



# The problem of existence of periodic solutions for neutral functional differential system with nonlinear difference operator<sup>☆</sup>

Shiping Lu<sup>a,b,\*</sup>, Lijuan Chen<sup>a</sup>

<sup>a</sup> College of Math. and Physics, NUIST, Nanjing 210044, PR China

<sup>b</sup> Department of Mathematics, Anhui Normal University, Wuhu 241000, Anhui, PR China

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## ABSTRACT

In this paper, the authors investigate some new properties of nonlinear difference operator in the first, which is associated with neutral functional differential equations. Then, by using these new properties, the problem of existence of periodic solutions for neutral functional differential system with nonlinear operator  $D_1(\varphi) = \varphi(0) - B\varphi(-\tau) - h(\varphi)$  is studied. The interesting thing is that the periodic solution to the system is allowed to be not differentiable.

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## 1. Introduction

In this paper, we study the following neutral functional differential equation with nonlinear difference operator:

$$\frac{d}{dt}D_1(x_t) = f(t, x_t), \quad (1.1)$$

where  $x_t = x_t(\theta)$ ,  $\theta \in [-\tau, 0]$ ,

$$D_1 : C([-\tau, 0], R^n) \rightarrow R^n, \quad D_1(\varphi) = \varphi(0) - B\varphi(-\tau) - h(\varphi), \quad (1.2)$$

$B = [b_{ij}]_{n \times n}$  is a real matrix,  $h \in C([-\tau, 0], R^n) \rightarrow R^n$ ,  $f \in C(R \times C([-\tau, 0], R^n), R^n)$  with  $f(t + \omega, \varphi) \equiv f(t, \varphi)$  for all  $\varphi \in C([-\tau, 0], R^n)$ , and for any bounded set  $\Omega \subset C([-\tau, 0], R^n)$ ,  $f([0, \omega] \times \Omega)$  is bounded in  $R^n$ ,  $\tau > 0$  is a constant. From the theory of neutral functional differential equations [1–4], any solution  $u(t)$  to Eq. (1.1) only implies that  $D_1(u_t) = u(t) - Bu(t - \tau) - h(u_t)$  is continuously differentiable in  $t$ , but, generally,  $u(t)$  may not be differentiable in  $t$ , which is essentially different from the corresponding case of retarded functional differential equations. In the foundation of theory of neutral functional differential equation, Jack Hale gave an important definition named stable difference operator  $D$  [1,2]: The linear difference operator  $D : C([-\tau, 0], R^n) \rightarrow R^n$ ,  $D(\varphi) = \varphi(0) - \int_{-\tau}^0 \varphi(\theta) d\mu(\theta)$  is called stable, if the zero solution of the difference equation  $Dy_t = 0$ ,  $y_0 = \varphi \in \{C([-\tau, 0], R^n) : D\varphi = 0\}$  is uniformly asymptotically stable. Under the condition that the linear difference operator  $D$  is stable, Jack Hale obtained that any  $\omega$ -periodic solution to the equation

$$\frac{d}{dt}D(x_t) = f(t, x_t) \quad (1.3)$$

has a continuous first derivative [1].

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\* Corresponding author at: College of Math. and Physics, NUIST, Nanjing 210044, PR China. Fax: +86 553 3828887.

E-mail address: lushiping88@sohu.com (S. Lu).

With the help of this good property of stable difference operator  $D$ , in the past years, many researchers [5–8] studied the problem of existence of periodic solutions for Eq. (1.3) by means of some fixed point theorems. Recently, under the condition that the linear difference operator  $D$  is allowed to be un-stable, the existence of periodic solutions for some kinds of neutral function differential equations is also studied by Shiping Lu and Weigao Ge in [9–12], and Jingli Ren et al. in [13,14], respectively. In [15], the authors studied a neutral functional differential system in the following form:

$$\frac{d}{dt}x(t) = A(t)x(t) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))), \quad (1.4)$$

where  $A(t)$  is a nonsingular  $n \times n$  matrix with real valued functions as its elements and  $A(t+T) \equiv A(t)$ . The functions  $Q: R \times R^n \rightarrow R^n$  and  $G: R \times R^n \rightarrow R^n$  are continuous in their respective arguments with  $Q(t+T, u) \equiv Q(t, u)$  and  $G(t+T, u) \equiv G(t, u)$  for all  $u \in R^n$ . By assuming that the linear system

$$y' = A(t)y \quad (1.5)$$

is noncritical, i.e., Eq. (1.5) has no periodic solution of periodic  $T$  except the trivial solution  $y = 0$ ; and by imposing some globally Lipschitz conditions on functions  $Q(t, x)$  and  $G(t, x, y)$ , the authors obtained that the following equation

$$\begin{aligned} & \frac{d}{dt}[x(t) - Q(t, x(t-g(t)))] \\ &= A(t)[x(t) - Q(t, x(t-g(t)))] + A(t)Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))) \end{aligned} \quad (1.6)$$

has a  $T$ -periodic solution  $u(t)$  via Krasnoselskii's fixed point theorem (see Theorem 2.2 in [15]). Then, the authors of [15] concluded that  $u(t)$  is just a periodic solution to Eq. (1.4) (see Lemma 2.1 in [15]). However, this conclusion may not be right. In fact, the necessary and sufficient condition for the  $T$ -periodic solution  $u(t)$  of Eq. (1.4) being a  $T$ -periodic solution of Eq. (1.6) is that  $u(t)$  and  $Q(t, u(t-g(t)))$  must be differentiable in  $t$ . But, the authors do not verify it.

In present paper, we study the properties of nonlinear difference operator  $D_1$  defined by (1.2) in the first, By using functional analysis theory, and under some conditions imposed on  $B$  and functional  $h$ , we obtain that for all  $e \in C(R, R^n)$  with  $e(t+\omega) \equiv e(t)$ , the equation  $D_1(x_t) = e(t)$ , i.e.,

$$x(t) - Bx(t-\tau) - h(x_t) = e(t)$$

has a unique continuous  $\omega$ -periodic solution  $x(t, e)$  satisfying

$$\max_{t \in [0, \omega]} |x(t, e)| \leq L \max_{t \in [0, \omega]} |e(t)|,$$

where  $L > 0$  is a constant independent of  $e$ . By means of this result and Mawhin's continuous theorem [16], we further investigate the problem of existence of periodic solutions for Eq. (1.1). Since the function  $h(\varphi)$  is only required to be continuous for  $\varphi \in C([-\tau, 0], R^n)$ . So any  $\omega$ -periodic solution of Eq. (1.1) may not have a first derivative, which is essentially differential from the cases of [9–15]. Furthermore, even if  $h \equiv 0$ , the  $D_1$ -operator is not required to be stable and symmetrical. So the work of present paper generalizes the corresponding ones of [2–8].

## 2. New properties of nonlinear operator $D_1$

For  $a = (a_1, a_2, \dots, a_n)^T \in C^n$  be a complex vector,  $|a| = (\sum_{i=1}^n |a_i|^2)^{1/2}$ , and for a complex matrix  $H = [h_{ij}]_{n \times n}$ ,  $|H| = (\sum_{i=1}^n \sum_{j=1}^n |h_{ij}|^2)^{1/2}$ . Let

$$C_\omega = \{x: x \in C(R, R^n), x(t+\omega) \equiv x(t), \text{ for all } t \in R\},$$

with the norm  $\|\varphi\|_{C_\omega} = \max_{t \in [0, \omega]} |\varphi(t)|$ . Clearly,  $C_\omega$  is a Banach space.

For the sake of investigating some properties of nonlinear difference operator  $D_1$ , we set

$$A: C_\omega \rightarrow C_\omega, \quad [Ax](t) := x(t) - Bx(t-\tau), \quad (2.1)$$

and

$$\Gamma: C_\omega \rightarrow C_\omega, \quad [\Gamma x](t) := [Ax](t) - h(x_t) = x(t) - Bx(t-\tau) - h(x_t). \quad (2.2)$$

From the definition of nonlinear operator  $D_1$  determined in (1.2), it is easy to see that for any  $e \in C_\omega$ , the existence of continuous  $\omega$ -periodic solution  $u(t)$  for the difference system

$$D_1(x_t) = e(t) \quad (2.3)$$

is equivalent to the existence of continuous  $\omega$ -periodic solution  $u(t)$  for the difference system

$$[\Gamma x](t) = e(t). \quad (2.4)$$

Thus, in order to investigate the existence of continuous  $\omega$ -periodic solutions to Eq. (2.3), it suffices for us to show that Eq. (2.4) has continuous  $\omega$ -periodic solutions. Moreover, we further study some properties of the continuous  $\omega$ -periodic solution for Eq. (2.4).

Since  $B = [b_{ij}]_{n \times n}$  is a real matrix, there must be a complex matrix  $U$  such that

$$UBU^{-1} = E_\lambda = \text{diag}(J_1, J_2, \dots, J_l), \quad (2.5)$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

with  $\sum_{i=1}^l n_i = n$ ,  $\{\lambda_i: i = 1, 2, \dots, l\}$  is the set of eigenvalues of matrix  $B$ .

**Lemma 2.1.** (See S.P. Lu et al. [12].) Suppose that the matrix  $U$  and the operator  $A$  is defined by (2.5) and (2.1), respectively, and for all  $i = 1, 2, \dots, l$ ,  $|\lambda_i| \neq 1$ . Then  $A$  has its inverse  $A^{-1}: C_\omega \rightarrow C_\omega$  with the following properties:

- (1)  $\|A^{-1}\| \leq |U^{-1}| |U| \sigma_0$ , where  $\sigma_0 = \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k}$ ;  
 (2) For all  $e \in C_\omega$ ,  $\int_0^\omega |[A^{-1}e](s)|^p ds \leq |U^{-1}|^p |U|^p \sigma_1 \int_0^\omega |e(s)|^p ds$ ,  $p \in [1, +\infty)$ , where

$$\sigma_1 = \begin{cases} [\sum_{i=1}^l \sum_{j=1}^{n_i} (\sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k})^2]^{1/2}, & p = 1, \\ n^{\frac{2-p}{2}} [\sum_{i=1}^l \sum_{j=1}^{n_i} (\sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k})^q]^{\frac{p}{q}}, & p \in (1, 2), \\ \sum_{i=1}^l \sum_{j=1}^{n_i} (\sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k})^2, & p = 2, \\ [\sum_{i=1}^l \sum_{j=1}^{n_i} (\sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k})^q]^{\frac{p}{q}}, & p \in (2, +\infty) \end{cases}$$

and  $q > 1$  is a constant with  $\frac{1}{p} + \frac{1}{q} = 1$ .

Now, we discuss the properties of nonlinear difference operator  $D_1$  determined by (1.2).

**Theorem 2.1.** Suppose that the following conditions are satisfied:

- (A1) The nonlinear functional  $h$  is satisfied with  $\varphi(0) = 0$  and  $|h(\varphi_1) - h(\varphi_2)| \leq l \max_{\theta \in [-\tau, 0]} |\varphi_1(\theta) - \varphi_2(\theta)|$  for all  $\varphi_1, \varphi_2 \in C([-\tau, 0], \mathbb{R}^n)$ , where  $l \in (0, +\infty)$  is a constant;  
 (A2) For all  $i = 1, 2, \dots, l$ ,  $|\lambda_i| \neq 1$ ;  
 (A3)  $l|U^{-1}| |U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k} < 1$ .

Then for any arbitrary  $e \in C_\omega$ , the equation  $D_1(x_t) = e(t)$  has a unique solution  $u^*$  in  $C_\omega$  with

$$\|u^*\|_{C_\omega} \leq \frac{|U^{-1}| |U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k}}{1 - l|U^{-1}| |U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1-|\lambda_i||^k}} \|e\|_{C_\omega}.$$

**Proof.** Considering the following operator equation:

$$x = Fx,$$

where

$$F: C_\omega \rightarrow C_\omega, \quad [Fx](t) = [A^{-1}(h(x(\cdot + \theta)) + e)](t), \quad \theta \in [-\tau, 0]. \quad (2.6)$$

For all  $x, y \in C_\omega$ ,

$$\begin{aligned} \|Fx - Fy\|_{C_\omega} &= \max_{t \in [0, \omega]} |[A^{-1}(h(x(\cdot + \theta)) + e)](t) - [A^{-1}(h(y(\cdot + \theta)) + e)](t)| \\ &\leq \|A^{-1}\| l \max_{t \in [0, \omega]} \max_{\theta \in [-\tau, 0]} |y(t + \theta) - x(t + \theta)| \\ &\leq \|A^{-1}\| l \|x - y\|_{C_\omega}, \end{aligned}$$

which together with  $l|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}<1$  yields the operator  $F$  has a unique fixed point  $u^*\in C_\omega$ , that is

$$[Au^*](t)=h(u_t^*)+e(t), \quad t\in R, \quad (2.7)$$

which together with (2.2) yields that  $u^*$  is a  $\omega$ -periodic solution to the equation

$$[\Gamma x](t)=e(t).$$

So the system

$$D_1(x_t)=e(t)$$

has a unique solution  $u^*$  in  $C_\omega$ . From (2.7), we see

$$\|u^*\|_{C_\omega}\leq\frac{|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}{1-l|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}\|e\|_{C_\omega}. \quad \square$$

**Remark 2.1.** If assumptions (A1)–(A3) are satisfied and  $e(t)\equiv a\in R^n$ , and let  $Fx=A^{-1}(h(x)+a)$ , then from the fact of  $A^{-1}b=(I-B)^{-1}b$  for all  $b\in R^n$ , we see  $F:R^n\rightarrow R^n$ . Arguing in the similar way as in the proof of Theorem 2.1, we can obtain that the equation

$$D_1(x_t)=a$$

has a unique solution  $u^*$  in  $R^n$  with

$$|u^*|\leq\frac{|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}{1-l|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}|a|.$$

**Theorem 2.2.** Suppose that assumptions (A1)–(A3) in Theorem 2.1 hold. Then the operator  $\Gamma$  has its inverse  $\Gamma^{-1}:C_\omega\rightarrow C_\omega$  with the following properties:

- (1)  $\|\Gamma^{-1}e\|_{C_\omega}\leq\frac{|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}{1-l|U^{-1}||U|\sum_{i=1}^l\sum_{j=1}^{n_i}\sum_{k=1}^j\frac{1}{|1-|\lambda_i||^k}}\|e\|_{C_\omega}$ , for all  $e\in C_\omega$ ;
- (2)  $\Gamma^{-1}$  is continuous on  $C_\omega$ .

**Proof.** The existence of  $\Gamma^{-1}:C_\omega\rightarrow C_\omega$  and conclusion (1) immediately follows from Theorem 2.1. Now, we begin to prove conclusion (2). Let  $f$  and  $g$  are arbitrary two functions in  $C_\omega$ , and  $x=\Gamma^{-1}f$ ,  $y=\Gamma^{-1}g$ . So from (2.6), we see

$$[Ax](t)=h(x_t)+f(t)=h(x(t+\theta))+f(t), \quad t\in[0,\omega], \quad \theta\in[-\tau,0]$$

and

$$[Ay](t)=h(y_t)+g(t)=h(y(t+\theta))+g(t), \quad t\in[0,\omega], \quad \theta\in[-\tau,0],$$

which implies that

$$\begin{aligned} |[A(x-y)](t)| &\leq |h(x(t+\theta))-h(y(t+\theta))|+|f(t)-g(t)| \\ &\leq l\|x-y\|_{C_\omega}+\|f-g\|_{C_\omega}, \quad t\in[0,\omega], \quad \theta\in[-\tau,0], \end{aligned}$$

i.e.,

$$\|A(x-y)\|_{C_\omega}\leq l\|x-y\|_{C_\omega}+\|f-g\|_{C_\omega}.$$

In view of

$$\|x-y\|_{C_\omega}=\|A^{-1}A(x-y)\|_{C_\omega}\leq\|A^{-1}\|\|A(x-y)\|_{C_\omega},$$

we have

$$\|x-y\|_{C_\omega}\leq\|A^{-1}\|(l\|x-y\|_{C_\omega}+\|f-g\|_{C_\omega}).$$

By using Lemma 2.1, we see

$$\|x - y\|_{C_\omega} \leq l|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k} \|x - y\|_{C_\omega} + |U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k} \|f - g\|_{C_\omega},$$

which together with assumption (A3) results in

$$\|x - y\|_{C_\omega} \leq \frac{|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k}}{1 - l|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k}} \|f - g\|_{C_\omega},$$

this is

$$\|\Gamma^{-1}f - \Gamma^{-1}g\|_{C_\omega} \leq \frac{|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k}}{1 - l|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k}} \|f - g\|_{C_\omega}.$$

Thus,  $\Gamma^{-1}$  is continuous on  $C_\omega$ .  $\square$

### 3. Main result

In this section, we will apply the new properties of nonlinear operator  $D_1$  obtained in Section 2 to study the existence of periodic solutions for Eq. (1.1). In order to do this, now, we recall Mawhin's continuation theorem in the first.

Let  $X$  and  $Y$  be real Banach spaces and let  $L : D(L) \subset X \rightarrow Y$  be a Fredholm operator with index zero. This means that  $X = \ker L \oplus X_1$  and  $Y = \operatorname{Im} L \oplus Y_1$ . Furthermore, let  $P : X \rightarrow \ker L$  and  $Q : Y \rightarrow Y_1$  be the continuous projectors. Clearly,  $\ker L \cap (D(L) \cap X_1) = \{0\}$ , thus the restriction  $L_P := L|_{D(L) \cap X_1}$  is invertible. Denote by  $K_P$  the inverse of  $L_P$ .

Let  $\Omega$  be an open bounded subset of  $X$  with  $D(L) \cap \Omega \neq \emptyset$ . A map  $N : \overline{\Omega} \rightarrow Y$  is said to be  $L$ -compact in  $\overline{\Omega}$ , if  $QN(\overline{\Omega})$  bounded and the operator  $K_P(I - Q)N : \overline{\Omega} \rightarrow X$  is compact.

**Lemma 3.1.** (See R.E. Gaines and J. Mawhin [16].) Suppose that  $X$  and  $Y$  are two Banach spaces, and  $L : D(L) \subset X \rightarrow Y$  is a Fredholm operator with index zero. Furthermore,  $\Omega \subset X$  is an open bounded set and  $N : \overline{\Omega} \rightarrow Y$  is  $L$ -compact on  $\overline{\Omega}$ . If all the following conditions hold:

- (1)  $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$ ;
- (2)  $Nx \notin \operatorname{Im} L, \forall x \in \partial\Omega \cap \ker L$ ;
- (3)  $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$ ,

where  $J : \operatorname{Im} Q \rightarrow \ker L$  is an isomorphism. Then equation  $Lx = Nx$  has a solution on  $\overline{\Omega} \cap D(L)$ .

Since operator  $D_1$  is nonlinear, Lemma 3.1 cannot be applied directly to study the existence of periodic solutions to Eq. (1.1). In order to overcome this difficulty, we take substituting  $y(t) = D_1 x_t$ , i.e.,  $y(t) = x(t) - Bx(t - \tau) - h(x_t) = [\Gamma x](t)$ . So if assumptions (A1)–(A3) in Theorem 2.1 hold, then by using Theorem 2.1, we have that the existence of  $\omega$ -periodic solutions to Eq. (1.1) is equivalent to the existence of  $\omega$ -periodic solutions to the following system:

$$y'(t) = f(t, [\Gamma^{-1}y]_t). \quad (3.1)$$

Bellow, we will investigate the existence of  $\omega$ -periodic solutions to Eq. (3.1) by means of Lemma 3.1 under the condition that assumptions (A1)–(A3) in Theorem 2.1 hold. Set  $X = Y = C_\omega$ ,

$$L : D(L) \cap X \rightarrow Y, \quad Ly = y', \quad (3.2)$$

where  $D(L) = \{y : y \in C_\omega^1\}$ .

$$N : C_\omega \rightarrow C_\omega, \quad [Ny](t) = f(t, [\Gamma^{-1}y]_t). \quad (3.3)$$

Clearly,  $\ker L = \mathbb{R}^n$ ,  $\operatorname{Im} L = \{y \in Y : \int_0^\omega y(s) ds = 0\}$ . So  $\operatorname{Im} L$  is closed in  $Y = C_\omega$  and  $\dim \ker L = \operatorname{codim} \operatorname{Im} L = n$ , and then operator  $L$  is a Fredholm operator with index zero.

Let projectors  $P : X \rightarrow \ker L$  and  $Q : Y \rightarrow \operatorname{Im} Q$  be defined by

$$Px = \frac{1}{\omega} \int_0^\omega x(s) ds, \quad Qy = \frac{1}{\omega} \int_0^\omega y(s) ds.$$

Obviously,  $\ker L = \operatorname{Im} P$ ,  $\ker Q = \operatorname{Im} L$ .

Taking operator  $K_P$  to represent the inverse of  $L|_{D(L) \cap \ker P}$ . By calculating, we see

$$K_P : \text{Im } L \subset C_\omega \rightarrow D(L) \cap \ker P, \quad [K_P z](t) = \int_0^t f(s, [\Gamma^{-1} z]_s) ds.$$

In view of Theorem 2.2, we see  $K_P$  is a completely continuous operator. Furthermore, from (3.2) and (3.3), one can easily see that  $N$  is  $L$ -compact on  $\overline{\Omega}$ . If set  $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $J(x) = x$ , then  $\deg\{JQN, \Omega \cap \ker L, 0\} = \deg\{\frac{1}{\omega} \int_0^\omega f(s, \Gamma^{-1}a) ds, \Omega \cap \mathbb{R}^n, 0\}$ . So by using Lemma 3.1, we have the following result:

**Theorem 3.1.** Suppose that operator  $D_1$  is determined by (1.2), and assumptions (A1)–(A3) in Theorem 2.1 hold. Furthermore,  $\Omega \subset X = C_\omega$  is an open bounded set. If all the following conditions hold:

(C1) For each  $\lambda \in (0, 1)$ , the equation

$$\frac{dy(t)}{dt} = \lambda f(t, [\Gamma^{-1} y]_t)$$

has no solution on  $\partial\Omega$ ;

(C2) The equation

$$\Delta(a) := \frac{1}{\omega} \int_0^\omega f(s, \Gamma^{-1}a) ds = 0$$

has no solution on  $\partial\Omega \cap \mathbb{R}^n$ ;

(C3) The degree

$$\deg\{\Delta, \Omega \cap \mathbb{R}^n, 0\} \neq 0.$$

Then Eq. (1.1) has at least one  $\omega$ -periodic solution on  $\overline{\Omega}$ .

Let  $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$  and  $\Omega_\rho = \{x \in C_\omega : \|x\|_{C_\omega} < \rho\}$ , where  $\rho > 0$  is a constant.

**Corollary 3.1.** Suppose that operator  $D_1$  is determined by (1.2), and assumptions (A1)–(A3) in Theorem 2.1 hold. If there is a constant  $\rho > 0$  such that all the following conditions hold:

(D1) For each  $\lambda \in (0, 1)$ , the equation

$$\frac{dy(t)}{dt} = \lambda f(t, [\Gamma^{-1} y]_t)$$

has no solution on  $\partial\Omega_\rho$ ;

(D2) The equation

$$\Delta(a) := \frac{1}{\omega} \int_0^\omega f(s, \Gamma^{-1}a) ds = 0$$

has no solution on  $\partial B_\rho$ ;

(D3) The Brouwer degree

$$\deg\{\Delta_1, \Gamma^{-1}(B_\rho), 0\} \neq 0,$$

where

$$\Delta_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \Delta_1(a) := \frac{1}{\omega} \int_0^\omega f(s, a) ds.$$

Then Eq. (1.1) has at least one  $\omega$ -periodic solution on  $\overline{\Omega}$ .

**Proof.** From Remark 2.1 and Theorem 2.2, we see  $\Gamma^{-1}(B_\rho)$  is a bounded, open and connected set in  $R^n$  with  $\partial\Gamma^{-1}(B_\rho) = \Gamma^{-1}(\partial B_\rho)$ . So the degree  $\deg\{\Delta, B_\rho, 0\} = \deg\{\Delta_1\Gamma^{-1}, B_\rho, 0\}$ , which is a Brouwer degree. Therefore,

$$\begin{aligned}\deg\{\Delta, B_\rho, 0\} &= \deg\{\Gamma^{-1}, B_\rho, 0\} \times \deg\{\Delta_1, \Gamma^{-1}(B_\rho), \Gamma^{-1}(0)\} \\ &= \deg\{\Gamma^{-1}, B_\rho, 0\} \times \deg\{\Delta_1, \Gamma^{-1}(B_\rho), 0\}.\end{aligned}\quad (3.4)$$

Let  $H(a, t) = \Gamma^{-1}(\frac{a}{1+t}) - \Gamma^{-1}(\frac{-ta}{1+t})$ ,  $\forall(x, t) \in B_\rho \times [0, 1]$ . It is continuous on  $B_\rho \times [0, 1]$  and  $0 \notin H(\partial B_\rho \times [0, 1])$ . For otherwise, there is a  $(a_0, t) \in \partial B_\rho \times [0, 1]$  such that

$$\Gamma^{-1}\left(\frac{a_0}{1+t}\right) = \Gamma^{-1}\left(\frac{-ta_0}{1+t}\right).$$

From Remark 2.1, we see  $\Gamma^{-1} : R^n \rightarrow R^n$  is  $1 \leftrightarrow 1$ . This implies

$$\frac{a_0}{1+t} = \frac{-ta_0}{1+t},$$

i.e.,  $a_0 = 0 \notin \partial B_\rho$ , which contradicts the assumption of  $a_0 \in \partial B_\rho$ . So we have

$$\begin{aligned}\deg\{\Gamma^{-1}, B_\rho, 0\} &= \deg\{H(\cdot, 0), B_\rho, 0\} \\ &= \deg\{H(\cdot, 1), \Omega \cap R^n, 0\} = \deg\{\alpha, B_\rho, 0\},\end{aligned}$$

where  $\alpha(a) = \Gamma^{-1}(\frac{a}{2}) - \Gamma^{-1}(\frac{-a}{2})$  is an odd function. By using Borsuk–Ulam theorem,  $\deg\{\Gamma^{-1}, \Omega \cap R^n, 0\} \neq 0$ , which together with (3.4) yields that assumption (C3) in Theorem 3.1 is equivalent to assumption (D3) in Corollary 3.1. In view of  $B_\rho = \Omega_\rho \cap R^n$ , the conclusion of Corollary 3.1 immediately follows from Theorem 3.1 and the above argument.  $\square$

**Remark 3.1.** If  $f(t, \varphi) = f(\varphi)$ , then

$$\Delta_1 : R^n \rightarrow R^n, \quad \Delta_1(a) := \frac{1}{\omega} \int_0^\omega f(a) ds = f(a).$$

So assumption (D3) is replaced by:

(D3)' The Brouwer degree  $\deg[f, \Gamma^{-1}(B_\rho), 0] \neq 0$ .

For illustrating the application of Corollary 3.1, we consider the following equation

$$\frac{d}{dt}(x(t) - Bx(t - \tau) - s(x(t - \tau))) = g(x(t - \tau)) + e(t), \quad (3.5)$$

where  $x(t) = (x_1(t), x_2(t))^T \in R^2$ ,  $\tau$  is a positive constant,  $B = \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix}$ ,  $s(x) = (\frac{9|x_2|}{20}, \frac{9|x_1|}{20})^T$ ,  $g(x) = (\frac{3}{2}x_2^3, \frac{3}{8}x_1^3)^T$ .  $e(t) = (\sin t, \cos t)^T$ . Clearly, we can chose  $U = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$  and  $U^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$  such that  $UBU^{-1} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$ , i.e.,

$$\lambda_1 = 4, \quad \lambda_2 = -4. \quad (3.6)$$

If set  $h(x_t) = s(x_t(-\tau)) = s(x(t - \tau))$  and  $f(t, x_t) = g(x_t(-\tau)) + e(t) = g(x(t - \tau)) + e(t)$ , then Eq. (3.5) is converted to the following equation:

$$\frac{d}{dt}D_1(x_t) = f(t, x_t).$$

Furthermore, we can chose  $l = \frac{9}{20}$  such that

$$\begin{aligned}|h(\varphi_1) - h(\varphi_2)| &= |s(\varphi_1(-\tau)) - s(\varphi_2(-\tau))| \leq l|\varphi_1(-\tau) - \varphi_2(-\tau)| \\ &\leq l \max_{\theta \in [-\tau, 0]} |\varphi_1(\theta) - \varphi_2(\theta)| \quad \text{for all } \varphi_1, \varphi_2 \in C([- \tau, 0], R^n),\end{aligned}\quad (3.7)$$

and

$$l|U^{-1}||U| \sum_{i=1}^l \sum_{j=1}^{n_i} \sum_{k=1}^j \frac{1}{|1 - |\lambda_i||^k} = \frac{3}{4} < 1. \quad (3.8)$$

From (3.6)–(3.8), it is easy to see that assumptions (A1)–(A3) in Theorem 2.1 hold. Suppose  $u \in C_{2\pi}$  is an arbitrary solution to the following equation

$$\frac{d}{dt}(x(t) - Bx(t - \tau) - s(x(t - \tau))) = \lambda g(x(t - \tau)) + \lambda e(t), \quad \lambda \in (0, 1),$$

then

$$\frac{d}{dt}(u(t) - Bu(t - \tau) - s(u(t - \tau))) = \lambda g(u(t - \tau)) + \lambda e(t), \quad \lambda \in (0, 1). \quad (3.9)$$

Multiplying both sides of Eq. (3.9) with  $u(t) - Bu(t - \tau) - s(u(t - \tau))$  and integrating it on  $[0, 2\pi]$ , we have

$$\int_0^{2\pi} [u(t) - Bu(t - \tau) - s(u(t - \tau))]^\top g(u(t - \tau)) dt = - \int_0^{2\pi} [u(t) - Bu(t - \tau) - s(u(t - \tau))]^\top e(t) dt,$$

which results in

$$\begin{aligned} \frac{3}{2} \int_0^{2\pi} |u(t)|^4 dt &\leq 3 \int_0^{2\pi} (|u_1(t)|^4 + |u_2(t)|^4) dt = \int_0^{2\pi} u^\top(t - \tau) B^\top g(u(t - \tau)) dt \\ &= \int_0^{2\pi} [u^\top(t) - s^\top(u(t - \tau))] g(u(t - \tau)) dt + \int_0^{2\pi} [u(t) - Bu(t - \tau) - s(u(t - \tau))]^\top e(t) dt \\ &< 2 \int_0^{2\pi} |u(t)|^2 dt + 10 \left( \int_0^{2\pi} |u(t)|^2 dt \right)^{1/2} \left( \int_0^{2\pi} |e(t)|^2 dt \right)^{1/2} \\ &\leq \frac{3(1+l)}{4} \int_0^{2\pi} |u(t)|^4 dt + (9+l)(2\pi)^{\frac{3}{4}} \left( \int_0^{2\pi} |u(t)|^4 dt \right)^{1/4}, \end{aligned}$$

i.e.,

$$\int_0^{2\pi} |u(t)|^4 dt < 2(24)^{\frac{4}{3}} \pi.$$

So

$$\begin{aligned} \int_0^{2\pi} |[\Gamma u](t)| dt &= \int_0^{2\pi} |u(t) - Bu(t - \tau) - s(u(t - \tau))| dt \\ &\leq (9+l) \int_0^{2\pi} |u(t)| dt \\ &\leq (9+l)(2\pi)^{\frac{3}{4}} \left( \int_0^{2\pi} |u(t)|^4 dt \right)^{1/4} \\ &< \frac{189}{5} \sqrt[3]{3\pi}. \end{aligned}$$

Thus, there is a point  $t_0 \in [0, 2\pi]$  such that

$$|[\Gamma u](t_0)| < \frac{189}{5} \sqrt[3]{3\pi}.$$

From (3.9), we see



$$\begin{aligned}
\int_0^{2\pi} |[(\Gamma u)'](t)| dt &\leq \int_0^{2\pi} |g(u(t))| dt + 2\pi \leq \frac{3}{2} \int_0^{2\pi} |u(t)|^3 dt + 2\pi \\
&\leq \frac{3}{2} (2\pi)^{1/4} \left( \int_0^{2\pi} |u(t)|^4 dt \right)^{3/4} + 2\pi \\
&< 74\pi,
\end{aligned}$$

and then

$$\|\Gamma u\|_{C_{2\pi}} \leq |[(\Gamma u)'](t_0)| + \int_0^{2\pi} |[(\Gamma u)'](t)| dt < 74\pi + \frac{189}{5} \sqrt[3]{3}\pi := M_0,$$

which together with Theorem 2.2 yields

$$\begin{aligned}
\|u\|_{C_{2\pi}} &= |\Gamma^{-1} \Gamma u|_{C_{2\pi}} \\
&\leq \frac{|U^{-1}| |U| [\frac{1}{|1-\lambda_1|} + \frac{1}{|1-\lambda_2|}]}{1 - |U^{-1}| |U| [\frac{1}{|1-\lambda_1|} + \frac{1}{|1-\lambda_2|}]} \|\Gamma u\|_{C_{2\pi}} \\
&< \frac{20}{3} M_0.
\end{aligned} \tag{3.10}$$

Let  $\Omega_\rho = \{x \in C_{2\pi} : \|x\|_{C_{2\pi}} < \frac{20}{3} M_0\}$ , then from (3.10), we see that assumption (D1) is satisfied. Suppose  $a \in R^n$  satisfying  $\Delta(a) = 0$ , i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} [g(\Gamma^{-1}a) + e(s)] ds = g(\Gamma^{-1}a) = 0.$$

By the definition of  $g$ , we have  $\Gamma^{-1}a = 0$ , this implies  $a = 0 \notin \partial B_\rho$ . So assumption (D2) is also satisfied. Take

$$H : \Gamma^{-1}(B_\rho) \times [0, 1], \quad H(x, \mu) = \mu x + (1 - \mu) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g(x),$$

then  $x^\top H(x, \mu) > 0$  for all  $(x, \mu) \in \Gamma^{-1}(B_\rho) \times [0, 1]$ . So

$$\begin{aligned}
\deg\{\Delta_1, \Gamma^{-1}(B_\rho), 0\} &= \deg\{g, B_\rho, 0\} \\
&= \deg\{H(\cdot, 0), \Gamma^{-1}(B_\rho), 0\} = \deg\{H(\cdot, 1), \Gamma^{-1}(B_\rho), 0\} \\
&= 1 \neq 0,
\end{aligned}$$

it follows that assumption (D3) holds. Thus, by using Corollary 3.1, we see Eq. (3.5) has a  $2\pi$ -periodic solution  $u_0(t)$ .

**Remark 3.2.** From Eq. (3.5), it is easy to see that the  $2\pi$ -periodic solution  $u_0(t)$  must be satisfied  $u_0(t) \neq c$ , which together with the condition of  $s(u_0(t)) = (\frac{9}{20}|u_{02}(t)|, \frac{9}{20}|u_{01}(t)|)^\top$  yields that  $s(u_0(t))$  may not be differentiable for  $t \in R$ . From Eq. (3.5) again, we see  $u_0(t) - Bu_0(t - \tau) - s(u_0(t - \tau))$  is differentiable for  $t \in R$ . So the  $2\pi$ -periodic solution  $u_0(t)$  may not be differentiable for  $t \in R$ , which is different from the corresponding results of [9–14].

**Remark 3.3.** Since there is no noncritical term  $A(t)x$  in Eq. (3.5) (i.e., the system  $y' = A(t)y$  has no  $\omega$ -periodic solution except the trivial solution  $y = 0$ ), and the term  $g(x) = (\frac{3}{2}x_2^3, \frac{3}{8}x_1^3)^\top$  in Eq. (1.5) does not satisfy globally Lipschitz condition, the main result of present paper is essentially different from the corresponding ones obtained for studying Eq. (1.6) in [15].

**Remark 3.4.** Since the eigenvalues of matrix  $B$  are  $\lambda_1 = 4$  and  $\lambda_2 = -4$ , even if  $h \equiv 0$ , the operator  $D_1$  associated to Eq. (3.5) such as  $D_1\varphi = \varphi(0) - B\varphi(-\tau)$  is un-stable. Furthermore, the matrix  $B$  is not symmetrical, so that the conclusion of Eq. (3.5) does not obtain by means of corresponding results of the known literature [5,7–11,13,14].

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