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The bounded complexity function versus the unbounded complexity function

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ABSTRACT

We construct two point-wise periodic flows which are equivalent, such that all the complexity functions of one flow are bounded while the other flow has an unbounded complexity function.

1. Introduction

We use a complexity function to measure how complex a system is. Let $T: X \to X$ be a continuous map on a compact metric space X and let $\mathscr{U}$ be an open cover of X. Set $\mathscr{U}^n = \bigvee_{i=0}^{n} T^{-i}\mathscr{U}$. We define the complexity function of $(T, \mathscr{U})$ as the non-decreasing function $c(\mathscr{U}, n, T) = \min\{\sharp\mathscr{C}\colon\ \mathscr{C}$ is a subcover of $\mathscr{U}^n\}$. It is clear that $h(T) = \sup_{\mathscr{U}} \limsup_{n\to\infty} \frac{1}{n} \log c(\mathscr{U}, n, T)$, where $h(T)$ denotes the topological entropy of T. For a continuous map with a positive topological entropy, the complexity function with a certain open cover must increase exponentially and thus tend to infinity. But in a zero entropy system, all of them might be bounded. To classify the zero entropy systems, we use the complexity function, which is finer than the entropy. By using time-1 map, we define the topological entropy and the complexity function for flow. More precisely, for a flow $\phi: X \times \mathbb{R} \to X$ we define $h(\phi) = h(\phi_1)$, and for an open cover $\mathscr{U}$ of X, we define $c(\mathscr{U}, n, \phi) = c(\mathscr{U}, n, \phi_1)$.

Two flows on compact spaces are equivalent if there exists a homeomorphism of the spaces that sends each orbit of one flow onto an orbit of the other flow while preserving the time orientation. We construct two equivalent flows both with the zero topological entropy, such that all the complexity functions of one flow are bounded while the other flow has an unbounded complexity function. It is known that the zero and the infinite topological entropy are invariants for equivalent flows without fixed points (see [3,4,7]), however, neither the zero topological entropy nor the infinite topological entropy is invariant for equivalent flows with fixed points (see [3,5,6]). Thus, the fixed points are crucial for the entropy of a flow. Nevertheless, the fixed points are not so crucial for the complexity functions of flows. We construct two pairs of equivalent flows with the zero topological entropy, such that all the complexity functions of one flow are bounded while the other

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flow has an unbounded complexity function, what's more, one pair has fixed points while the other does not. We state our theorems as follows.

Theorem 1.1. *There exists a compact metric space M and a pair of equivalent flows $\Phi, \Psi : M \times \mathbb{R} \to M$ without fixed points on M, such that the following properties hold*:

1. *For any open cover $\mathscr{U}$ on M, there exists $K > 0$ such that $c(\mathscr{U}, n, \Phi) < K$ for any $n \in \mathbb{N}$.*
2. *There exists a finite open cover $\mathscr{U}_0$ on M, such that $c(\mathscr{U}_0, n, \Psi) \to \infty$ as $n \to \infty$.*

Let $\phi : X \times \mathbb{R} \to X$ be a flow on a compact metric space X with fixed points. If $\overline{\{x \in X \mid x \text{ is not a fixed point}\}}$ contains a fixed point of ϕ, we call ϕ a flow with *nontrivial fixed points*. We call a flow *point-wise periodic* if every point is periodic.

Theorem 1.2. *There exists a compact metric space M and a pair of equivalent point-wise periodic flows $\Phi : M \times \mathbb{R} \to M$ and $\Psi : M \times \mathbb{R} \to M$ with nontrivial fixed points, such that the following properties hold*:

1. *For any open cover $\mathscr{U}$ on M, there exists $K > 0$ such that $c(\mathscr{U}, n, \Phi) < K$ for any $n \in \mathbb{N}$.*
2. *There exists a finite open cover $\mathscr{U}_0$ on M, such that $c(\mathscr{U}_0, n, \Psi) \to \infty$ as $n \to \infty$.*

Lemma 1.3. *(See [1].) Let $T : X \to X$ be a homeomorphism on a compact metric space X. Then the following two statements are equivalent*:

1. *(X, T) is equicontinuous, that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that for any pair of points $x, y \in X$ with $d(x, y) < \delta$ and any $n \in \mathbb{N}$, it holds that $d(T^n(x), T^n(y)) < \varepsilon$.*
2. *For any finite open cover $\mathscr{U}$ of X, $c(\mathscr{U}, n, T)$ is bounded.*

According to this lemma, we will prove the two theorems by constructing a pair of equivalent flows, in which, the time-1 map of one flow is equicontinuous, while the other is not.

We will prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. In Section 4, we will refine the Bowen–Walters metric [2] and construct a delicate example of a point-wise periodic suspension flow with an unbounded complexity function.

2. The proof of Theorem 1.1

2.1. The construction of two flows without fixed points

In this subsection, we construct two equivalent flows without fixed points. Let $I = [0, 1]$. Let $\mathbb{S}^1$ denote a circle with circumference 1. We denote by $M = I \times \mathbb{S}^1 = \{(x, u) \mid x \in I,\ u \in \mathbb{S}^1\}$ the quotient space of $\{(x, u) \mid x \in I,\ u \in \mathbb{R}\}$ under the equivalent relation $(x, u) \sim (x, u + 1)$. Under the Riemannian metric d, M is a compact metric space. On M, we define a flow Φ by

$$\Phi(x, u, t) = \big(x, u + t \ (\mathrm{mod}\ 1)\big), \quad \text{for } t \in \mathbb{R}.$$

And we define another flow Ψ by

$$\Psi(x, u, t) = \left(x, u + \frac{t}{1 + x} \ (\mathrm{mod}\ 1)\right), \quad \text{for } t \in \mathbb{R}.$$

Ψ is equivalent to Φ, since the transformation $id : M \to M$, $(x, u) \mapsto (x, u)$ maps each orbit of Ψ onto an orbit of Φ preserving the time orientation.

2.2. The equicontinuity versus the non-equicontinuity

By Lemma 1.3, to prove Theorem 1.1, it suffices to show that one of the two flows constructed in Section 2.1 is equicontinuous, while the other is not.

It is clear that Φ_1 is equicontinuous. By Lemma 1.3, for any finite open cover $\mathscr{U}$ of M, $c(\mathscr{U}, n, \Phi)$ is bounded. It remains to prove that (M, Ψ_1) is not equicontinuous. Let $\varepsilon_0 = \frac{1}{3}$. For any $\delta > 0$, we take an irrational number $y \in (0, \delta)$. It is clear that $d((0, 0), (y, 0)) < \delta$. However, we will prove that there is $u \in \mathbb{N}$ such that $d(\Psi_u(0, 0), \Psi_u(y, 0)) > \varepsilon_0$. Although $(0, 0)$ is not a fixed point of Ψ, it is a fixed point under Ψ_1. The orbit of $(y, 0)$ under Ψ_1 is a rotation on $\mathbb{S}^1$ with the rotation number $\frac{1}{1+y}$, which is irrational as well as y. So this orbit is dense in $\mathbb{S}^1$. Thus, there is $u \in \mathbb{N}$ such that $\Psi_u(y, 0) \in \{y\} \times [\frac{3}{8}, \frac{5}{8}]$. As a result,

$$d\big(\Psi_u(0, 0), \Psi_u(y, 0)\big) \geqslant \left(y^2 + \left(\frac{3}{8}\right)^2\right)^{\frac{1}{2}} > \frac{3}{8} > \frac{1}{3},$$

which implies that (M, Ψ_1) is non-equicontinuous. By Lemma 1.3, there exists a finite open cover $\mathscr{U}_0$ on M, such that $c(\mathscr{U}_0, n, \Psi_1) \to \infty$ as $n \to \infty$, which gives rise to Theorem 1.1. $\square$

Remark 1. By adding some isolated fixed points to Φ and Ψ, respectively, one can extend Theorem 1.1 to the flows with fixed points, but these fixed points are trivial.

Remark 2. There are some point-wise periodic flows without fixed points, such as the flows in the proof of Theorem 1.1. By time reparameterization, one can decelerate a flow near a given point to get a nontrivial fixed point in order to extend Theorem 1.1. However, the decelerated flows are not point-wise periodic, because the orbit of any other point on the original orbit of the given point is no longer periodic.

3. The proof of Theorem 1.2

3.1. The construction of two flows with fixed points

In this subsection, we construct a pair of equivalent flows with fixed points on a disk $D = \{\rho e^{i\pi\theta} \mid \rho \in [0, 1], \theta \in [0, 2)\}$. We define a flow Φ by

$$\Phi(x, t) = \rho e^{i\pi(\theta+t)}, \quad \forall x = \rho e^{i\pi\theta} \in D,\ t \in \mathbb{R}.$$

And we define another flow Ψ by

$$\Psi(x, t) = \rho e^{i\pi(\theta+\frac{t}{\rho})}, \quad \forall x = \rho e^{i\pi\theta} \in D,\ t \in \mathbb{R}.$$

We note that 0 is the unique fixed point for both Φ and Ψ. Moreover, Ψ is equivalent to Φ, since the transformation $id : M \to M$, $(x, u) \mapsto (x, u)$ maps each orbit of Ψ onto an orbit of Φ preserving the time orientation. We still use Riemann metric in this example.

3.2. The equicontinuity versus the non-equicontinuity

By Lemma 1.3, to prove Theorem 1.2, it suffices to show that one of the two flows constructed in Section 3.1 is equicontinuous, while the other is not.

Notice that for any $x = \rho_x e^{i\pi\theta_x}$, $y = \rho_y e^{i\pi\theta_y} \in D$ and $t \in \mathbb{N}$,

$$d\big(\Phi(x, t), \Phi(y, t)\big) = \|x - y\| \cdot \big\|e^{i\pi t}\big\| = \|x - y\| = d(x, y).$$

So Φ_1 is equicontinuous. To get the non-equicontinuity of (D, Ψ_1), it remains to prove that for any $\delta > 0$ we can find $x, y \in D$ and $u \in \mathbb{N}$ such that $d(x, y) < \delta$ while $d(\Psi(x, u), \Psi(y, u)) > 1$. For a $\delta > 0$ given arbitrarily, we take $x = \frac{1}{2}$ and $y = \frac{p}{q} \in (\frac{1}{2}, \frac{1}{2} + \delta)$ such that $p, q \in \mathbb{N}$ and q is odd. Here, p and q are not necessarily coprime. Clearly, $d(x, y) < \delta$. However, if we take $u = p$, then it holds that $\Psi(x, u) = xe^{i2\pi p} = x = \frac{1}{2}$, while $\Psi(y, u) = ye^{i\pi\frac{q}{p}\cdot p} = -y = -\frac{p}{q}$. So $d(\Psi(x, u), \Psi(y, u)) = \frac{1}{2} + \frac{p}{q} > 1$. As a result, Ψ_1 is not equicontinuous, which gives rise to Theorem 1.2. $\square$

4. An example of a suspension flow with an unbounded complexity function

In this section, we construct a delicate example of a point-wise periodic suspension flow with an unbounded complexity function.

4.1. The construction of the flow Ψ

In this subsection, we will construct a point-wise periodic suspension flow Ψ. Set $\Sigma_2 = \{(\cdots s_{-1}\bar{s}_0 s_1 \cdots) \mid s_i = 0 \text{ or } 1,\ \forall i \in \mathbb{Z}\}$. A point $(\cdots s_{-1}\bar{s}_0 s_1 \cdots)$ in Σ_2 indicates a 2-sided infinite sequence, in which each position is assigned a word 0 or 1 and the 0th position is marked with a bar. We use the metric topology in this space:

$$\|x - y\| = \sum_{n=-\infty}^{\infty} |x_n - y_n| 2^{-|n|}, \quad \forall x, y \in \Sigma_2.$$

Clearly, $(\Sigma_2, \|\cdot\|)$ is a compact metric space. Moreover, we set

$$\sigma : \Sigma_2 \to \Sigma_2, \quad \sigma(\cdots s_{-1}\bar{s}_0 s_1 \cdots) = (\cdots s_{-1} s_0 \bar{s}_1 s_2 \cdots).$$

We choose a series of periodic points of σ as follows. Take $x_2 = (\cdots 011\bar{0}11011\cdots)$, a 2-sided sequence repeating the word 011. Take $x_3 = (\cdots 0111\bar{0}1110111\cdots)$, a 2-sided sequence repeating the word 0111. In general, we take

$$x_n = (\cdots 0\underbrace{1\cdots 1}_{n}\bar{0}\underbrace{1\cdots 1}_{n}0\underbrace{1\cdots 1}_{n}0\underbrace{1\cdots 1}_{n}\cdots),$$

a 2-sided sequence repeating the word $0\underbrace{1\cdots 1}_{n}$. We denote these periodic orbits by $X_2, \dots, X_n, \dots$, that is, $X_n = \mathrm{Orb}(x_n, \sigma)$, $n \geqslant 2$. We denote by X the disjoint union of all the X_i's ($i \geqslant 2$) together with a point x_∞. Moreover, we define on X a metric d_X as follows:

$$d_X(x, y) = \begin{cases} \frac{1}{n^2}\|x-y\| & \text{for } x, y \in X_n; \\ d_X(y, x) = \sum_{i=n}^{p} \frac{3}{i^2} & \text{for } x \in X_n,\ y \in X_p,\ n < p; \\ \sum_{i=n}^{\infty} \frac{3}{i^2} & \text{for } x = x_\infty,\ y \in X_n. \end{cases} \tag{4.1}$$

We claim that (X, d_X) is a compact metric space. Suppose there is an open cover $\mathscr{U}$ of X, in which an element U covers x_∞, what's more, U also contains a small ball $B(x_\infty, b)$ for some $b > 0$. Note that there exists $n \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \frac{3}{i^2} < b$, then $\bigcup_{i=n}^{\infty} X_i \subset B(x_\infty, b) \subset U$. Since $X_2, \dots, X_{n-1}$ are compact, so is their disjoint union, and then a subset $\mathscr{U}_0 \subset \mathscr{U}$ with only finite element can cover $\bigcup_{i=2}^{n-1} X_i$. As a result, $\mathscr{U}_0 \cup \{U\}$ can cover X, which implies that X is compact under d_X. Moreover, under this metric, it is clear that $\mathrm{diam}(X_i) < 1$, $i \geqslant 2$.

On this compact metric space (X, d_X), we define $T : X \to X$ by taking $T|_{X_i} = \sigma$ for $i \in \mathbb{N}$ and $T(x_\infty) = x_\infty$. Obviously, $h(T) = \sup_i h(T|_{X_i}) = 0$. Now we construct our flows. Let r_i be a positive continuous function on X_i. We define a quotient space $X_i^{r_i} = \{(x, u) \mid 0 \leqslant u \leqslant r_i(x),\ x \in X_i\}$ by the equivalent relation $(x, r_i(x)) \sim (T(x), 0)$. We define a flow Ψ_i by

$$\Psi_i(x, u, t) = (x, u + t), \quad \forall t \in [-u, r_i(x) - u).$$

Then $X_i^{r_i}$ is a compact metric space and Ψ_i is a flow on it. Set $I_n = \{x \in \Sigma_2 \mid x(k) = 1,\ -n+1 \leqslant k \leqslant n-1\}$, $n \in \mathbb{N}$. From now on, we take

$$r_i(x) = \begin{cases} i^n \cdot n \cdot 4 \cdot 3^n & \text{for } x \in I_n \setminus I_{n+1}; \\ 12i & \text{for } x \notin I_1. \end{cases} \tag{4.2}$$

By the definition of X_i, $I_n \cap X_i = \emptyset$, $\forall n > \frac{i}{2}$. Let X_∞ denote $\{x_\infty\} \times \mathbb{S}^1$, namely, the quotient space of $\{(x_\infty, u) \mid 0 \leqslant u \leqslant 1\}$ under the equivalent relation $(x_\infty, 0) \sim (x_\infty, 1)$. And let W denote the disjoint union of all the $X_i^{r_i}$'s and X_∞. Then we can define a flow Ψ on W as

$$\Psi|_{X_i^{r_i}} = \Psi_i \quad \text{and} \quad \Psi\big((x_\infty, u), t\big) = (x_\infty, u), \quad \forall t \in \mathbb{R}. \tag{4.3}$$

Obviously, Ψ is a point-wise periodic flow. From now on, in the whole section, $r(x)$ stands for $r_i(x)$ for $x \in X_i$, $2 \leqslant i < \infty$. We recall that $r(x) \geqslant 1$ for all x and $r(x_\infty) = 1$.

4.2. An adapted Bowen–Walters distance and its properties

In this subsection, we introduce a distance d on W, which is an adaption of Bowen–Walters distance (see [2]), and then we give some delicate properties of d, see Proposition 4.1, Proposition 4.2 and Corollary 4.3.

For a point $(x, h) \in W$, we call $\frac{h}{r(x)}$ its ratio. If two points (x, h) and $(y, l) \in W$ have the same ratio, i.e. $\frac{h}{r(x)} = \frac{l}{r(y)} = t \in [0, 1]$, then we say that there is a *horizontal segment* with ending points (x, h) and (y, l) (or a horizontal segment between (x, h) and (y, l)) and call the common ratio t of (x, h) and (y, l) the ratio of the horizontal segment. We define the length of the horizontal segment by

$$\rho_{\mathrm{hor}}(x, y, t) = (1-t) d_X(x, y) + t d_X\big(T(x), T(y)\big). \tag{4.4}$$

Clearly, $\rho_{\mathrm{hor}}(x, y, 0) = d_X(x, y)$ and $\rho_{\mathrm{hor}}(x, y, 1) = d_X(T(x), T(y))$. If $x \in X_i$ and $y \in X_j$ with $2 \leqslant i < j \leqslant \infty$, then by (4.1), the formula (4.4) can be simplified to

$$\rho_{\mathrm{hor}}(x, y, t) = d_X(x, y) = d_X\big(T(x), T(y)\big) = \sum_{l=i}^{j} \frac{3}{l^2}. \tag{4.5}$$

Therefore, if $x, y \in X_i$ and $z \in X_j$ with $j \neq i$, then for $(x, t \cdot r(x))$, $(y, t \cdot r(y))$ and $(z, t \cdot r(z))$, it holds that $\rho_{\mathrm{hor}}(x, y, t) \leqslant \rho_{\mathrm{hor}}(x, z, t)$.

For (x,t), (y,s), where $x, y \in X \setminus \{x_\infty\}$ are in the same orbit of Ψ, we name the *vertical segment* with ending points (x,t) and (y,s) (or between the two points) after the shortest orbit segment, and thus we define the length of the vertical segment between (x,t) and (y,s) by

$$\rho_{\mathrm{ver}}\big((x,t),(y,s)\big) = \inf\big\{|u| \;\big|\; \Psi_u(x,t) = (y,s),\ u \in \mathbb{R}\big\}. \tag{4.6}$$

For (x_∞, s) and (x_∞, t), the shortest arc in $\mathbb{S}^1$ is called the vertical segment with ending points (x_∞, t) and (x_∞, s) (or between the two points), and thus the length of the vertical segment is defined by

$$\rho_{\mathrm{ver}}\big((x_\infty,t),(x_\infty,s)\big) = \min\big\{|t-s|, 1-|t-s|\big\}. \tag{4.7}$$

We denote a path by a series of horizontal or vertical segments $\Gamma_1, \Gamma_2, \ldots$, where one ending point of Γ_i coincides with that of Γ_{i+1}, one by one.

Finally, for points $(x,h), (y,l) \in W$, the distance $d((x,h),(y,l))$ is defined as the infimum of the length of paths between (x,h) and (y,l) composed of a finite number of horizontal and vertical segments. A path connecting two points (x,h) and $(y,l) \in W$ is called the shortest one if its length is $d((x,h),(y,l))$. Clearly, d is a metric on W, and W is compact under this metric d. Actually, X_∞ is also compact by the definition of d. For a given open cover $\mathscr{U}$ of W, there are finite many elements $U_1, \ldots, U_n$ such that $\bigcup_{i=1}^n U_i \supset X_\infty$. Thus, the union $\bigcup_{i=1}^n U_i$ covers a b-neighborhood of X_∞ for some real $b > 0$. It is clear that there are only finite $X_k^{r_k}$'s ($k \in \mathbb{N}$) outside this b-neighborhood, whose union is compact and can be covered by a finite subcover $\mathscr{U}'$ of $\mathscr{U}$. Consequently, W can be covered by $\mathscr{U}' \cup \{U_1, \ldots, U_n\}$, a finite subcover of $\mathscr{U}$. So W is compact.

Proposition 4.1 below shows that $d((x,0),(y,0)) = d_X(x,y)$ for any $x, y \in X$, meaning that d restricted on X coincides with d_X. Actually, d has more delicate properties as in the following propositions.

Proposition 4.1. *For two points $(x,h), (y,l) \in W$ with the common ratio $\frac{h}{r(x)} = \frac{l}{r(y)} \in [0,1]$, the horizontal segment between them is the shortest path.*

Proof. To prove this proposition, we consider the following two cases.

Case 1. Suppose $x \in X_i$ and $y \in X_j$ for $i \neq j$, $i, j = 2, 3, \ldots, \infty$.

Without loss of generality, we assume $i < j \leqslant +\infty$. For a given path Γ which consists of some horizontal segments $\Gamma_1, \ldots, \Gamma_n$ and some vertical segments, we pick up such horizontal segments that the two ending points of each horizontal segment are not in the same X_k^r. This can be done since $i \neq j$. Without loss of generality, we suppose $\Gamma_1, \ldots, \Gamma_n$ are all such horizontal segments, and suppose that Γ_1 is between a point in $X_{i_1}^r$ and a point in $X_{i_2}^r$, Γ_2 is between a point in $X_{i_2}^r$ and a point in $X_{i_3}^r, \ldots, \Gamma_n$ is between a point in $X_{i_n}^r$ and a point in $X_{i_{n+1}}^r$, where $i = i_1 < i_2 < \cdots < i_{n+1} = j$. By (4.1) and (4.5), the sum of the length of these horizontal segments, $\sum_{p=1}^n \sum_{q=i_p}^{i_{p+1}} \frac{3}{q^2}$, is clearly bigger than or equal to $\sum_{l=i}^j \frac{3}{l^2} = d_X(x,y)$, the length of the unique horizontal segment between (x,h) and (y,l). So the proposition holds in Case 1.

Case 2. Suppose that x, y belongs to the same X_i, $2 \leqslant i \leqslant +\infty$.

We will prove that, between (x,h) and (y,l), there always exists a path shorter than a given path with more than one horizontal segment. Let us consider a path Γ connecting (x,h) and (y,l) and consider the situation that there exists a horizontal segment between certain (p,h') and (q,l') such that the ratio $t' = \frac{h'}{r(p)} = \frac{l'}{r(q)}$ is bigger than $t = \frac{h}{r(x)} = \frac{l}{r(y)}$ (the situation $t' < t$ is similar). There are only finite many horizontal segments in Γ, among which we choose one horizontal segment with the biggest ratio. Denote by (a,h_2) and (b,l_2), the two ending points of the chosen horizontal segment; denote by (a,h_1) and (a,h_2) with $h_2 > h_1$, the two ending points of the leading vertical segment; denote by (b,l_2) and (b,l_1) with $l_2 > l_1$, the two ending points of the trailing vertical segment. Without loss of generality, we suppose that $a, b \in X_i$ and $\frac{h_1}{r(a)} \geqslant \frac{l_1}{r(b)}$. Take $l_2' = \frac{h_1}{r(a)} \cdot r(b)$ and denote the ratios $t_1 = \frac{h_2}{r(a)} = \frac{l_2}{r(b)}$, $t_2 = \frac{h_1}{r(a)} = \frac{l_2'}{r(b)}$. Then we consider two paths Γ_1 and Γ_2 between (a,h_1) and (b,l_1), see Fig. 1. Γ_1 consists of three segments: a vertical segment between (a,h_1) and (a,h_2), a horizontal segment between (a,h_2) and (b,l_2), a vertical segment between (b,l_2) and (b,l_1). Γ_2 consists of two segments: a horizontal segment between (a,h_1) and (b,l_2'), a vertical segment between (b,l_2') and (b,l_1). By (4.4) and (4.6), the length of the Γ_1 is

$$L_1 = h_2 - h_1 + (1-t_1)d_X(a,b) + t_1 d_X\big(T(a),T(b)\big) + l_2 - l_1,$$

while the length of Γ_2 is

$$L_2 = (1-t_2)d_X(a,b) + t_2 d_X\big(T(a),T(b)\big) + l_2' - l_1.$$

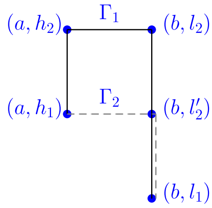

Fig. 1. Two paths connecting (a, h_1) and (b, l_1).

Thus,

$$\begin{aligned} L_1 - L_2 &= (t_1 - t_2)\big(r(a) + r(b) + d_X(a, b) - d_X\big(T(a), T(b)\big)\big) \\ &> (t_1 - t_2)(2 - \operatorname{diam} X_i) \\ &= (t_1 - t_2)\left(2 - \frac{3}{i^2}\right) \\ &> 0. \end{aligned}$$

If Γ_1 is replaced with Γ_2 in Γ, then we get a new path between (x, h) and (y, l), which is clearly shorter than Γ. Therefore, given a path connecting (x, h) and (y, l) with horizontal segments, there is an adapted shorter path such that all its horizontal segments share the same ratio t. Now we consider a path whose horizontal segments share the ratio $t = \frac{h}{r(x)} = \frac{l}{r(y)}$. We note that between two points $(p, h'), (q, l') \in X_i^r$ with the ratio $\frac{h'}{r(p)} = \frac{l'}{r(q)} = t$ the vertical segment is longer than the horizontal segment, because

$$\rho_{\mathrm{ver}}\big(\big(p, h'\big), \big(q, l'\big)\big) \geqslant \min\big\{(1 - t)r(p) + t \cdot r(q), (1 - t)r(q) + t \cdot r(p)\big\} \geqslant 1,$$

while

$$\rho_{\mathrm{hor}}(p, q, t) = (1 - t)d_X(p, q) + t d_X\big(T(p), T(q)\big) \leqslant \operatorname{diam} X_i \leqslant \frac{3}{i^2} < 1,$$

see (4.1), (4.4), (4.6) and recall that $i \geqslant 2$. Therefore, given a path connecting (x, h) and (y, l), there is an adapted shorter path consisting of horizontal segments with the same ratio t. Note that for three points (p_1, k_1), (p_2, k_2) and $(p_3, k_3) \in X_i^r$ with the same ratio $\frac{k_1}{r(p_1)} = \frac{k_2}{r(p_2)} = \frac{k_3}{r(p_3)} = t$, the triangle inequality holds:

$$\begin{aligned} &\rho_{\mathrm{hor}}(p_1, p_2, t) + \rho_{\mathrm{hor}}(p_2, p_3, t) - \rho_{\mathrm{hor}}(p_3, p_1, t) \\ &\quad = (1 - t)\big(d_X(p_1, p_2) + d_X(p_2, p_3) - d_X(p_3, p_1)\big) \\ &\qquad + t\big(d_X\big(T(p_1), T(p_2)\big) + d_X\big(T(p_2), T(p_3)\big) - d_X\big(T(p_3), T(p_1)\big)\big) \\ &\quad \geqslant 0. \end{aligned}$$

Therefore, given a path connecting (x, h) and (y, l), there is an adapted shorter path consisting of exactly one horizontal segment. We conclude by definition that the horizontal segment is the shortest path between (x, h) and (y, l) with $\frac{h}{r(x)} = \frac{l}{r(y)}$, which gives rise to Proposition 4.1. $\square$

Proposition 4.2. *Let (x, h), (y, l) be two points in W and suppose there is a path with exactly one horizontal segment such that the length of this path coincides with $d((x, h), (y, l))$. Then among all such paths, there is a path whose horizontal segment has the ratio $\frac{h}{r(x)}$ or $\frac{l}{r(y)}$.*

Proof. Without loss of generality, we suppose $t_1 = \frac{h}{r(x)} \geqslant \frac{l}{r(y)} = t_3$. For a given path Γ_2 with exactly one horizontal segment whose length is $d((x, h), (y, l))$, we denote by t_2 the ratio of its horizontal segment. If $t_2 > t_1$, then between (x, h) and $(y, t_1 r(y))$ we can replace the part of Γ_2 with the unique horizontal segment, which is the shortest path between (x, h) and $(y, t_1 r(y))$ by Proposition 4.1. Consequently, we can get a path between (x, h) and (y, l) strictly shorter than Γ_2, which contradicts the assumption of Γ_2. So, $t_2 \leqslant t_1$. Similarly, we can prove $t_2 \geqslant t_3$. Then it follows that $t_1 \geqslant t_2 \geqslant t_3$. By (4.4) and (4.6), the length of Γ_2 is

$$L_2 = (1 - t_2)d_X(x, y) + t_2 d_X\big(T(x), T(y)\big) + (t_1 - t_2)r(x) + (t_2 - t_3)r(y).$$

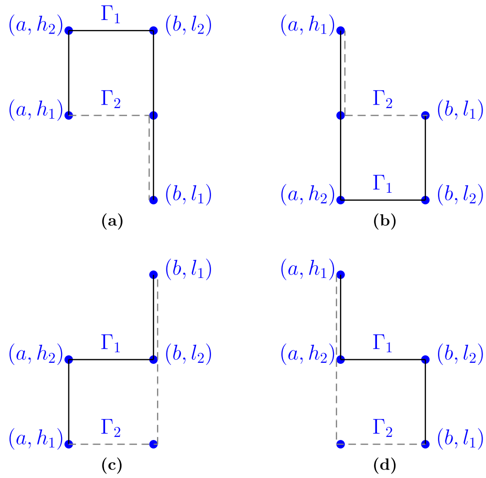

Fig. 2. The closed polygonal curves are 4 examples of the possible parts in non-typical paths.

We take two paths, one of which named Γ_1 consists of a horizontal segment between (x,h) and $(y,t_1r(y))$ and a vertical segment between $(y,t_1r(y))$ and (y,l), while the other named Γ_3 consists of a vertical segment between (x,h) and $(x,t_3r(x))$ and a horizontal segment between $(x,t_3r(x))$ and (y,l). The length of Γ_1 is

$$L_1 = (1-t_1)d_X(x,y) + t_1 d_X\big(T(x),T(y)\big) + (t_1 - t_3)r(y),$$

while the length of Γ_3 is

$$L_3 = (1-t_3)d_X(x,y) + t_3 d_X\big(T(x),T(y)\big) + (t_1 - t_3)r(x).$$

Thus, we have that

$$\begin{aligned}(L_1-L_2)(L_3-L_2) &= \big[(t_1-t_2)\big(d_X\big(T(x),T(y)\big) - d_X(x,y)\big) + (t_1-t_2)r(y) - (t_1-t_2)r(x)\big] \\ &\quad \cdot \big[(t_2-t_3)\big(d_X(x,y) - d_X\big(T(x),T(y)\big)\big) - (t_2-t_3)r(y) - (t_2-t_3)r(x)\big] \\ &= -(t_1-t_2)(t_2-t_3)\big(d_X\big(T(x),T(y)\big) - d_X(x,y) + r(y) + r(x)\big)^2 \\ &\leqslant 0.\end{aligned}$$

On the other hand, since Γ_2 is assumed to be the shortest path between (x,h) and (y,l), it holds that $(L_1-L_2)(L_3-L_2) \geqslant 0$. As a result, $(L_1-L_2)(L_3-L_2)=0$, i.e. either L_1 or L_3 is equal to $L_2 = d((x,h),(y,l))$.

If $L_1 = L_2$, we take Γ_1 as the shortest path, which contains a horizontal segment with the ratio $\frac{h}{r(x)}$. Otherwise, we take Γ_3 as the shortest path containing a horizontal segment with the ratio $\frac{l}{r(y)}$, which completes the proof of Proposition 4.2. □

Recall that a path connecting two points (x,h) and $(y,l) \in W$ is said to be a shortest one if its length is $d((x,h),(y,l))$, then we have the following corollary.

Corollary 4.3. *Suppose there is a shortest path Γ connecting (x,h) and $(y,l) \in W$. Then none of its parts consists of the three segments together listed in* (1) *or* (2)*, where,* $(a,h_1), (a,h_2), (b,l_2), (b,l_1) \in W$ *(see* (a) *and* (b) *of Fig.* 2*).*

(1) *A vertical segment between (a,h_1) and (a,h_2), a horizontal segment between (a,h_2) and (b,l_2), and a vertical segment between (b,l_2) and (b,l_1), where $h_2 > h_1$, $l_2 > l_1$;*
(2) *a vertical segment between (a,h_1) and (a,h_2), a horizontal segment between (a,h_2) and (b,l_2), and a vertical segment between (b,l_2) and (b,l_1), where $h_2 < h_1$, $l_2 < l_1$.*

Moreover, there is such a shortest path, an adapted path of Γ, that none of its parts consists of the three segments together listed in (3) *or* (4)*, where,* $(a,h_1), (a,h_2), (b,l_2), (b,l_1) \in W$ *(see* (c) *and* (d) *of Fig.* 2*).*

(3) *A vertical segment between (a,h_1) and (a,h_2), a horizontal segment between (a,h_2) and (b,l_2), and a vertical segment between (b,l_2) and (b,l_1), where $h_2 > h_1$, $l_2 < l_1$;*
(4) *a vertical segment between (a,h_1) and (a,h_2), a horizontal segment between (a,h_2) and (b,l_2), and a vertical segment between (b,l_2) and (b,l_1), where $h_2 < h_1$, $l_2 > l_1$.*

Proof. The closed polygonal curve, denoted by Γ_1, between (a, h_1) and (b, l_1) in each plot of Fig. 2, consists of two vertical segments and one horizontal segment. If the shortest path Γ contains Γ_1, then Γ_1 is clearly a shortest path between (a, h_1) and (b, l_1). Otherwise we replace Γ_1 with a shorter path between (a, h_1) and (b, l_1) and get a new path between (x, h) and (y, l) shorter than Γ, which is a contradiction.

First, if Γ_1 is the closed polygonal curve in (a) or (b) of Fig. 2, we can replace it with the open one as in the proof of Proposition 4.1 and get a shorter path between (a, h_1) and (b, l_1). It contradicts the fact that Γ_1 is the shortest path.

Second, if Γ_1 is the closed polygonal curve in (c) or (d) of Fig. 2, then by Proposition 4.2, it is the shortest path between (a, h_1) and (b, l_1) while the open polygonal curve with only one vertical segment in the same plot is also a shortest path between (a, h_1) and (b, l_1). Replacing Γ_1 by the open polygonal curve, we get a new shortest path. There are only finite many segments in Γ, so we can repeat this process and get a shortest path between (x, h) and (y, l) containing no part as in (c) or (d) of Fig. 2. $\square$

Definition 4.4. If a path satisfies one of the three conditions below, then we call it a non-typical path, otherwise we call it a typical one.

A. The path contains one of the closed polygonal curves in Fig. 2;
B. the path contains a part consisting of horizontal segments only, and the number of the segments is bigger than 1;
C. the path contains a part consisting of vertical segments only, and the number of the segments is bigger than 1.

Given two points $(x, h), (y, l) \in W$ and a non-typical path Γ between them, if Γ contains a part as one of the closed polygonal curves in Fig. 2, then we can replace the closed polygonal curve with the open one in the same plot and get a new path such that its length does not exceed that of Γ. We note that the new path has one less vertical segment and one less non-typical part than Γ. Inductively, Γ can be adjusted to a typical path whose length is less than or equal to that of Γ. Therefore, for these two points, the distance $d((x, h), (y, l))$ is the infimum of the length of the typical paths between (x, h) and (y, l) consisting of a finite number of horizontal and vertical segments. So, we will search for the shortest paths only among the typical paths.

4.3. The non-equicontinuity of Ψ

We show in this subsection that the time-1 map of Ψ is not equicontinuous, i.e. there is an $\varepsilon_0 > 0$, such that for any $\delta > 0$, there exist two points $(x, h), (y, l) \in W$ and $u \in \mathbb{N}$ satisfying that $d((x, h), (y, l)) < \delta$ while $d(\Psi_u(x, h), \Psi_u(y, l)) > \varepsilon_0$. In the following, we take $\varepsilon_0 = \frac{1}{3}$.

For any $\delta > 0$, we take an integer $i \geqslant 6$ large enough such that the diameter d_i' of X_i under d_X is less than δ. For such an i, we choose $x = (\cdots \underbrace{1 \cdots 1}_{i} 0 \underbrace{\bar{1} \cdots 1}_{i} 0 \underbrace{1 \cdots 1}_{i} 0 \cdots) \in X_i$. When i is even, $i = 2m$ for some $m \in \mathbb{N}$, we denote $y = (\cdots 0 \underbrace{1 \cdots \bar{1}}_{m} \underbrace{1 \cdots 1}_{m} 0 \cdots) \in X_i$. When i is odd, $i = 2m + 1$ for some $m \in \mathbb{N}$, we denote $y = (\cdots 0 \underbrace{1 \cdots 1}_{m} \bar{1} \underbrace{1 \cdots 1}_{m} 0 \cdots) \in X_i$. Recall that $r(z)$ stands for $r_i(z)$ for $z \in X_i$, $i \in \mathbb{N}$, then by (4.2), it is clear that $r(x) = \min_{X_i} r$ and $r(y) = \max_{X_i} r$. We take an integer $u = 11i$. Note that $x \in I_1 \setminus I_2$ and thus $r(x) = i \cdot 4 \cdot 3 = 12i > u$. We see that $\Psi_u(x, 0) = (x, u)$ and $\Psi_u(y, 0) = (y, u)$, because $r(y) > r(x) > 11i = u$. The fact $d_X(x, y) < \operatorname{diam}(X_i) < \delta$ implies that $d((x, 0), (y, 0)) < \delta$. Now we estimate the distance between (x, u) and (y, u). Recall the distance $d((x, u), (y, u))$ is the infimum of the length of typical paths between (x, u) and (y, u) composed of a finite number of horizontal and vertical segments, thus, from now on we consider the typical paths between (x, u) and (y, u) only.

The situation that there is a shortest path between (x, u) and (y, u) with only one horizontal segment will be dealt with in Case 1 below. For the other situation, that is, there is not a shortest path between (x, u) and (y, u) with only one horizontal segment, we make the following claim.

Claim. *There exists a shortest path $\Gamma(z)$, between (x, u) and (y, u), consisting of a horizontal segment with ending points (x, u) and (z, t_1), a vertical segment with ending points (z, t_1) and (z, t_2), and another horizontal segment with ending points (z, t_2) and (y, u), where $z \in X_i$, $t_1 = \frac{u}{r(x)} > t_2 = \frac{u}{r(y)}$ (see Fig. 5).*

Proof. For any path Γ between (x, u) and (y, u), we put the ratios of the horizontal segments in sequential order, from (x, u) to (y, u). If they are not in descending order, i.e. there is a horizontal segment whose ratio is bigger (smaller) than the ratios of the leading one and the trailing one, by Corollary 4.3(1) (Corollary 4.3(2)), Γ is non-typical. It is a contradiction.

Now we consider a path Γ between (x, u) and (y, u), the ratios of the horizontal segments are in descending order. If there are more than two horizontal segments in Γ, then we take the horizontal segment, named Γ_2, with the second largest ratio, and denote the ending points of Γ_2 by (a, h_2) and (b, l_2). Moreover, we take the leading vertical segment Γ_1 with ending points (a, h_1) and (a, h_2), and the trailing vertical segment Γ_3 with ending points (b, l_2) and (b, l_1), see Fig. 3. It is clear that $h_1 > h_2$ and $l_2 > l_1$, because the ratios in Γ are in descending order. The subpath consisting of Γ_1, Γ_2 and Γ_3 is non-typical by Corollary 4.3(4). Thus Γ is non-typical. It is a contradiction.

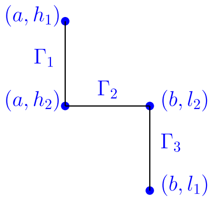

Fig. 3. A simple path of this form cannot be a part of any typical shortest path.

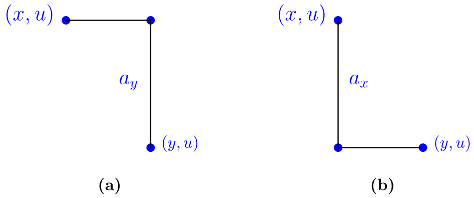

Fig. 4. The paths illustrating Case 1.1 and Case 1.2.

And then, we consider a path Γ from (x, u) to (y, u), with only two horizontal segments Γ_1 and Γ_2. If there is a leading vertical segment between (x, u) and an ending point of Γ_1 or a trailing vertical segment between an ending point of Γ_2 and (y, u), then by Corollary 4.3(4), Γ is non-typical, which is also a contradiction.

So, we consider the paths between (x, u) and (y, u) consisting of two horizontal segments and one vertical segment. For some $z \in X$, we take one such path $\Gamma(z)$, which consists of a horizontal segment between (x, u) and $(z, t_1 r(z))$, a vertical segment between $(z, t_1 r(z))$ and $(z, t_2 r(z))$ and a horizontal segment between $(z, t_2 r(z))$ and (y, u), see Fig. 5. By (4.4), the length of the path is

$$\rho_{\mathrm{hor}}(x, z, t_1) + \rho_{\mathrm{hor}}(y, z, t_2) + (t_1 - t_2) r(z).$$

If $z \in X_j$, $j \neq i$ (without loss of generality, we suppose $i < j$), we can easily get a point $z' \in X_i$ such that $r(z') \leqslant r(z)$ and

$$\rho_{\mathrm{hor}}\big(x, z', t_1\big) + \rho_{\mathrm{hor}}\big(y, z', t_2\big) < 2\frac{3}{i^2} < 2\sum_{k=i}^{j} \frac{3}{k^2} = \rho_{\mathrm{hor}}(x, z, t_1) + \rho_{\mathrm{hor}}(y, z, t_2).$$

This means that the path $\Gamma(z')$ for $z' \in X_i$ is shorter than $\Gamma(z)$ for $z \notin X_i$. Thus, we focus on the paths with all ending points in X_i^r.

We note that $\sharp X_i = i + 1$ and thus there are $i - 1$ different paths $\Gamma(z)$, because each point in X_i determines one such path. We choose the shortest one among them. The chosen one denoted still by $\Gamma(z)$ is then a shortest path between (x, u) and (y, u) with exactly two horizontal segments. We complete the proof of the claim. $\square$

Now we continue to show the non-equicontinuity of Ψ. Obviously,

$$\frac{r(y)}{r(x)} = \frac{i^n \cdot n \cdot 4 \cdot 3^n}{12i} = i^{n-1} \cdot n \cdot 3^{n-1} > 2 \quad \text{for } n \geqslant 2.$$

Case 1. We deal with the distance between (x, u) and (y, u) in the case that the shortest path consists of exactly one horizontal segment and one vertical segment. We discuss in two subcases, depending on the position of the horizontal segments.

Case 1.1. (See (a) of Fig. 4.)

The horizontal segment is between (x, u) and $(y, u + a_y)$, where $a_y > 0$. Then it holds that $\frac{u}{r(x)} = \frac{u + a_y}{r(y)}$, i.e. $\frac{r(x)}{r(y)} = \frac{u}{u + a_y} = A$. The choice of x and y together with the fact $i > 6$ gives that

$$d\big((x, u), (y, u)\big) > a_y = u\left(\frac{1}{A} - 1\right) > 11i(2 - 1) > 11 \cdot 6 = 66 > \frac{1}{3} = \varepsilon_0.$$

Case 1.2. (See (b) of Fig. 4.)

The horizontal segment is between $(x, u - a_x)$ and (y, u), where $a_x > 0$. Then it holds that $\frac{u - a_x}{r(x)} = \frac{u}{r(y)}$, i.e. $\frac{r(x)}{r(y)} = \frac{u - a_x}{u} = A$. Consequently, we have

$$d\big((x, u), (y, u)\big) > a_x = u(1 - A) > 11i\left(1 - \frac{1}{2}\right) > 11 \cdot 6\left(1 - \frac{1}{2}\right) = 33 > \frac{1}{3} = \varepsilon_0.$$

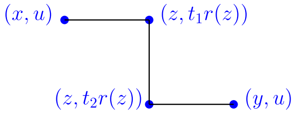

Fig. 5. The path illustrating Case 2.

Case 2. For the situation other than Case 1, by the claim, we estimate the distance between (x,u) and (y,u) using a shortest path $\Gamma(z)$ consisting of two horizontal segments and one vertical segment, see Fig. 5.

Since $\Gamma(z)$ is the shortest path between (x,u) and (y,u), the length of $\Gamma(z)$ is

$$\rho_{\mathrm{hor}}(x,z,t_1)+\rho_{\mathrm{hor}}(y,z,t_2)+(t_1-t_2)r(z).$$

Therefore,

$$d\big((x,u),(y,u)\big)>(t_1-t_2)r(z)\geqslant t_1-t_2=\frac{11i}{r(x)}-\frac{11i}{r(y)}>11i\cdot\frac{1}{2}\cdot\frac{1}{12i}=\frac{11}{24}>\frac{1}{3}=\varepsilon_0.$$

In conclusion, (W,Ψ_1) is not an equicontinuous system. Applying Lemma 1.3, we complete the construction of the suspension flow with an unbounded complexity function. □

5. Remarks

Remark 3. When we talk about how complex a flow is, we talk about, as usual, how complex its time-1 map is. The topological entropy is an invariant for equivalent discrete systems. However, even the extreme entropies, 0 and ∞, are not invariants for equivalent flows. Ohno gave a pair of equivalent topological flows with the zero entropy and a positive entropy respectively [3], while Sun, Young and Zhou gave a pair of equivalent smooth flows with the zero entropy and a positive entropy respectively [5]. What's more, Sun and Zhang constructed a pair of equivalent flows with the zero entropy and the infinite entropy respectively [6], reaching the two extremes of the entropy. The results in the present paper reach the two extremes of the complexity functions.

Remark 4. Fixed points do not play important roles in the complexity functions of equivalent flows, unlike the case of the topological entropy (recall that the zero entropy and the infinity topological entropy are invariants for equivalent flows without fixed points, [4]).

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