Delta shock waves in chromatography equations<sup>☆</sup>Hongjun Cheng<sup>a,b,\*</sup>, Hanchun Yang<sup>a</sup><sup>a</sup> Department of Mathematics, Yunnan University, Kunming 650091, PR China<sup>b</sup> Dianchi College, Yunnan University, Kunming 650228, PR China

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## ABSTRACT

The previous investigations on delta shock waves were mostly focused on those with Dirac delta function in only one state variable. In this paper, we obtain another kind from the nonlinear chromatography equations, in which the Dirac delta functions develop simultaneously in both state variables. It is strictly proved to satisfy the system in the sense of distributions. The generalized Rankine–Hugoniot relation and entropy condition are clarified. The numerical results completely coinciding with the theoretical analysis are presented.

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## 1. Introduction

Besides the classical waves: rarefaction wave, shock and contact discontinuity, there exists the nonclassical one, delta shock wave, for hyperbolic systems of conservation laws. A delta shock wave is a generalization of an ordinary shock wave. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a Dirac delta function with the discontinuity as its support. It is more compressive than an ordinary shock wave in the sense that more characteristics enter the discontinuity line. From the physical point of view, a delta shock wave represents the process of concentration of the mass and formation of the universe.

In the past two decades, the research of such a kind of solution has become very active. A great many excellent achievements have been obtained for various hyperbolic systems by many authors. For instance, Korchinski [1], Keyfitz and Kranzer [2,3], LeFloch [4], Joseph [5], Tan, Zhang and Zheng [6], Sheng and Zhang [7], Li and Zhang [8], Li, Yang and Zhang [9], Cheng, Liu and Yang [10], Yang [11], Guo, Sheng and Zhang [12], Chen and Liu [13,14], Danilov and Shelkovich [15], etc. However, it can be noticed that the investigations were mostly focused on the case when only one state variable develops the Dirac delta function and the others are functions with a bounded variation. Thus, an interesting topic is the delta shock waves with the Dirac delta functions in more than one state variables.

To reach such a kind of delta shock wave, in this paper, we focus our attention on the Riemann problem for the following nonlinear chromatography equations

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left\{ \left( 1 + \frac{1}{1-u+v} \right) u \right\} = 0, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left\{ \left( 1 + \frac{1}{1-u+v} \right) v \right\} = 0 \end{cases} \quad (1.1)$$

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with initial condition

$$(u, v)(t = 0, x) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_+, v_+), & x > 0, \end{cases} \quad (1.2)$$

where  $u$  and  $v$  are the non-negative functions of the variables  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ , which express the concentrations of the species to be separated, and  $1 - u + v > 0$ . It is easy to see that (1.1) belongs to the Temple class, i.e., the shock curves coincide with the rarefaction curves in phase plane due to the special form [16,17]. A distinctive feature for (1.1) is just that the delta shock wave with Dirac delta functions in both  $u$  and  $v$  will appear in solutions.

The more general nonlinear chromatography equations read

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial}{\partial t} \left\{ \left( 1 + \frac{a}{1 - u + v} \right) u \right\} = 0, \\ \frac{\partial v}{\partial x} + \frac{\partial}{\partial t} \left\{ \left( 1 + \frac{b}{1 - u + v} \right) v \right\} = 0, \end{cases} \quad (1.3)$$

where  $a$  and  $b$ , with  $b > a > 0$ , are constants. It is noted that  $x$  and  $t$  in (1.3) are opposite to those in physical hyperbolic systems (e.g. gas dynamics) and in most hyperbolic systems literature. Mazzotti et al. [18,19] predicted theoretically the occurrence of the delta shock wave, in which the extreme mass concentrations appear in both species, and provided the detailed experimental evidence. The delta-shock phenomenon originates in the synergistic-competitive behavior of the two species as described by the nonlinear chromatography. There exist some differences between (1.1) and (1.3). System (1.3) is of a mixed type, i.e., it is hyperbolic in the region where  $\{a(1 + v) + b(1 - u)\}^2 - 4ab(1 - u + v) > 0$  and elliptic in the remaining part of it, while (1.1) is always hyperbolic in the whole composition space. For delta-shock solutions, the denominator  $1 - u + v$  still contains a weighted Dirac delta function for (1.3) while it is only a function with a bounded variation for (1.1).

We also notice that another system of nonlinear chromatography equations in the form

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u}{1 + u + v} \right) = 0, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left( \frac{v}{1 + u + v} \right) = 0 \end{cases} \quad (1.4)$$

was considered in [20–22], etc. Different from (1.1) and (1.3), delta shock waves do not develop in solutions to (1.4).

For (1.1) and (1.2), we check that it is solvable in the class of solutions consisting of classical waves of constant states, shocks, rarefaction waves and contact discontinuities except for the case  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ , for which, however, we find that the delta shock waves appear. A distinctive feature for this delta-shock solution is that weighted Dirac delta functions develop in both state variables  $u$  and  $v$  simultaneously. Moreover, we strictly prove it satisfy the system in the sense of distributions. Furthermore, we clarify the generalized Rankine–Hugoniot relation, which allows for the exact determination of the speed of propagation of the singularity and for the rate of change of its strength as it travels in space, and entropy condition, which is an overcompressive one and guarantees the uniqueness of solution. We also simulate the delta shock waves by employing the NT scheme [23]. The numerical results are completely coincident with the theoretical analysis.

The organization of this paper is as follows. In Section 2, the classical elementary waves are solved and by which some Riemann solutions are constructed. In Section 3, a delta-shock solution is obtained for the case  $-u_- + v_- \leq 0 \leq -u_+ + v_+$  and proven to satisfy system (1.1) in the sense of distributions. In Section 4, the generalized Rankine–Hugoniot relation and entropy condition are derived and applied to solving the Riemann problem again. In Section 5, the numerical results of delta shock waves are presented.

## 2. Classical elementary waves and some Riemann solutions

We study the Riemann problem (1.2) for system (1.1). The characteristic roots and corresponding right characteristic vectors of (1.1) are

$$\begin{cases} \lambda_1 = 1 + \frac{1}{1 - u + v}, & r_1 = (1, 1)^T, \\ \lambda_2 = 1 + \frac{1}{(1 - u + v)^2}, & r_2 = (u, v)^T, \end{cases} \quad (2.1)$$

satisfying

$$\nabla \lambda_1 \cdot r_1 \equiv 0, \quad \nabla \lambda_2 \cdot r_2 = -2 \cdot \frac{-u + v}{(1 - u + v)^3}. \quad (2.2)$$

So system (1.1) is nonstrictly hyperbolic,  $\lambda_1$  is always linearly degenerate, and  $\lambda_2$  is genuinely nonlinear if  $u \neq v$  and linearly degenerate if  $u = v$ .

Since (1.1) and (1.2) remain invariant under a uniform expansion of coordinates  $t \rightarrow \alpha t$ ,  $x \rightarrow \alpha x$ ,  $\alpha > 0$ , the solution is only connected with  $\xi = x/t$ . Thus we should seek the self-similar solution

$$(u, v)(t, x) = (u, v)(\xi), \quad \xi = x/t, \quad (2.3)$$

for which system (1.1) becomes

$$\begin{cases} -\xi \frac{du}{d\xi} + \frac{d}{d\xi} \left\{ \left( 1 + \frac{1}{1-u+v} \right) u \right\} = 0, \\ -\xi \frac{dv}{d\xi} + \frac{d}{d\xi} \left\{ \left( 1 + \frac{1}{1-u+v} \right) v \right\} = 0 \end{cases} \quad (2.4)$$

and initial condition (1.2) changes into the infinity boundary condition

$$(u, v)(\pm\infty) = (u_{\pm}, v_{\pm}). \quad (2.5)$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary values in the infinity.

For smooth solutions, (2.4) is equivalent to

$$\begin{pmatrix} 1 + \frac{1}{1-u+v} + \frac{u}{(1-u+v)^2} - \xi & -\frac{u}{(1-u+v)^2} \\ \frac{v}{(1-u+v)^2} & 1 + \frac{1}{1-u+v} - \frac{v}{(1-u+v)^2} - \xi \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = 0, \quad (2.6)$$

which provides either the general solution (constant state)

$$(u, v)(\xi) = \text{Const.}, \quad (2.7)$$

or singular solution, which is a wave of the first characteristic family,

$$\begin{cases} \xi = 1 + \frac{1}{1-u+v}, \\ d(-u+v) = 0, \end{cases} \quad (2.8)$$

or rarefaction wave, which is a wave of the second characteristic family,

$$\begin{cases} \xi = 1 + \frac{1}{(1-u+v)^2}, \\ \frac{du}{dv} = \frac{u}{v}. \end{cases} \quad (2.9)$$

Integrating (2.8) leads to

$$\begin{cases} \xi = 1 + \frac{1}{1-u+v}, \\ -u+v = -u_- + v_-, \end{cases} \quad (2.10)$$

which is actually a contact discontinuity (see (2.13)). We also integrate (2.10) and take the requirement  $\lambda_2(u_-, v_-) < \lambda_2(u, v)$  into account to obtain

$$R: \begin{cases} \xi = 1 + \frac{1}{(1-u+v)^2}, \\ \frac{u}{v} = \frac{u_-}{v_-}, \quad -u+v < -u_- + v_-. \end{cases} \quad (2.11)$$

Thus, for a given state  $(u_-, v_-)$ , all possible states which can be connected to  $(u_-, v_-)$  on the right by a rarefaction wave must be located on the curve  $R(u_-, v_-)$ :  $u/v = u_-/v_-$  ( $-u+v < -u_- + v_-$ ) in the  $(u, v)$ -plane, see Fig. 1.

Let us turn to bounded discontinuity solutions. If  $\xi = \omega$  is a bounded discontinuity, then the Rankine–Hugoniot relation

$$\begin{cases} -\omega[u] + \left[ \left( 1 + \frac{1}{1-u+v} \right) u \right] = 0, \\ -\omega[v] + \left[ \left( 1 + \frac{1}{1-u+v} \right) v \right] = 0 \end{cases} \quad (2.12)$$

holds, where and in the following,  $[p] = p_- - p$  denotes the jump of  $p$  across the discontinuity. By solving (2.12), we obtain contact discontinuity, which is a wave of the first characteristic family,

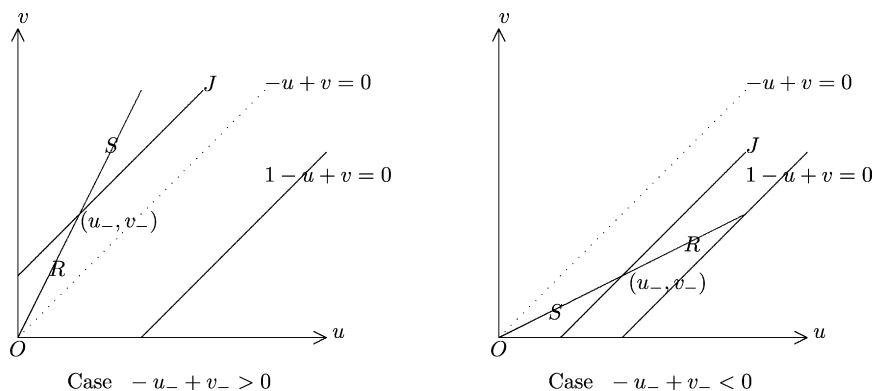


Fig. 1. Curves of elementary waves.

$$J: \omega = 1 + \frac{1}{1 - u_- + v_-} = 1 + \frac{1}{1 - u + v} \quad (2.13)$$

and shock wave, which is a wave of the second characteristic family,

$$S: \begin{cases} \omega = 1 + \frac{1}{(1 - u_- + v_-)(1 - u + v)}, \\ \frac{u}{v} = \frac{u_-}{v_-}. \end{cases} \quad (2.14)$$

The stability condition (entropy condition) for shock can be defined as “three incoming, one outgoing”, which means that three of the characteristic lines on both sides of shock, two  $\lambda_2$  and one  $\lambda_1$ , are incoming with respect to the shock, while the remaining one,  $\lambda_1$ , is outgoing. The stability condition implies that

$$0 < -u_- + v_- < -u + v \quad (2.15)$$

or

$$-u_- + v_- < -u + v < 0. \quad (2.16)$$

As before, the states  $(u, v)$  located on the curve  $J(u_-, v_-)$ :  $-u + v = -u_- + v_-$  in the  $(u, v)$ -plane can be connected to  $(u_-, v_-)$  on the right by a contact discontinuity, and the states  $(u, v)$  located on the curve  $S(u_-, v_-)$ :  $u/v = u_-/v_-$  ( $-u + v > -u_- + v_- > 0$  or  $0 > -u + v > -u_- + v_-$ ) in the  $(u, v)$ -plane can be connected to  $(u_-, v_-)$  on the right by a shock, see Fig. 1.

Using these classical waves, one can construct the solutions of (1.1) and (1.2) by the analysis method in phase plane. From (2.11), (2.13) and (2.14), it is clear that when  $-u_+ + v_+ = -u_- + v_-$  or  $u_- v_+ = u_+ v_-$ , the Riemann solutions contain a single classical wave. For the other cases, we can construct the solutions as follows:

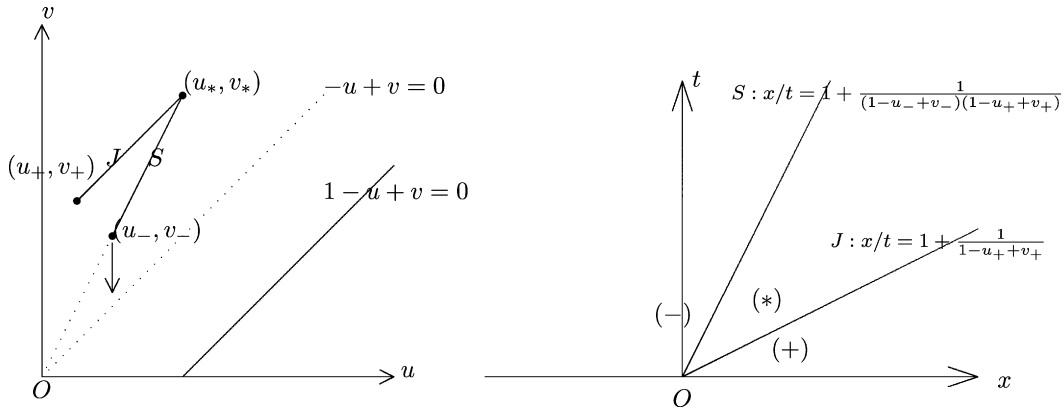
- (1) when  $0 < -u_- + v_- < -u_+ + v_+$ , the solution is  $S + J$ ;
- (2) when  $0 \leq -u_+ + v_+ < -u_- + v_-$ , the solution is  $R + J$ ;
- (3) when  $-u_+ + v_+ < 0 < -u_- + v_-$ , the solution is  $R + R$ ;
- (4) when  $-u_+ + v_+ < -u_- + v_- \leq 0$ , the solution is  $J + R$ ;
- (5) when  $-u_- + v_- < -u_+ + v_+ < 0$ , the solution is  $J + S$ .

### 3. Delta-shock solution

In the last section, by the analysis method in phase plane, we have constructed the solutions for (1.1) and (1.2) by using classical waves except for the cases  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ . However, for this rest case, one can easily check that the solution cannot be constructed by applying these classical waves. In fact, at this moment, the delta shock wave with Dirac delta functions in both  $u$  and  $v$  will occur.

For the boundary case  $-u_- + v_- = 0 < -u_+ + v_+$ , let us consider the limit of solution  $(u, v)(\xi)$  when  $u_+, v_+$  and  $u_-, v_-$  are fixed,  $-u_+ + v_+ > 0$  and  $-u_- + v_- \rightarrow 0^+$ . When  $0 < -u_- + v_- < -u_+ + v_+$ , the solution is  $S + J$  shown in Fig. 2. The intermediate state  $(u_*, v_*)$  satisfies

$$-u_* + v_* = -u_+ + v_+, \quad \frac{u_*}{v_*} = \frac{u_-}{v_-},$$

Fig. 2.  $-u_+ + v_+ > 0$ ,  $-u_- + v_- \rightarrow 0^+$ .

which give

$$u_* = \frac{(-u_+ + v_+)u_-}{-u_- + v_-}, \quad v_* = \frac{(-u_+ + v_+)v_-}{-u_- + v_-}.$$

Therefore

$$\lim_{-u_- + v_- \rightarrow 0^+} u_* = +\infty, \quad \lim_{-u_- + v_- \rightarrow 0^+} v_* = +\infty. \quad (3.1)$$

Moreover, at this time, the speed of shock wave  $S$  tends to that of the contact discontinuity  $J$ , which means that  $S$  and  $J$  coincide to form a new type of nonlinear hyperbolic wave.

Next, let us calculate the total quantities of  $u$ ,  $v$  and  $-u + v$  between  $S$  and  $J$  as  $-u_- + v_- \rightarrow 0^+$ . From the first equation of (2.4), we have

$$\begin{aligned} 0 &= \int_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} -\xi \, du + d\left\{\left(1 + \frac{1}{1-u+v}\right)u\right\} \\ &= -(\xi u) \Big|_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} + \int_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} u \, d\xi \\ &\quad + \left\{\left(1 + \frac{1}{1-u+v}\right)u\right\} \Big|_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0}, \end{aligned}$$

which gives that if  $u_- \neq 0$

$$\lim_{-u_- + v_- \rightarrow 0^+} \int_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} u(\xi) \, d\xi = \int_{1+\frac{1}{1-u_++v_+}-0}^{1+\frac{1}{1-u_++v_+}+0} u(\xi) \, d\xi = \frac{-u_+ + v_+}{1 - u_+ + v_+} u_- \neq 0. \quad (3.2)$$

Similarly, from the second equation of (2.4), we can reach that if  $u_- \neq 0$

$$\lim_{-u_- + v_- \rightarrow 0^+} \int_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} v(\xi) \, d\xi = \int_{1+\frac{1}{1-u_++v_+}-0}^{1+\frac{1}{1-u_++v_+}+0} v(\xi) \, d\xi = \frac{-u_+ + v_+}{1 - u_+ + v_+} u_- \neq 0. \quad (3.3)$$

And we can calculate

$$\lim_{-u_- + v_- \rightarrow 0^+} \int_{\xi=1+\frac{1}{(1-u_++v_+)(1-u_-+v_-)}-0}^{\xi=1+\frac{1}{1-u_++v_+}+0} (-u + v)(\xi) \, d\xi \equiv 0. \quad (3.4)$$

The equalities (3.2) and (3.3) show that  $u(\xi)$  and  $v(\xi)$  possess the same singularity as a weighed Dirac delta function at  $\xi = 1 + \frac{1}{1-u_+ + v_+}$  while (3.4) implies that  $-u + v$  has a bounded variation. Moreover, the inequality

$$\lambda_2(u_+, v_+) < \lambda_1(u_+, v_+) = \sigma = 1 + \frac{1}{1-u_+ + v_+} < \lambda_1(u_-, v_-) = \lambda_2(u_-, v_-) \quad (3.5)$$

is satisfied, where  $\sigma$  is the propagation speed. We call such a kind of nonlinear hyperbolic wave a delta shock wave to system (1.1).

For the other boundary case  $-u_- + v_- < 0 = -u_+ + v_+$ , we consider the limit of solution  $(u, v)(\xi)$  when  $u_-, v_-$  and  $u_+$  are fixed,  $-u_- + v_- < 0$  and  $-u_+ + v_+ \rightarrow 0^-$ . Similarly, we also have a delta shock wave with

$$\lambda_2(u_+, v_+) = \lambda_1(u_+, v_+) < \sigma = 1 + \frac{1}{1-u_- + v_-} = \lambda_1(u_-, v_-) < \lambda_2(u_-, v_-), \quad (3.6)$$

and

$$\int_{\sigma-0}^{\sigma+0} u(\xi) d\xi = \int_{\sigma-0}^{\sigma+0} v(\xi) d\xi = -\frac{-u_- + v_-}{1-u_- + v_-} u_+ \neq 0, \quad \int_{\sigma-0}^{\sigma+0} (-u + v)(\xi) d\xi \equiv 0. \quad (3.7)$$

For the case  $-u_- + v_- < 0 < -u_+ + v_+$ , we suggest that the solution of the Riemann problem is also a delta shock wave with the speed  $\sigma = 1 + \frac{1}{(1-u_- + v_-)(1-u_+ + v_+)}$  satisfying

$$\lambda_2(u_+, v_+) < \lambda_1(u_+, v_+) < \sigma < \lambda_1(u_-, v_-) < \lambda_2(u_-, v_-). \quad (3.8)$$

And we have that if  $u_- v_+ - u_+ v_- \neq 0$

$$\begin{aligned} \int_{\sigma-0}^{\sigma+0} u(\xi) d\xi &= -\sigma(u_- - u_+) + \left(1 + \frac{1}{1-u_- + v_-}\right)u_- - \left(1 + \frac{1}{1-u_+ + v_+}\right)u_+ \\ &= -\left\{1 + \frac{1}{(1-u_- + v_-)(1-u_+ + v_+)}\right\}(u_- - u_+) \\ &\quad + \left(1 + \frac{1}{1-u_- + v_-}\right)u_- - \left(1 + \frac{1}{1-u_+ + v_+}\right)u_+ \\ &= \frac{u_- v_+ - u_+ v_-}{(1-u_- + v_-)(1-u_+ + v_+)} \neq 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \int_{\sigma-0}^{\sigma+0} v(\xi) d\xi &= -\sigma(v_- - v_+) + \left(1 + \frac{1}{1-u_- + v_-}\right)v_- - \left(1 + \frac{1}{1-u_+ + v_+}\right)v_+ \\ &= -\left\{1 + \frac{1}{(1-u_- + v_-)(1-u_+ + v_+)}\right\}(v_- - v_+) \\ &\quad + \left(1 + \frac{1}{1-u_- + v_-}\right)v_- - \left(1 + \frac{1}{1-u_+ + v_+}\right)v_+ \\ &= \frac{u_- v_+ - u_+ v_-}{(1-u_- + v_-)(1-u_+ + v_+)} \neq 0, \end{aligned} \quad (3.10)$$

and

$$\int_{\sigma-0}^{\sigma+0} (-u + v)(\xi) d\xi = -\int_{\sigma-0}^{\sigma+0} u(\xi) d\xi + \int_{\sigma-0}^{\sigma+0} v(\xi) d\xi \equiv 0. \quad (3.11)$$

All in all, for the case  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ , we can construct the Riemann solutions by using a delta shock wave, which is a discontinuity at  $\xi = \sigma$  and satisfies

$$\sigma = 1 + \frac{1}{(1-u_- + v_-)(1-u_+ + v_+)} \quad (3.12)$$

and

$$\lambda_2(u_+, v_+) \leq \lambda_1(u_+, v_+) \leq \sigma \leq \lambda_1(u_-, v_-) \leq \lambda_2(u_-, v_-). \quad (3.13)$$

It is clear that the classical solutions in Section 2 satisfy (1.1) in the sense of distributions. Now we show that the delta shock wave also satisfies (1.1) in the sense of distributions.

**Definition 1.** A pair of  $(u, v)$  constitutes a solution of (1.1) in the sense of distributions if it satisfies

$$\begin{cases} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{ \varphi_t + \left( 1 + \frac{1}{1-u+v} \right) \varphi_x \right\} u \, dx \, dt = 0, \\ \int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{ \varphi_t + \left( 1 + \frac{1}{1-u+v} \right) \varphi_x \right\} v \, dx \, dt = 0 \end{cases} \quad (3.14)$$

for all the test functions  $\varphi(t, x) \in C_0^\infty(R^+ \times R^1)$ .

**Definition 2.** A two-dimensional weighted delta function  $w(s)\delta_L$  supported on a smooth curve  $L$  parameterized as  $t = t(s)$ ,  $x = x(s)$  ( $c \leq s \leq d$ ) is defined by

$$\langle w(s)\delta_L, \varphi(t, x) \rangle = \int_c^d w(s) \varphi(t(s), x(s)) \, ds$$

for all the test functions  $\varphi \in C_0^\infty(R^2)$ .

When  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ , based on the results (3.2)–(3.4), (3.7) and (3.9)–(3.11), we can take the delta shock wave as follows

$$(u, v)(t, x) = \begin{cases} (u_-, v_-), & x < \sigma t, \\ (w_0 t, w_0 t) \delta(x - \sigma t), & x = \sigma t, \\ (u_+, v_+), & x > \sigma t, \end{cases} \quad (3.15)$$

where

$$\begin{aligned} \sigma &= 1 + \frac{1}{(1-u_- + v_-)(1-u_+ + v_+)}, \\ w_0 &= \frac{u_- v_+ - u_+ v_-}{(1-u_- + v_-)(1-u_+ + v_+)}. \end{aligned} \quad (3.16)$$

Moreover, due to the conclusion that  $-u + v$  has a bounded variation, we make the arrangement on the discontinuous line  $x = \sigma t$  as follows

$$\left( 1 + \frac{1}{1-u+v} \right) \Big|_{x=\sigma t} = \sigma. \quad (3.17)$$

Thus we can multiply a delta function and a Heaviside function in a manner similar to that in [6,11], etc.

**Theorem 1.** The delta shock wave defined by (3.15) with (3.16) and (3.17) satisfies (1.1) in the sense of distributions.

**Proof.** For any test function  $\varphi(t, x) \in \mathcal{D}(R^+ \times R)$ , we have

$$\begin{aligned} I_1 &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{ \varphi_t + \left( 1 + \frac{1}{1-u+v} \right) \varphi_x \right\} u \, dx \, dt \\ &= \int_0^{+\infty} \int_{-\infty}^{\sigma t} \left\{ \varphi_t + \left( 1 + \frac{1}{1-u_- + v_-} \right) \varphi_x \right\} u_- \, dx \, dt + \int_0^{+\infty} \int_{\sigma t}^{+\infty} \left\{ \varphi_t + \left( 1 + \frac{1}{1-u_+ + v_+} \right) \varphi_x \right\} u_+ \, dx \, dt \\ &\quad + \int_0^{+\infty} \{ \varphi_t(t, \sigma t) + \sigma \varphi_x(t, \sigma t) \} w_0 t \, dt \end{aligned}$$

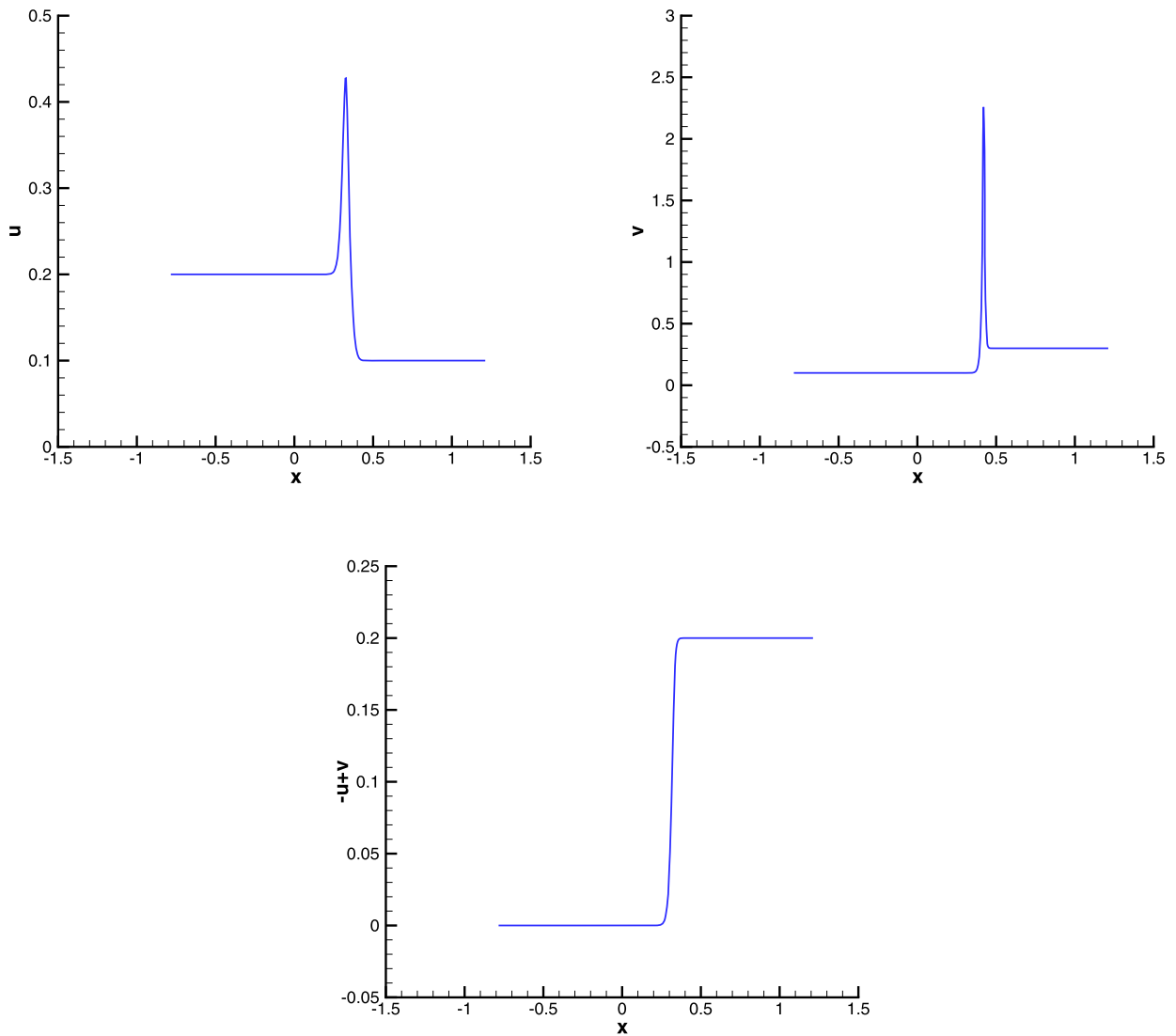


Fig. 3. Numerical results for the case  $-u_- + v_- = 0 < -u_+ + v_+$ .

$$\begin{aligned}
 &= \int_0^{+\infty} \left\{ -[u]\sigma + \left[ \left( 1 + \frac{1}{1-u+v} \right) u \right] - w_0 \right\} \varphi \, dt \\
 &= 0.
 \end{aligned}$$

Similarly, the second equation of (3.14) can be proven. The proof is complete.  $\square$

#### 4. Generalized Rankine–Hugoniot relation for delta shock waves

In this section, we will clarify the generalized Rankine–Hugoniot relation and entropy condition for the delta shock waves, and then apply them to solving the Riemann problem (1.1) and (1.2) for the case  $-u_- + v_- \leq 0 \leq -u_+ + v_+$  again.

Let us seek a solution of (1.1) in the form

$$(u, v)(t, x) = \begin{cases} (u_-, v_-)(t, x), & x < x(t), \\ (w(t), w(t))\delta(x - x(t)), & x = x(t), \\ (u_+, v_+)(t, x), & x > x(t), \end{cases} \quad (4.1)$$

where  $(u_-, v_-)$  and  $(u_+, v_+)(t, x)$  are piecewise smooth solutions of (1.1),  $x(t) \in C^1$ ,  $w(t) \in C^1$ , and  $\delta(\cdot)$  is the standard Dirac measure. If (4.1) satisfies



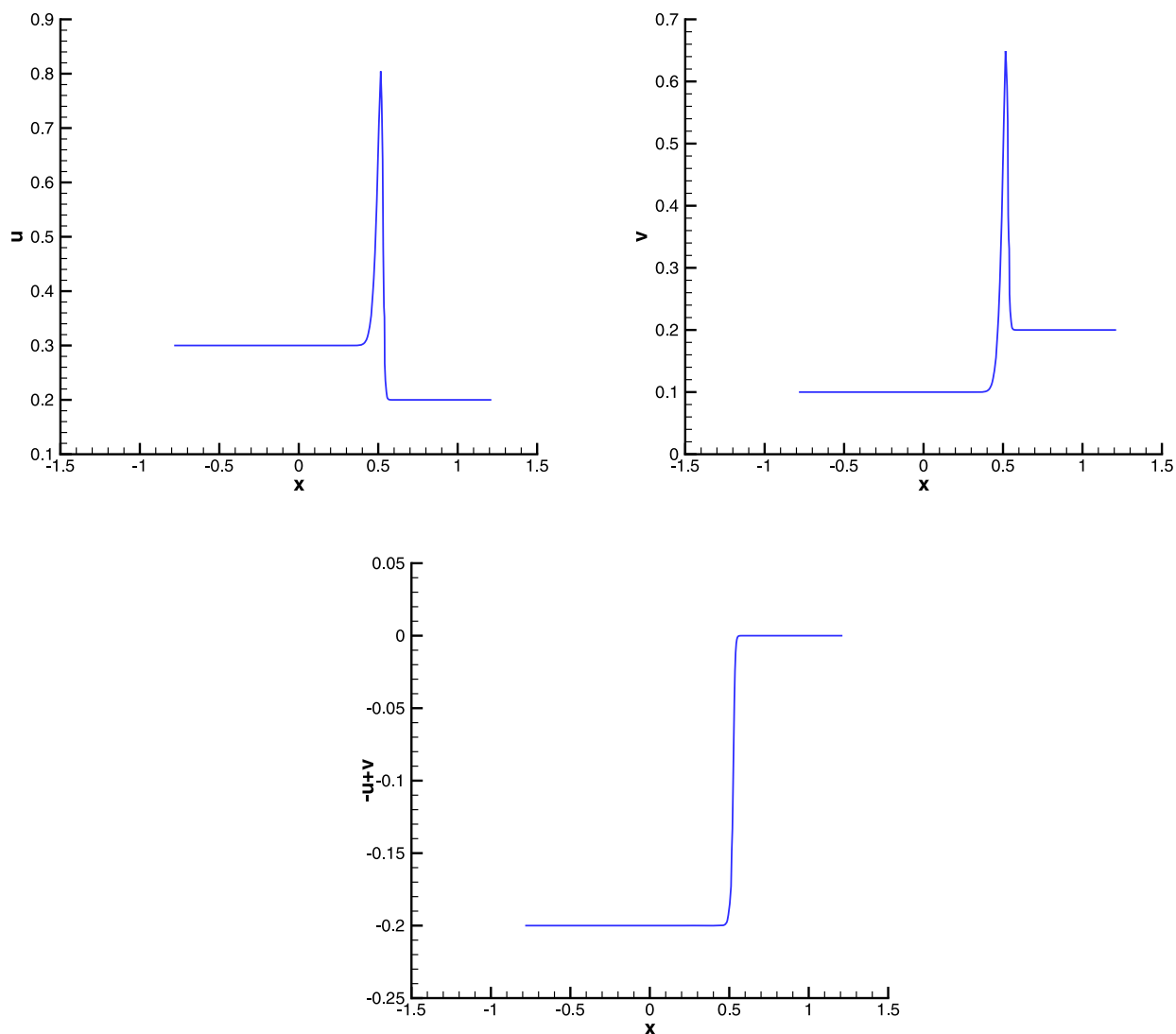


Fig. 4. Numerical results for the case  $-u_- + v_- < 0 = -u_+ + v_+$ .

$$\begin{cases} \frac{dx(t)}{dt} = \sigma, \\ \frac{dw(t)}{dt} = -\sigma[u] + \left[ \left( 1 + \frac{1}{1-u+v} \right) u \right], \\ \frac{dw(t)}{dt} = -\sigma[v] + \left[ \left( 1 + \frac{1}{1-u+v} \right) v \right] \end{cases} \quad (4.2)$$

and

$$\left( 1 + \frac{1}{1-u+v} \right) \Big|_{x=x(t)} = \sigma, \quad (4.3)$$

then the solution defined in (4.1) satisfies (1.1) in the sense of distributions. The proof is similar to that of Theorem 1, so we omit it.

The relation (4.2)–(4.3) is called the generalized Rankine–Hugoniot relation. It describes the relationship among the location, propagation speed, weights and assignment of  $(1 + \frac{1}{1-u+v})$  on its discontinuity.

In addition to the generalized Rankine–Hugoniot relation (4.2)–(4.3), to guarantee uniqueness, the discontinuity satisfies

$$\lambda_2(l_+u(t), l_+v(t)) \leq \lambda_1(l_+u(t), l_+v(t)) \leq \sigma \leq \lambda_1(l_-u(t), l_-v(t)) \leq \lambda_2(l_-u(t), l_-v(t)), \quad (4.4)$$

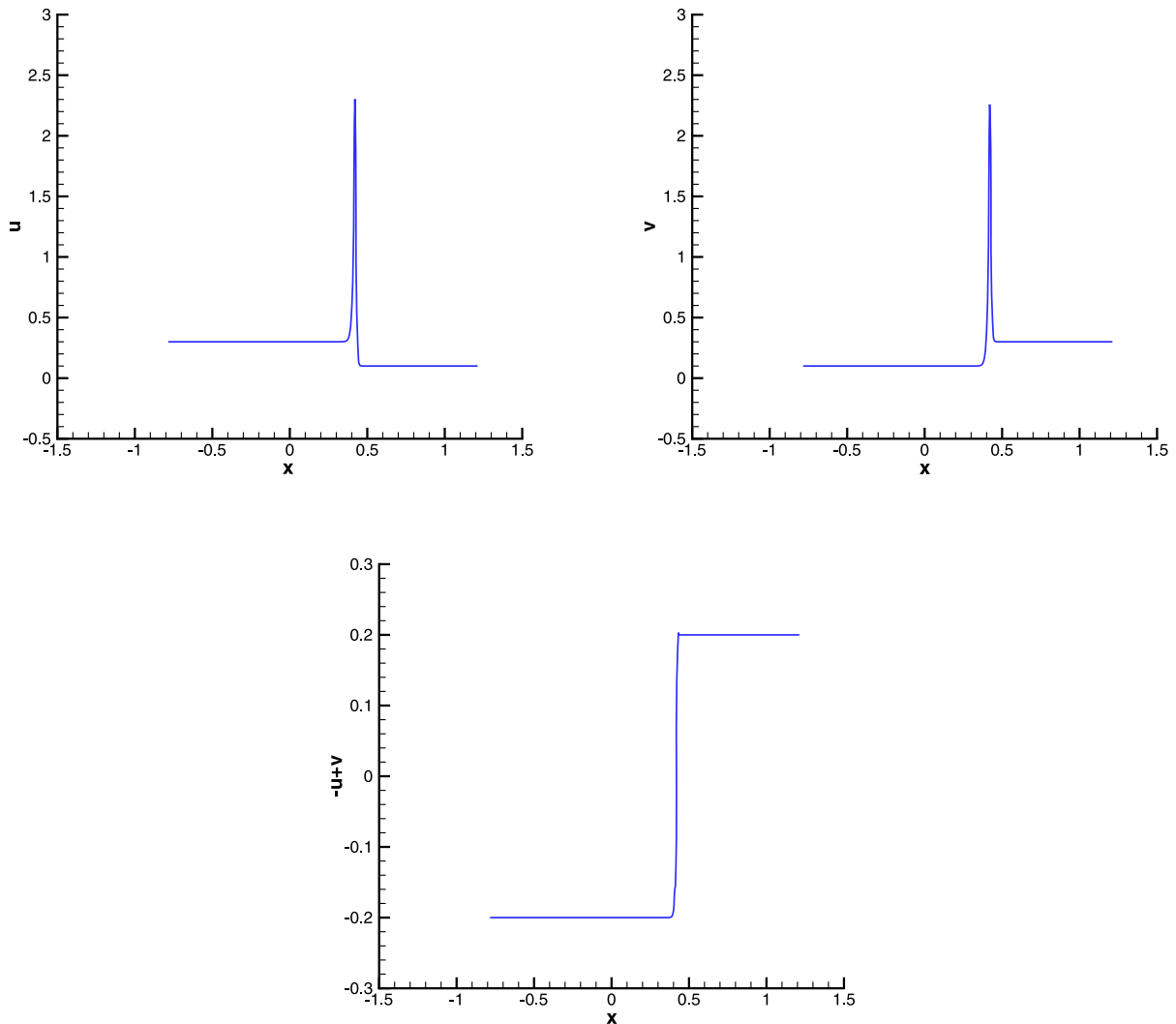


Fig. 5. Numerical results for the case  $-u_- + v_- < 0 < -u_+ + v_+$ .

where  $l_-u(t)$ ,  $l_-v(t)$  and  $l_+u(t)$ ,  $l_+v(t)$  are the respective left- and right-hand limit values of  $u(t, x)$ ,  $v(t, x)$  on the discontinuity curve. Condition (4.4) is called the entropy condition. It is overcompressive and means that all the characteristic lines on both sides of the discontinuity are not out-coming.

A discontinuity satisfying (4.2)–(4.3) and (4.4) will be called a delta shock wave to system (1.1), symbolized by  $\delta$ .

In what follows, the generalized Rankine–Hugoniot relation will be applied in particular to Riemann problem (1.1) and (1.2) for the case  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ . At this moment, the Riemann problem is reduced to solving (4.2)–(4.3), where  $(u_-, v_-)(t, x) = (u_-, v_-)$  and  $(u_+, v_+)(t, x) = (u_+, v_+)$ , with initial data

$$t = 0: \quad x(0) = 0, \quad w(0) = 0. \quad (4.5)$$

After a simple calculation, one can easily obtain that

$$\begin{cases} \sigma = 1 + \frac{1}{(1 - u_- + v_-)(1 - u_+ + v_+)}, \\ x = \sigma t, \\ w(t) = \frac{u_- v_+ - u_+ v_-}{(1 - u_- + v_-)(1 - u_+ + v_+)} t, \\ \left(1 + \frac{1}{1 - u + v}\right)\Big|_{x=\sigma t} = \sigma. \end{cases} \quad (4.6)$$

Since  $-u_- + v_- \leq 0 \leq -u_+ + v_+$ , we can check

$$\lambda_2(u_+, v_+) \leq \lambda_1(u_+, v_+) \leq \sigma \leq \lambda_1(u_-, v_-) \leq \lambda_2(u_-, v_-), \quad (4.7)$$

which means that the entropy condition (4.4) is valid. All these are completely coincident with the results in Section 3.

## 5. Numerical simulations for delta shock waves

In this section, we present some representative numerical results for the delta shock waves mentioned in this paper. Many more numerical tests have been performed to make sure that what are presented are not numerical artifacts. To discretize the system, we employ the Nessyahu–Tadmor scheme [23] with  $500 \times 500$  cells and  $CFL = 0.475$ .

For the boundary case  $-u_- + v_- = 0 < -u_+ + v_+$ , we take the initial data as follows

$$u_- = 0.20, \quad v_- = 0.20, \quad u_+ = 0.10, \quad v_+ = 0.30,$$

and present the numerical results at  $t = 0.5$  in Fig. 3.

For the boundary case  $-u_- + v_- < 0 = -u_+ + v_+$ , we take

$$u_- = 0.30, \quad v_- = 0.10, \quad u_+ = 0.20, \quad v_+ = 0.20$$

as initial data, and give the numerical results at  $t = 0.5$  in Fig. 4.

For the case  $-u_- + v_- < 0 < -u_+ + v_+$ , the initial data is taken as

$$u_- = 0.30, \quad v_- = 0.10, \quad u_+ = 0.10, \quad v_+ = 0.30.$$

And the numerical results at  $t = 0.5$  are shown in Fig. 5.

From Figs. 3–5, one can clearly observe that both state variables  $u$  and  $v$  develop a weighted Dirac delta function, that is, both species develop the extreme mass concentrations, while  $-u + v$  always has a bounded variation and develops a classical shock. All these are in complete agreement with the theoretical analysis.

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