s1}} \cdots y_n^{\kappa_{sn}}\big)^{a_s} \\ &= \rho^1(t,y)^{a_1} \cdots \rho^s(t,y)^{a_s}. \end{aligned} \tag{23}$$

Clearly, we can conclude that $\Psi_l(t,y)$ is a polynomial function of $\rho^1(t,y), \dots, \rho^s(t,y)$. The lemma is proved. □

**Lemma 2.** *Assume system* (6) *has $s$ ($1 \leqslant s < n$) nontrivial formal integrals $\Psi^1(t,y), \dots, \Psi^s(t,y)$ which are functionally independent in $y$. If any nontrivial homogeneous integral $\Psi_q(t,y)$ of the system* (12) *is an analytic function of $\Psi^1_{l_1}(t,y), \dots, \Psi^s_{l_s}(t,y)$, then any other nontrivial formal integral $\Psi(t,y)$ of system* (6) *must be a function of $\Psi^1(t,y), \dots, \Psi^s(t,y)$, i.e.,*

$$\Psi(t,y) = \mathcal{H}\big(\Psi^1(t,y), \dots, \Psi^s(t,y)\big),$$

*where $\mathcal{H}$ is a formal series.*

**Proof.** System (6) is transformed to

$$\dot{u} = Bu + \varepsilon \tilde{h}(t, u, \varepsilon), \tag{24}$$

under the transformation

$$y = \varepsilon u, \tag{25}$$

where $\tilde{h}(t, u, \varepsilon) = O(\|u\|^2)$ and $\tilde{h}(t + T, u, \varepsilon) = \tilde{h}(t, u, \varepsilon)$.

The integral $\Psi^i(t, y)$ $(i = 1, \ldots, s)$ of system (6) can be rewritten as follows

$$\begin{aligned}\tilde{\Psi}^i(t, u, \varepsilon) = \Psi^i(t, \varepsilon u) &= \Psi^i_{l_i}(t, \varepsilon u) + \sum_{j=1}^{\infty} \Psi^i_{l_i+j}(t, \varepsilon u) \\ &= \varepsilon^{l_i}\left(\Psi^i_{l_i}(t, u) + \sum_{j=1}^{\infty} \varepsilon^j \Psi^i_{l_i+j}(t, u)\right),\end{aligned} \tag{26}$$

where $\Psi^i_{l_i+j}(t, u)$ is a homogeneous function of degree $l_i + j$ in $u$ with $T$-periodic function in $t$ as coefficients.

Similarly, other nontrivial formal integral $\Psi(t, y)$ of system (6) reads

$$\begin{aligned}\tilde{\Psi}(t, u, \varepsilon) = \Psi(t, \varepsilon u) &= \Psi_q(t, \varepsilon u) + \sum_{j=1}^{\infty} \Psi_{q+j}(t, \varepsilon u) \\ &= \varepsilon^q\left(\Psi_q(t, u) + \sum_{j=1}^{\infty} \varepsilon^j \Psi_{q+j}(t, u)\right),\end{aligned} \tag{27}$$

where $\Psi_{q+j}(t, u)$ is a homogeneous function of degree $q + j$ in $u$ with $T$-periodic function in $t$ as coefficients.

Evidently, $\tilde{\Psi}^1(t, u, \varepsilon), \ldots, \tilde{\Psi}^s(t, u, \varepsilon)$ and $\tilde{\Psi}(t, u, \varepsilon)$ are integrals of the system (24), therefore $\Psi^1_{l_1}(t, u), \ldots, \Psi^s_{l_s}(t, u)$ and $\Psi_q(t, u)$ are homogeneous integrals of linear system

$$\dot{u} = Bu. \tag{28}$$

By the assumption of the lemma, there exists an analytic function $\mathcal{G}$ such that

$$\Psi_q(t, y) = \mathcal{G}\big(\Psi^1_{l_1}(t, y), \ldots, \Psi^s_{l_s}(t, y)\big), \tag{29}$$

under the transformation (25), we have

$$\begin{aligned}\varepsilon^q \Psi_q(t, u) &= \Psi_q(t, \varepsilon u) \\ &= \mathcal{G}\big(\Psi^1_{l_1}(t, \varepsilon u), \ldots, \Psi^s_{l_s}(t, \varepsilon u)\big) \\ &= \mathcal{G}\big(\varepsilon^{l_1}\Psi^1_{l_1}(t, u), \ldots, \varepsilon^{l_s}\Psi^s_{l_s}(t, u)\big).\end{aligned} \tag{30}$$

By (29) and (30), we obtain

$$\mathcal{G}\big(\varepsilon^{l_1}\Psi^1_{l_1}(t, u), \ldots, \varepsilon^{l_s}\Psi^s_{l_s}(t, u)\big) = \varepsilon^q \mathcal{G}\big(\Psi^1_{l_1}(t, u), \ldots, \Psi^s_{l_s}(t, u)\big). \tag{31}$$

Let $\mathcal{G}^{(0)} = \mathcal{G}$. Then the function

$$\tilde{\Psi}^{(1)}(t, u, \varepsilon) = \tilde{\Psi}(t, u, \varepsilon) - \mathcal{G}^{(0)}\big(\tilde{\Psi}^1(t, u, \varepsilon), \ldots, \tilde{\Psi}^s(t, u, \varepsilon)\big)$$

is obviously an integral of system (24).

By (26), (27), (29) and (31), it is easy to see that the function $\tilde{\Psi}^{(1)}(t, u, \varepsilon)$ is at least of $q + 1$ order with respect to $\varepsilon$, and it can be rewritten as

$$\tilde{\Psi}^{(1)}(t, u, \varepsilon) = \varepsilon^{q_1}\left(\Psi^{(1)}_{q_1}(t, u) + \sum_{j=1}^{\infty} \varepsilon^j \Psi^{(1)}_{q_1+j}(t, u)\right),$$

where $q_1 \geqslant q + 1$ is an integer, $\Psi^{(1)}_{q_1+j}(t, u)$ is a homogeneous form of degree $q_1 + j$ in $u$ with $T$-periodic coefficients in $t$.

It is not difficult to prove that $\Psi_{q_1}^{(1)}(t,u)$ is a homogeneous integral of the system (28). According to the assumptions of the lemma and by a direct computation, we have $\Psi_{q_1}^{(1)}(t,u) = \mathcal{G}^{(1)}(\Psi_{l_1}^1(t,u), \ldots, \Psi_{l_s}^s(t,u))$. So the function

$$\tilde{\Psi}^{(2)}(t,u,\varepsilon) = \tilde{\Psi}^{(1)}(t,u,\varepsilon) - \mathcal{G}^{(1)}\big(\tilde{\Psi}^1(t,u,\varepsilon), \ldots, \tilde{\Psi}^s(t,u,\varepsilon)\big)$$

is also an integral of the system (24) which is at least of $q_1 + 1$ degree with respect to $\varepsilon$.

By repeating infinitely this process, we obtain

$$\tilde{\Psi}(t,u,\varepsilon) = \sum_{j=0}^{\infty} \mathcal{G}^{(j)}\big(\tilde{\Psi}^1(t,u,\varepsilon), \ldots, \tilde{\Psi}^s(t,u,\varepsilon)\big),$$

which is equivalent to the fact that

$$\Psi(t,y) = \sum_{j=0}^{\infty} \mathcal{G}^{(j)}\big(\Psi^1(t,y), \ldots, \Psi^s(t,y)\big) = \mathcal{H}\big(\Psi^1(t,y), \ldots, \Psi^s(t,y)\big),$$

for a certain formal series $\mathcal{H}(\Psi^1(t,y), \ldots, \Psi(t,y))$. This completes the proof. $\square$

With the help of Lemmas 1 and 2, we give the following proof of Theorem 2.

**Proof of Theorem 2.** According to Corollary 1, if system (2) has $s$ $(1 \leqslant s < n)$ nontrivial formal integrals $\Phi^1(t,x), \ldots, \Phi^s(t,x)$ in a neighborhood of a constant solution, then system (6) admits $s$ nontrivial formal integrals $\Psi^1(t,y), \ldots, \Psi^s(t,y)$ in a neighborhood of $y = 0$. In order to obtain the result, we just need to prove that for any nontrivial formal integral $\Phi(t,x)$ of system (2), integral $\Psi(t,y) = \Phi(t, Q(t)y)$ of system (6) is a formal series of $\Psi^1(t,y), \ldots, \Psi^s(t,y)$. From Lemma 2, if we already have that any nontrivial homogeneous integral $\Psi_q(t,y)$ of the linear system (12) is an analytic function of the integrals $\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)$, then the proof of the theorem is completed.

By Lemma 1, any nontrivial homogeneous integral of the linear system (12) is a polynomial function of $\rho^1(t,y), \ldots,$ $\rho^s(t,y)$, so there exist polynomial function $\mathcal{T}$ and $\mathcal{F}_i$ such that

$$\begin{aligned} &\Psi_q(t,y) = \mathcal{T}\big(\rho^1(t,y), \ldots, \rho^s(t,y)\big), \\ &\Psi_{l_i}^i(t,y) = \mathcal{F}_i\big(\rho^1(t,y), \ldots, \rho^s(t,y)\big), \quad i = 1,2,\ldots,s. \end{aligned} \tag{32}$$

Since $\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)$ are functionally independent in $y$, the matrix

$$\frac{\partial(\Psi_{l_1}^1, \ldots, \Psi_{l_s}^s)}{\partial(y_1, \ldots, y_n)} = \frac{\partial(\mathcal{F}_1, \ldots, \mathcal{F}_s)}{\partial(\rho^1, \ldots, \rho^s)} \cdot \frac{\partial(\rho^1, \ldots, \rho^s)}{\partial(y_1, \ldots, y_n)}$$

is full-ranked, therefore, the matrices

$$\frac{\partial(\rho^1, \ldots, \rho^s)}{\partial(y_1, \ldots, y_n)}, \quad \frac{\partial(\mathcal{F}_1, \ldots, \mathcal{F}_s)}{\partial(\rho^1, \ldots, \rho^s)}$$

are full-ranked (nondegenerated). By the Inverse Function Theorem we get from (32) that

$$\rho^i(t,y) = \mathcal{T}_i\big(\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)\big), \quad i = 1,2,\ldots,s,$$

where $\mathcal{T}_i$ is indeed an analytic function. So we have

$$\begin{aligned} \Psi_q(t,y) &= \mathcal{T}\big(\rho^1(t,y), \ldots, \rho^s(t,y)\big) \\ &= \mathcal{T}\big(\mathcal{T}_1\big(\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)\big), \ldots, \mathcal{T}_s\big(\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)\big)\big) \\ &= \mathcal{G}\big(\Psi_{l_1}^1(t,y), \ldots, \Psi_{l_s}^s(t,y)\big). \end{aligned}$$

The theorem is proved. $\square$

## 4. Examples

**Example 1.** Consider the following periodic system

$$\begin{cases} \dot{x}_1 = \left(-1 + \dfrac{3}{2}\cos t\right)x_1 + f_1(t,x), \\ \dot{x}_2 = \left(-1 - \dfrac{3}{2}\sin t\right)x_2 + f_2(t,x), \\ \dot{x}_3 = \sqrt{2}x_3 + f_3(t,x), \end{cases} \tag{33}$$

where $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, $f_i(t,x)$ (for $i = 1,2,3$) are $2\pi$-periodic functions in $t$ and $f_i(t,x) = O(\|x\|^2)$.

We obtain easily that the characteristic multipliers of system (33) are

$$\mu_1 = e^{-2\pi}, \qquad \mu_2 = e^{-2\pi}, \qquad \mu_3 = e^{2\sqrt{2}\pi}.$$

Since there is no resonant condition of the type

$$e^{(-k_1-k_2+\sqrt{2}k_3)2\pi} = 1, \quad k_1,k_2,k_3 \in \mathbb{Z}^+,\ |k_1| + |k_2| + |k_3| \neq 0$$

is fulfilled, according to Theorem 1, then periodic system (33) does not have any nontrivial formal first integral in a neighborhood of $x = 0$.

**Example 2.** To illustrate Theorem 2, we consider the following periodic system

$$\begin{cases} \dot{x}_1 = (\alpha + 3\sin t)x_1 + f(t,x), \\ \dot{x}_2 = (\beta - 2\cos t)x_2 + g(t,x), \\ \dot{x}_3 = (1 + \sin t)x_3, \\ \dot{x}_4 = (-1 - \cos t)x_4, \\ \dot{x}_5 = (-1 + 2\sin t)x_5, \end{cases} \tag{34}$$

where $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5$, $f(t,x)$ and $g(t,x)$ are $2\pi$-periodic functions in $t$ and $f(t,x) = O(\|x\|^2)$, $g(t,x) = O(\|x\|^2)$.

It is not difficult to know that the characteristic multipliers of system (34) are

$$\mu_1 = e^{2\alpha\pi}, \qquad \mu_2 = e^{2\beta\pi}, \qquad \mu_3 = e^{2\pi}, \qquad \mu_4 = e^{-2\pi}, \qquad \mu_5 = e^{-2\pi},$$

and $\Phi_1 = e^{\cos t + \sin t}x_3x_4$ and $\Phi_2 = e^{3\cos t}x_3x_5$ are two nontrivial formal integrals of system (34) and they are functionally independent. According to Theorem 2, we can conclude that any other nontrivial formal first integral of system (34) is a smooth function of $\Phi_1$ and $\Phi_2$ if the rank of the set

$$\Theta = \left\{k = (k_1,k_2,k_3,k_4,k_5) \in \mathbb{Z}_+^5\colon\ \mu_1^{k_1}\mu_2^{k_2}\cdots\mu_5^{k_5} = 1\right\}$$

is equal to 2. Obviously, this is equivalent to that for any $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3 \in \mathbb{Z}^+$,

$$e^{2\pi(\alpha\tilde{k}_1 + \beta\tilde{k}_2 \pm \tilde{k}_3)} \neq 1,$$

since $(0,0,1,1,0)$ and $(0,0,1,0,1)$ are two generating elements of set $\Theta$.

## Acknowledgment

The authors would like to express their sincere thanks to the reviewer for careful reading of the manuscript and valuable comments.

## References

[1] C.C. Chicone, Ordinary Differential Equations with Applications, Springer-Verlag, New York, 1999.
[2] J. Chen, Y.F. Yi, X. Zhang, First integrals and normal forms the germs of analytic vector fields, J. Differential Equations 245 (2008) 1167–1184.
[3] S.D. Furta, On non-integrability of general systems of differential equations, Z. Angew. Math. Phys. 47 (1996) 112–131.
[4] A. Goriely, Integrability, partial integrability, and nonintegrability for systems of ordinary differential equations, J. Math. Phys. 37 (1996) 1871–1893.
[5] K.H. Kwek, Y. Li, S.Y. Shi, Partial integrability for general nonlinear systems, Z. Angew. Math. Phys. 54 (2003) 26–47.
[6] W.G. Li, J. Llibre, X. Zhang, Local first integrals of differential systems and diffeomorphisms, Z. Angew. Math. Phys. 54 (2003) 235–255.
[7] W.G. Li, J. Llibre, X. Zhang, Extension of Floquet's Theory to nonlinear periodic differential systems and embedding diffeomorphisms in differential flows, Amer. J. Math. 124 (2002) 107–127.
[8] J. Moulin-Ollagnier, A. Nowicki, J.M. Strelcyn, On the nonexistence of constants of derivations: the proof of a theorem of Jouanolou and its development, Bull. Sci. Math. 119 (1995) 195–223.
[9] A. Nowicki, On the nonexistence of rational first integrals for systems of linear differential equations, Linear Algebra Appl. 235 (1996) 107–120.

[10] H. Poincaré, Sur l'intégrations des équations différentielles du premier order et du premier degré I and II, Rend. Circ. Mat. Palermo 5 (1891) 161–191; Rend. Circ. Mat. Palermo 11 (1897) 193–239.
[11] S.Y. Shi, W.Z. Zhu, B.F. Liu, Non-existence of first integrals in Laurent polynomial ring for general semi-quasihomogeneous systems, Z. Angew. Math. Phys. 57 (2006) 723–733.
[12] S.Y. Shi, On the nonexistence of rational first integrals for nonlinear systems and semiquasihomogeneous systems, J. Math. Anal. Appl. 335 (2007) 125–134.
[13] H. Yoshida, Necessary condition for the non-existence of algebraic first integral I, II, Celestial Mech. 31 (1983) 363–379, 381–399.
[14] X. Zhang, Local first integrals for systems of differential equations, J. Phys. A 36 (2003) 12243–12253.
[15] X. Zhang, Analytic normalization of analytic integrable systems and the embedding flows, J. Differential Equations 244 (2008) 1080–1092.
[16] S.L. Ziglin, Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I, II, Funct. Anal. Appl. 16 (1983) 181–189; Funct. Anal. Appl. 17 (1983) 6–17.