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Areas of attraction for nonautonomous differential equations on finite time intervals

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ABSTRACT

In this article, new concepts of (exponential) attractivity for nonautonomous differential equations on a finite time interval are introduced. Due to nonuniqueness of finite-time objects, areas of attraction are studied rather than solutions which are attractive. A sufficient and necessary condition for the existence of such areas is presented, which makes use of a time-varying Riemannian metric. Several examples illustrate the theoretical results and definitions.

1. Introduction

The work on attractivity and stability of dynamical systems by the Russian engineer and mathematician Aleksandr Lyapunov at the end of the 19th century was fundamental for the initiation of the so-called *Qualitative Theory of Dynamical Systems* and revolutionised the way scientists think about dynamical processes. This theory is based on asymptotic convergence properties of solutions when time tends to infinity and is successfully applied to many real world applications which are modelled by autonomous dynamical processes.

While the dynamical systems theory concentrates almost exclusively on autonomous dynamical systems, *nonautonomous* dynamical systems have experienced a renewed and steadily growing interest in the last twenty years, stimulated also by synergetic effects of disciplines which have developed relatively independent for some time (such as control theory [10], random dynamics [2,21] and nonautonomous differential and difference equations [25,24]). The importance of nonautonomous dynamical systems is illustrated by the fact that autonomous theory serves only as a theory with slowly (adiabatically) time-varying parameters, where the convergence to the long-term asymptotic limit is very fast in comparison to the timescale of the parameter variation. As a consequence, the classical theory is irrelevant for the huge class of real world applications, where one typically observes rapid changes of parameters including economics (e.g., stock markets [19]), environmental studies (e.g., climate change modelling [1]) or health care studies (e.g., seizure prediction [23]). In these cases, the interesting dynamical behavior manifests itself on a *finite* time interval rather than on an unbounded interval using asymptotic properties.

A mathematical theory for finite-time dynamics was fast-paced for applications in fluid dynamics and oceanography, where a nonautonomous differential equation describes the time-dependent velocity field around an airfoil or of a stretch

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of ocean surface. In order to analyse patterns which influence transport phenomena in such systems, George Haller gave a precise definition of a *Lagrangian coherent structure* [17], a concept which was frequently used before on a descriptive level. Crucial for this definition is the concept of a *finite-time Lyapunov exponent*, which yields a notion of finite-time hyperbolicity. Further studies concerned different aspects of the theory and extensions to higher dimensions [6,7,4,5,11,22,29,27].

The theory of finite-time Lyapunov exponents yields a corresponding theory of attractivity, where Lagrangian coherent structures can serve as boundaries of attraction areas. While in this theory, attractivity is supposed to occur at every instance within the time domain under consideration (leading to well-defined objects such as stable and unstable manifolds [5,7,17]), we develop a concept of attraction in this paper, which allows that points near an attractive solution move away from it, provided they return before the end of the time period. It is clear that this is *weaker* compared to the above *strong* concept of attraction, but the main result of this paper relates these two different points of view: it implies that an area of (weak) attraction can be characterised by (strong) attraction when changing the metric in time, with the additional condition that the metrics at both initial and final time coincide.

We like to emphasise that an important difference to the classical Lyapunov theory for infinite time intervals is given by the nonuniqueness of finite-time objects. This means in turn that single solutions (with a certain stability behavior) do not play a special role any more. For this reason, we will mainly discuss areas of attraction consisting of attractive solutions rather than single attractive solutions and their respective domain of attraction in this paper. One aim of this paper is to compare *areas of attraction* with *domains of attraction* of solutions contained in it. These objects do not differ in the study of infinite time intervals, but we illustrate that they are different in our situation. One of our results shows that the area of attraction contains the domains of attraction of solutions within it, provided that the area of attraction is convex at the initial time. We give an example of a non-convex area of attraction, where this property does not hold true.

Finally, we would like to remark that the study of domains of attraction in ordinary differential equations plays a fundamental role for the deep understanding of the global asymptotic behavior also in the infinite time context, and there have been several approaches to compute such domains. One classical concept is Zubov's method (see [30]), which is based on a partial differential equation whose solution determines the entire domain of attraction of an attractive equilibrium. Another concept was developed by Göran Borg in 1960 (see [8]). His approach admitted the determination of a subset of the domain of attraction of a periodic orbit, and it is a main advantage that for the formulation of his criterion no special information concerning the periodic orbit is needed. The criterion is local and makes only use of the fact that adjacent solutions approach each other in forward time. In Borg's article, this approach has been formulated in terms of the standard Euclidean metric. However, Borg's criterion was extended already in the 1960s in [18] and [28] by employing a general Riemannian metric, and moreover, [12] has shown that Borg's criterion is both sufficient and necessary for the existence of an exponentially stable periodic orbit; it is always possible to construct an appropriate Riemannian metric. The improvement of Borg's criterion became a subject of several studies. In particular, it was extended to periodic and almost periodic equations, as well as to non-smooth systems (see [13,15,14]). In this article, we will provide an appropriate version of Borg's criterion for the study of nonautonomous differential equations on finite time intervals, which will give a characterisation of areas of attraction.

This paper is organised as follows. After some notational preparations, the notion of finite-time attractivity is introduced in Section 2. Section 3 is devoted to the definition of domains and areas of attraction. The finite-time analogue of Borg's criterion is then treated in the next two sections: First we prove that a certain condition is sufficient to determine the area of attraction in Section 4, and then we show that this condition is also necessary in Section 5. In Section 6 we compare areas of attractions with domains of attraction of solutions within, and we discuss examples including the relation of the newly introduced finite-time definitions to the respective infinite-time ones and the nonautonomous logistic equation.

Notation and basic setup. We denote by $\mathbb{R}$ (or $\mathbb{C}$, respectively) the set containing all real (or complex, respectively) numbers and by $\mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$, respectively) the set of all real (or complex, respectively) $n\times n$ matrices, and we write the symbol $\mathbb{1}$ for the unit matrix. For $C\in\mathbb{C}^{n\times n}$, the conjugate complex matrix is denoted by $\bar{C}$, and $C^*=\overline{C^T}$ denotes the adjoint matrix. The standard scalar product for $v,w\in\mathbb{C}^n$ is denoted by $\langle v,w\rangle=\overline{v^T}w$ and $\|v\|:=\sqrt{\langle v,v\rangle}$ denotes the Euclidean norm.

Denote by $B_\varepsilon(x_0)=\{x\in\mathbb{R}^n\colon\ \|x-x_0\|<\varepsilon\}$ the ε-neighborhood of a point $x_0\in\mathbb{R}^n$. For arbitrary nonempty sets $A,B\subset\mathbb{R}^n$ and $x\in\mathbb{R}^n$, let $\operatorname{dist}(x,A):=\inf\{\|x-y\|\colon\ y\in A\}$ be the *distance* of x to A and $\operatorname{dist}(A,B):=\sup\{\operatorname{dist}(x,B)\colon\ x\in A\}$ be the *Hausdorff semi-distance* of A to B.

2. Finite-time attractivity

We consider the finite time interval $\mathbb{I}:=[0,T]$ of given length $T>0$ and a nonautonomous differential equation

$$\dot{x}=f(t,x), \tag{2.1}$$

where $f:\mathbb{I}\times\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be continuously differentiable. The general solution of this equation is denoted by $\varphi:\mathbb{I}\times\mathbb{I}\times\mathbb{R}^n\to\mathbb{R}^n$, i.e., $\varphi(\cdot,\tau,\xi)$ is the solution to the initial value problem (2.1), $x(\tau)=\xi$. We assume that $\varphi(t,\tau,\xi)$ exists for all $t\in\mathbb{I}$.

A subset M of $\mathbb{I}\times\mathbb{R}^n$ is called a *nonautonomous set* if for all $t\in\mathbb{I}$, the so-called *t-fibres* $M(t):=\{x\in\mathbb{R}^n\colon\ (t,x)\in M\}$ are nonempty. We call M *connected*, *compact* or *open* if all fibres are connected, compact or open, respectively. A nonautonomous set M is said to be *positively invariant* if $\varphi(t,\tau,M(\tau)) \subset M(t)$ for all $\tau, t \in \mathbb{I}$ with $t \geqslant \tau$, and it is called *invariant* if $\varphi(t,\tau,M(\tau)) = M(t)$ for all $\tau, t \in \mathbb{I}$.

We make use of the following notions of finite-time attractivity from [26]. Note that in contrast to the infinite-time dynamics the choice of the norm is crucial. In the following we choose $\|\cdot\|$ to be the Euclidean norm, although the definition can be generalised to other norms.

Definition 2.1 *(Finite-time attractivity).* Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be a solution of (2.1).

(i) μ is called *attractive on* $\mathbb{I}$ if there exists an $\eta > 0$ such that

$$\big\|\varphi(T,0,x) - \mu(T)\big\| < \big\|x - \mu(0)\big\| \quad \text{for all } x \in B_\eta\big(\mu(0)\big) \setminus \big\{\mu(0)\big\}.$$

(ii) μ is called *exponentially attractive on* $\mathbb{I}$ if

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T,0,B_\eta\big(\mu(0)\big)\big), \big\{\mu(T)\big\}\big) < 1,$$

and the negative number

$$\frac{1}{T} \ln\left(\limsup_{\eta \searrow 0} \frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T,0,B_\eta\big(\mu(0)\big)\big), \big\{\mu(T)\big\}\big)\right)$$

is called *rate of exponential attraction*.

The following two nonautonomous examples illustrate the notions of both attractivity and exponential attractivity.

Example 2.2. We consider the nonautonomous linear differential equation

$$\dot{x} = a(t)x, \quad \text{where } x \in \mathbb{R},$$

and $a : [0,T] \to \mathbb{R}$ is a continuous function. If $A := \int_0^T a(t)\,dt < 0$, then the trivial solution $x(t) = 0$ is exponentially attractive with rate of exponential attraction $\frac{A}{T}$. This follows from

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T,0,B_\eta(0)\big), \{0\}\big) = \frac{1}{\eta} \eta e^{\int_0^T a(t)\,dt} = e^A < 1$$

and $\frac{1}{T} \ln e^A = \frac{A}{T}$. Note also that due to the linearity of the equation, this does not only hold for the trivial solution but for every solution.

It follows easily from the definitions that exponential attractivity implies attractivity. The following example shows that the converse statement does not hold.

Example 2.3. Consider the differential equation

$$\dot{x} = a(t)x^3$$

where $a : [0,T] \to \mathbb{R}$ is a continuous function. If $A := \int_0^T a(t)\,dt < 0$, then the trivial solution is attractive on $[0,T]$, but not exponentially attractive. This follows basically from the representation

$$\varphi(T,0,\xi) = \frac{\xi}{\sqrt{1 - 2A\xi^2}}.$$

The trivial solution is attractive, since $\sqrt{1-2A\xi^2} > 1$ for all $\xi \neq 0$ as $A < 0$. However,

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T,0,B_\eta(0)\big), \{0\}\big) = \limsup_{\eta \searrow 0} \frac{1}{\sqrt{1 - 2A\eta^2}} = 1,$$

which proves that we do not have exponential attractivity.

Remark 2.4. Consider the differential equation (2.1) on the infinite interval $\mathbb{R}_0^+$, i.e., having a right-hand side $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$. If a solution $\mu : \mathbb{R}_0^+ \to \mathbb{R}^n$ is exponentially attractive on each interval $[0,T]$ for $T > 0$ and the corresponding rates of exponential attraction are bounded away from 0, then it is easy to see that μ is exponentially attractive in the sense of Lyapunov. If the rates of exponential attraction are not bounded away from zero, then even attractivity of μ in the sense of Lyapunov cannot be concluded as the trivial solution of the example $\dot{x} = \min\{-1+t, 0\}x$ shows.

On the other hand, if a solution is exponentially attractive in the sense of Lyapunov for infinite times, then in general, there might be time intervals $[0, T]$ where the solution is not attractive. The example

$$\dot{x} = \begin{cases} -x, & (t,x) \in \mathbb{R} \times \mathbb{R} \text{ with } x \geqslant e^{-t} \text{ or } x \leqslant 0, \\ -e^{-t} - 2(x - e^{-t}), & (t,x) \in \mathbb{R} \times \mathbb{R} \text{ with } \frac{1}{2}e^{-t} \leqslant x \leqslant e^{-t}, \\ 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \text{ with } x \leqslant \frac{1}{2}e^{-t}, \end{cases}$$

shows even that the trivial solution can be exponentially attractive for infinite times, but for each finite time interval, the trivial solution is not attractive.

The following proposition characterises (exponential) attractivity by means of the time-T map.

Proposition 2.5. *Denote by $F_T : \mathbb{R}^n \to \mathbb{R}^n$ the time-T map of* (2.1)*, which is defined by $F_T(x) := \varphi(T, 0, x)$. Moreover, let $\mu : \mathbb{I} \to \mathbb{R}^n$ be a solution of* (2.1)*. Then the following statements hold.*

(i) *If μ is attractive on $\mathbb{I}$, then $\lambda \leqslant 1$ holds for all eigenvalues λ of the matrix $DF_T(\mu(0))^T DF_T(\mu(0))$,*
(ii) *μ is exponentially attractive on $\mathbb{I}$ if and only if $\lambda < 1$ holds for all eigenvalues λ of $DF_T(\mu(0))^T DF_T(\mu(0))$. The rate of exponential attraction is given by*

$$\frac{1}{2T} \ln \lambda_{\max},$$

where $\lambda_{\max}$ is the largest eigenvalue of $DF_T(\mu(0))^T DF_T(\mu(0))$.

Proof. (i) Let λ be an eigenvalue of $DF_T(\mu(0))^T DF_T(\mu(0))$ with $\lambda > 1$, and let v be a corresponding eigenvector with $\|v\| = 1$. Thus, $\|DF_T(\mu(0))v\|^2 = v^T DF_T(\mu(0))^T DF_T(\mu(0))v = \lambda v^T v = \lambda$. We will show that μ is not attractive. Taylor's Theorem implies that

$$F_T\big(\mu(0) + \varepsilon v\big) - F_T\big(\mu(0)\big) = \varepsilon DF_T\big(\mu(0)\big)v + \psi(\varepsilon),$$

where $\lim_{\varepsilon \to 0} \frac{\psi(\varepsilon)}{\varepsilon} = 0$. Choose $\varepsilon_0 > 0$ so small that $\|\psi(\varepsilon)\| \leqslant (\sqrt{\lambda} - 1)\varepsilon$ holds for all $0 < \varepsilon < \varepsilon_0$. Hence,

$$\big\|\varphi\big(T, 0, \mu(0) + \varepsilon v\big) - \mu(T)\big\| \geqslant \varepsilon\big(\big\|DF_T\big(\mu(0)\big)v\big\| - (\sqrt{\lambda} - 1)\big) = \varepsilon = \big\|\big(\mu(0) + \varepsilon v\big) - \mu(0)\big\|.$$

This contradicts Definition 2.1 for all $x = \mu(0) + \varepsilon v$ with $\varepsilon \in (0, \varepsilon_0)$ and thus shows that μ is not attractive.

(ii) We consider $\mu(t)$ and the solution starting in $\mu(0) + w$. Using Taylor's Theorem, we obtain

$$\varphi\big(T, 0, \mu(0) + w\big) - \mu(T) = F_T\big(\mu(0) + w\big) - F_T\big(\mu(0)\big) = DF_T\big(\mu(0)\big)w + \psi(w),$$

where $\lim_{\|w\| \to 0} \frac{\psi(w)}{\|w\|} = 0$. Thus,

$$\limsup_{\|w\| \to 0} \frac{\varphi(T, 0, \mu(0) + w) - \mu(T)}{\|w\|} = \limsup_{\|w\| \to 0} \frac{DF_T(\mu(0))w}{\|w\|}$$

and

$$\limsup_{\|w\| \to 0} \frac{\|\varphi(T, 0, \mu(0) + w) - \mu(T)\|}{\|w\|} = \limsup_{\|w\| \to 0} \frac{\|DF_T(\mu(0))w\|}{\|w\|} = \sqrt{\lambda_{\max}}, \tag{2.2}$$

where $\lambda_{\max}$ is the largest eigenvalue of $DF_T(\mu(0))^T DF_T(\mu(0))$, since $DF_T(\mu(0))^T DF_T(\mu(0))$ is symmetric. Now

$$\frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T, 0, B_\eta\big(\mu(0)\big)\big), \big\{\mu(T)\big\}\big) = \sup_{\|w\| < \eta} \frac{\|\varphi(T, 0, \mu(0) + w) - \mu(T)\|}{\|w\|} \frac{\|w\|}{\eta}. \tag{2.3}$$

From (2.3) we can conclude

$$\frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T, 0, B_\eta\big(\mu(0)\big)\big), \big\{\mu(T)\big\}\big) \leqslant \sup_{\|w\| < \eta} \frac{\|\varphi(T, 0, \mu(0) + w) - \mu(T)\|}{\|w\|},$$

which implies with (2.2) that

$$\limsup_{\eta \searrow 0} \frac{1}{\eta} \operatorname{dist}\big(\varphi\big(T, 0, B_\eta\big(\mu(0)\big)\big), \big\{\mu(T)\big\}\big) \leqslant \limsup_{\|w\| \to 0} \frac{\|\varphi(T, 0, \mu(0) + w) - \mu(T)\|}{\|w\|} = \sqrt{\lambda_{\max}}.$$

Furthermore, (2.3) and (2.2) yield for all $\theta \in (0,1)$

$$\frac{1}{\eta}\operatorname{dist}\bigl(\varphi\bigl(T,0,B_\eta\bigl(\mu(0)\bigr)\bigr),\bigl\{\mu(T)\bigr\}\bigr) \geqslant \sup_{\|w\|=\theta\eta}\frac{\|\varphi(T,0,\mu(0)+w)-\mu(T)\|}{\|w\|}\theta$$

and

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\operatorname{dist}\bigl(\varphi\bigl(T,0,B_\eta\bigl(\mu(0)\bigr)\bigr),\bigl\{\mu(T)\bigr\}\bigr) \geqslant \limsup_{\|w\|\to 0}\frac{\|\varphi(T,0,\mu(0)+w)-\mu(T)\|}{\|w\|}\theta = \theta\sqrt{\lambda_{\max}}.$$

Since this inequality holds for all $\theta \in (0,1)$, we have

$$\limsup_{\eta\searrow 0}\frac{1}{\eta}\operatorname{dist}\bigl(\varphi\bigl(T,0,B_\eta\bigl(\mu(0)\bigr)\bigr),\bigl\{\mu(T)\bigr\}\bigr) \geqslant \sqrt{\lambda_{\max}}.$$

This finishes the proof of this proposition. □

Example 2.6. We consider again Example 2.2 and use Proposition 2.5 to determine the rate of exponential attraction. The time-T map is given by $F_T(x) = e^A x$, where $A = \int_0^T a(t)\,dt < 0$. This gives $DF_T(x)^T DF_T(x) = e^{2A}$ for all solutions starting at $\mu(0) = x$. The eigenvalue is $\lambda = \lambda_{\max} = e^{2A}$, which fulfils $\lambda_{\max} < 1$ since $A < 0$. The rate of exponential attraction is given by

$$\frac{1}{2T}\ln\lambda_{\max} = \frac{A}{T}.$$

Furthermore, we consider also Example 2.3 where the time-T map is given by $F_T(\xi) = \frac{\xi}{\sqrt{1-2A\xi^2}}$, where $A = \int_0^T a(t)\,dt < 0$. This gives $DF_T(\xi) = \frac{1}{(1-2A\xi^2)^{3/2}}$. For the trivial solution $\mu(t) = 0$ this implies

$$DF_T(0) = 1,$$

and thus the eigenvalue of $DF_T(0)^T DF_T(0)$ is 1. Hence, Proposition 2.5(ii) shows that the trivial solution is not exponentially attractive.

Lyapunov functions are supposed to decrease along solutions. In our context of finite time intervals, however, we also have to link the Lyapunov function to the Euclidean metric at the starting and end time of the finite time interval.

Definition 2.7. Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be a solution of (2.1). A continuously differentiable function $V : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}$ is called a *strict finite-time Lyapunov function* for μ if

$$V(0,x) = \bigl\|x-\mu(0)\bigr\|^2, \qquad V(T,x) = \bigl\|x-\mu(T)\bigr\|^2 \quad \text{for all } x \in \mathbb{R}^n \tag{2.4}$$

and

$$V'(t,x) := \bigl\langle \nabla_x V(t,x), f(t,x)\bigr\rangle + \frac{\partial V}{\partial t}(t,x) < 0 \quad \text{for all } (t,x) \in U \setminus \bigl\{\bigl(t,\mu(t)\bigr)\colon t \in \mathbb{I}\bigr\},$$

where the nonautonomous set U is a neighborhood of $\{(t,\mu(t))\colon t \in \mathbb{I}\}$ in $\mathbb{I} \times \mathbb{R}^n$.

Attractive solutions on a finite time interval can be characterised by strict finite-time Lyapunov functions.

Theorem 2.8. *The existence of a strict finite-time Lyapunov function for a solution μ implies that μ is attractive on $\mathbb{I}$. Conversely, if a solution μ is attractive on $\mathbb{I}$, then there exists a strict finite-time Lyapunov function.*

Proof. Let μ be a solution, V be a strict finite-time Lyapunov function for μ and U be a neighborhood as specified in Definition 2.7. Because of the continuous dependence on initial conditions, there exists an invariant neighborhood $U' \subset U$ of μ, and there is a $\beta > 0$ such that $(0, B_\beta(\mu(0))) \subset U'$. Then for all $x \in B_\beta(\mu(0)) \setminus \{\mu(0)\}$, we have

$$\begin{aligned}\bigl\|\varphi(T,0,x)-\mu(T)\bigr\|^2 &= V\bigl(T,\varphi(T,0,x)\bigr) = V(0,x) + \int_0^T V'\bigl(t,\varphi(t,0,x)\bigr)\,dt \\ &< V(0,x) = \bigl\|x-\mu(0)\bigr\|^2,\end{aligned} \tag{2.5}$$

which shows the first part of the theorem.

Now assume conversely that the solution μ is attractive on $\mathbb{I}$. We define the function V for $t \in \mathbb{I}$ by the linear interpolation between the values at 0 and T of this trajectory, i.e.,

$$V(t,x) = \left\|\varphi(0,t,x) - \mu(0)\right\|^2 + \left(\left\|\varphi(T,t,x) - \mu(T)\right\|^2 - \left\|\varphi(0,t,x) - \mu(0)\right\|^2\right)\frac{t}{T}. \tag{2.6}$$

The function V is obviously continuously differentiable and the property (2.4) of Definition 2.7 is satisfied. Since μ is attractive, there exists a $\beta > 0$ such that

$$\left\|\varphi(T,0,x) - \mu(T)\right\| < \left\|x - \mu(0)\right\| \quad \text{for all } x \in B_\beta\big(\mu(0)\big) \setminus \big\{\mu(0)\big\}. \tag{2.7}$$

Define the invariant set $U = \{(t,x) \in \mathbb{I} \times \mathbb{R}^n\colon\ \varphi(0,t,x) \in B_\beta(\mu(0))\}$. To calculate the orbital derivative $V'(t,x)$ note that the orbital derivative of $w(t,x) = \varphi(t_0,t,x)$ for fixed $t_0 \in \mathbb{I}$ is zero. Indeed, by the semiflow property we have

$$\begin{aligned} w'(t,x) &= \frac{\mathrm{d}}{\mathrm{d}\theta} w\big(t+\theta, \varphi(t+\theta,t,x)\big)\bigg|_{\theta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi\big(t_0, t+\theta, \varphi(t+\theta,t,x)\big)\bigg|_{\theta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(t_0,t,x)\bigg|_{\theta=0} \\ &= 0. \end{aligned}$$

Applying this with $w(t,x) = \varphi(0,t,x)$ and $w(t,x) = \varphi(T,t,x)$ to (2.6), the only non-zero contribution comes from the last factor $\frac{t}{T}$ in (2.6) and thus we obtain

$$V'(t,x) = \frac{1}{T}\left(\left\|\varphi(T,t,x) - \mu(T)\right\|^2 - \left\|\varphi(0,t,x) - \mu(0)\right\|^2\right) < 0 \quad \text{by (2.7)}$$

for all $(t,x) \in U \setminus \{(t,\mu(t))\colon\ t \in \mathbb{I}\}$. This finishes the proof of this theorem. $\square$

3. Domains and areas of attraction

In this section, we introduce the notions of both a domain and an area of attraction. While an area of attraction does not depend on a special solution, the concept of a domain of attraction relies on a given attractive solution.

Definition 3.1. Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be an attractive solution on $\mathbb{I}$. Then a connected and invariant nonautonomous set $G_\mu \subset \mathbb{I} \times \mathbb{R}^n$ is called *domain of attraction of* μ if

$$\left\|\varphi(T,0,x) - \mu(T)\right\| < \left\|x - \mu(0)\right\| \quad \text{for all } x \in G_\mu(0) \setminus \big\{\mu(0)\big\},$$

G_μ is maximal (with respect to set inclusion) and contains the graph of μ.

Such a maximal set always exists and is uniquely determined. In fact, the connected component of the set $\{x \in \mathbb{R}^n\colon \|\varphi(T,0,x) - \mu(T)\| < \|x - \mu(0)\|\}$ which contains $\mu(0)$ is the 0-fibre of the domain of attraction of μ. Moreover, due to the continuity of the general solution, the domain of attraction is an open nonautonomous set.

In addition to the domain of attraction of an attractive solution, we also consider so-called areas of attraction which are not based on a special attractive solution.

Definition 3.2. A connected and invariant nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}^n$ is called

(i) *area of attraction* if all solutions in G are attractive,
(ii) *area of exponential attraction* if all solutions in G are exponentially attractive.

Remark 3.3.

(i) An area of (exponential) attraction G is fully determined by its 0-fibre $G(0)$, since it is invariant.
(ii) A connected component of the set of all (exponentially) attractive solutions is an area of (exponential) attraction. It is a maximal area of attraction, i.e., there is no proper superset which is also an area of attraction.
(iii) The rate of exponential attraction depends continuously on the initial value of the solution at time 0. Indeed, by Proposition 2.5(ii), the rate of exponential attraction can be characterised by the largest eigenvalue of the Jacobian of the time-T map. Since the map and the eigenvalues vary continuously with respect to the initial value, the rate of exponential attraction depends continuously on the initial value as well. Thus, for any compact subset of an area of exponential attraction, the rate of exponential attraction is bounded away from 0.

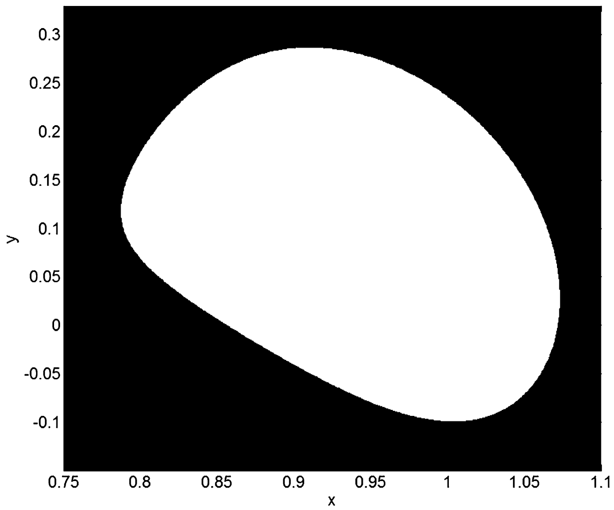

Fig. 1. The 0-fibre of the domain of attraction of $\nu(t) = \varphi(t, 0, (\frac{1}{2}, 0))$ for $T = 1$ is the plane $\mathbb{R}^2$ without the interior of the white ellipse-like set shown in the figure. It includes the maximal area of exponential attraction.

The following example shows that the (maximal) area of attraction and the domain of attraction of solutions within it are different sets in general. In the example, the area of attraction is a subset of the domains of attraction of two solutions contained in it, more precisely the solutions starting in $(0, 0)$ and $(1/2, 0)$. We will later show that this holds in general, provided that the 0-fibre of the (maximal) area of attraction is convex, cf. Theorem 6.1.

Example 3.4. Let $c := \frac{2\pi}{1-e^{-2}}$, and consider the planar system

$$\dot{x} = -x - cy\left(x^2 + y^2\right),$$
$$\dot{y} = -y + cx\left(x^2 + y^2\right),$$

which can be represented in the polar coordinates $x = r\cos\phi$ and $y = r\sin\phi$ by

$$\dot{r} = -r,$$
$$\dot{\phi} = cr^2.$$

We show that

(i) for any $T > 0$ the domain of attraction of the trivial solution $\mu(t) = \varphi(t, 0, (0, 0))$ is given by $\mathbb{I} \times \mathbb{R}^2$,
(ii) for any $T > 0$ the maximal area of exponential attraction G is determined by

$$G(0) = \left\{(x, y) \in \mathbb{R}^2\colon \sqrt{x^2 + y^2} < \sqrt{e^T/c}\right\},$$

(iii) for $T = 1$, the maximal area of exponential attraction G is determined by

$$G(0) = \left\{(x, y) \in \mathbb{R}^2\colon \sqrt{x^2 + y^2} < \sqrt{e/c} \approx 0.6116\right\},$$

and the domain of attraction of the solution starting in $(1/2, 0)$ is a proper superset of the maximal area of exponential attraction, cf. Fig. 1.

To show (i), note that the solution flow for a given initial value (r_0, ϕ_0) in polar coordinates is given by

$$r(t) = e^{-t} r_0,$$
$$\phi(t) = \phi_0 + c\int_0^t e^{-2\tau} r_0^2 \,\mathrm{d}\tau = \phi_0 - \frac{1}{2} c r_0^2\left(e^{-2t} - 1\right),$$

which immediately implies that the trivial solution is exponentially attractive on any finite time interval with domain of attraction $\mathbb{I} \times \mathbb{R}^2$, cf. (i).

To show (ii) we now seek to determine the maximal area of exponential attraction including the trivial solution. The time-T map F_T, which maps an initial point (x, y) at time 0 to the point $F_T(x, y)$ at a given time $T > 0$, is given by

$$F_T(x, y) = e^{-T} \begin{pmatrix} \cos(\rho r^2) & -\sin(\rho r^2) \\ \sin(\rho r^2) & \cos(\rho r^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.1}$$

where $\rho = \frac{c}{2}(1 - e^{-2T})$ and $r = \sqrt{x^2 + y^2}$. We use Proposition 2.5(ii) to determine whether the solution starting in (x, y) is exponentially attractive. It is sufficient to calculate $DF_T(x, y)$ only for the special case $x = r$ and $y = 0$, since DF_T does not depend on the initial angle of (x, y). We obtain

$$DF_T(r, 0) = e^{-T} \begin{pmatrix} \cos(\rho r^2) - 2\rho r^2 \sin(\rho r^2) & -\sin(\rho r^2) \\ \sin(\rho r^2) + 2\rho r^2 \cos(\rho r^2) & \cos(\rho r^2) \end{pmatrix},$$

which yields

$$DF_T(r, 0)^T DF_T(r, 0) = e^{-2T} \begin{pmatrix} 1 + 4\rho^2 r^4 & 2\rho r^2 \\ 2\rho r^2 & 1 \end{pmatrix}.$$

The eigenvalues $\lambda_{1,2}$ of this matrix are given by

$$\lambda_{1,2} = e^{-2T}\bigl(1 + 2\rho^2 r^4 \pm 2\sqrt{\rho^2 r^4 + \rho^4 r^8}\bigr).$$

The second eigenvalue λ_2 is less then 1 in all cases, since $0 < 1 + 2\rho^2 r^4 - 2\sqrt{\rho^2 r^4 + \rho^4 r^8} \leqslant 1$. For λ_1, one has

$$\begin{aligned} \lambda_1 < 1 \quad &\Leftrightarrow \quad e^{-2T}\bigl(1 + 2\rho^2 r^4 + 2\sqrt{\rho^2 r^4 + \rho^4 r^8}\bigr) < 1 \\ &\Leftrightarrow \quad 2\sqrt{\rho^2 r^4 + \rho^4 r^8} < e^{2T} - \bigl(1 + 2\rho^2 r^4\bigr) \\ &\Leftrightarrow \quad 4\bigl(\rho^2 r^4 + \rho^4 r^8\bigr) < e^{4T} - 2e^{2T}\bigl(1 + 2\rho^2 r^4\bigr) + 1 + 4\rho^2 r^4 + 4\rho^4 r^8 \\ &\Leftrightarrow \quad \rho^2 r^4 < \frac{(e^T - e^{-T})^2}{4} = \sinh^2(T) \\ &\Leftrightarrow \quad r^2 < \frac{\sinh(T)}{\rho} = \frac{e^T}{c}. \end{aligned}$$

Note that $r^2 < \frac{\sinh(T)}{\rho}$ implies $4\rho^2 r^4 < e^{2T} - 2 + e^{-2T}$, and thus, $e^{2T} > 1 + 2\rho^2 r^4$ for $T > 0$. Hence, solutions starting in (x, y) are exponentially attractive on $\mathbb{I}$ if and only if one has $x^2 + y^2 < \frac{e^T}{c}$, which shows (ii).

For (iii) we assume that $T = 1$, e.g., $\mathbb{I} = [0, 1]$. In this case, (ii) shows that solutions starting in (x, y) are exponentially attractive on $\mathbb{I}$ if one has $x^2 + y^2 < \frac{e}{c}$ and the maximal area of exponential attraction G is given by the disk

$$G(0) = \bigl\{(x, y) \in \mathbb{R}^2\colon \sqrt{x^2 + y^2} < \sqrt{e/c} \approx 0.6116\bigr\}.$$

We consider now the domain of attraction of the solution $\nu(t)$ starting in $\nu(0) = (\frac{1}{2}, 0)$. Since $T = 1$, we have $\rho = \pi$. Due to (3.1), we have

$$F_1(x, y) = e^{-1} \begin{pmatrix} \cos(\pi(x^2 + y^2))x - \sin(\pi(x^2 + y^2))y \\ \sin(\pi(x^2 + y^2))x + \cos(\pi(x^2 + y^2))y \end{pmatrix}.$$

In particular, $F_1(\frac{1}{2}, 0) = \frac{1}{4}\sqrt{2}e^{-1}(1, 1)$. The 0-fibre of the domain of attraction is shown in Fig. 1.

This planar system will be discussed again in Example 4.4.

The situation is different in the case of infinite time intervals, where domain of attraction and area of attraction are almost the same.

Remark 3.5. Consider the differential equation (2.1) on the infinite interval $\mathbb{R}_0^+$, i.e., having a right-hand side $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$, and let $\mu : \mathbb{R}_0^+ \to \mathbb{R}^n$ be an attractive solution in the sense of Lyapunov. Then the domain of attraction of μ, given by $\{(\tau, \xi)\colon \lim_{t\to\infty} \|\varphi(t, \tau, \xi) - \mu(t)\| = 0\}$ does not necessarily coincide with a maximal area of attraction (consider any negative solution of the example $\dot{x} = |x|$), but one can prove that the interior of the domain of attraction coincides with a maximal area of attraction. For this, one needs to show that solutions lying in the boundary of the domain of attraction are not attractive in the sense of Lyapunov.

4. Sufficiency

In this section, we provide a sufficient condition for a nonautonomous set to be an area of attraction. We will see in the next section, that this condition is also necessary. Crucial for what follows is the notion of a Riemannian metric.

Definition 4.1. A continuously differentiable function $M : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is called *Riemannian metric* if $M(t,x)$ is a symmetric and positive definite matrix for each $(t,x) \in \mathbb{I} \times \mathbb{R}^n$.

Note that $\langle v, w\rangle_M := v^T M(t,x) w$ defines a scalar product for $v, w \in \mathbb{R}^n$ and each $(t,x) \in \mathbb{I} \times \mathbb{R}^n$, if M is a Riemannian metric.

Given a Riemannian metric M, we define

$$L_M(t,x;w) := w^T \left(M(t,x) D_x f(t,x) + \frac{1}{2} M'(t,x) \right) w$$

and

$$L_M(t,x) := \max_{w\in\mathbb{R}^n,\, w^T M(t,x) w = 1} L_M(t,x;w),$$

where $M'(t,x)$ denotes the matrix with entries

$$m_{ij} = \frac{\partial M_{ij}(t,x)}{\partial t} + \sum_{k=1}^{n} \frac{\partial M_{ij}(t,x)}{\partial x_k} f_k(t,x).$$

Note that M' is the orbital derivative of M, i.e., $M'(t,x) = \frac{\mathrm{d}}{\mathrm{d}\tau} M(\tau, \varphi(\tau,t,x))|_{\tau=t}$.

The quantity $L_M(t,x)$ measures the rate of approach of adjacent trajectories; if $L_M(t,x) < 0$, then all trajectories adjacent to the one passing through (t,x) approach the one passing through (t,x). This condition was first established by Borg [8] and is known as Borg's criterion.

Theorem 4.2. *Consider the differential equation* (2.1)*, let $G \subset \mathbb{I} \times \mathbb{R}^n$ be a nonempty, connected, compact and invariant nonautonomous set and M be a Riemannian metric such that $M(0,x) = M(T,x) = \mathbb{1}$ for all $x \in \mathbb{R}^n$. Assume that there exists a $\nu > 0$ such that*

$$L_M(t,x) \leqslant -\nu \quad \text{for all } (t,x) \in G.$$

Then G is an area of exponential attraction. In particular, for all $\gamma < \nu$, there exists a $\delta > 0$ such that for all $x_0 \in G(0)$, we have

$$\left\| \varphi(T,0,x_0) - \varphi(T,0,\xi) \right\| \leqslant e^{-\gamma T} \|x_0 - \xi\| \quad \text{for all } \xi \in B_\delta(x_0), \tag{4.1}$$

i.e., all solutions lying in G are exponentially attractive such that the rate of exponential attraction is $\leqslant -\nu$.

Proof. The proof is divided into three parts. First, we prove some technical inequalities, and in the second step, we introduce a distance Γ with respect to a given reference solution. Finally, we show that this distance decreases exponentially.

Part 1. The matrix $M(t,x)$ is symmetric and positive definite for all $(t,x) \in G$. Hence, the smallest eigenvalue $\lambda(t,x)$ of $M(t,x)$ is positive, and since the eigenvalues depend continuously on (t,x), there are $0 < \lambda_- \leqslant \lambda_+ < \infty$ such that

$$\lambda_- \|y\|^2 \leqslant y^T M(t,x) y \leqslant \lambda_+ \|y\|^2 \tag{4.2}$$

and

$$\left\| M(t,x) y \right\| \leqslant \lambda_+ \|y\| \tag{4.3}$$

hold for all $y \in \mathbb{R}^n$ and all $(t,x) \in G$; note that G is compact in $\mathbb{R} \times \mathbb{R}^n$. Let $\gamma < \nu$, and define $k := 1 - \frac{\gamma}{\nu} > 0$. The derivative $D_x f(t,x)$ is continuous and thus uniformly continuous on G. Hence, there exists a $\tilde{\delta} > 0$ such that

$$\left\| D_x f(t,x) - D_x f(t,\xi) \right\| \leqslant k\nu \frac{\lambda_-}{\lambda_+} \tag{4.4}$$

holds for all $(t,x) \in G$ and $\xi \in \mathbb{R}^n$ with $\|\xi - x\| \leqslant \tilde{\delta}$. We set $\delta := \frac{\tilde{\delta}}{2}\sqrt{\lambda_-/\lambda_+}$.

Part 2. We fix $x_0 \in G(0)$ and $\xi \in \mathbb{R}^n$ with $\|\xi - x_0\| \leqslant \delta$. We denote by $\mu : \mathbb{I} \to \mathbb{R}^n$ the solution starting in $(0,x_0)$, i.e., $\mu(t) = \varphi(t,0,x_0)$ for all $t \in \mathbb{I}$. We define the distance

$$\Gamma(t) := \sqrt{\big(\varphi(t,0,\xi) - \mu(t)\big)^T M\big(t,\mu(t)\big)\big(\varphi(t,0,\xi) - \mu(t)\big)} \quad \text{for all } t \in \mathbb{I}.$$

Note that by (4.2), we have

$$\sqrt{\lambda_-}\big\|\varphi(t,0,\xi)-\mu(t)\big\| \leqslant \Gamma(t) \leqslant \sqrt{\lambda_+}\big\|\varphi(t,0,\xi)-\mu(t)\big\|.$$

In the following, we only consider the nontrivial case $\xi \neq x_0$. Then we have $\Gamma(t) \neq 0$ for all $t \in \mathbb{I}$, and we set

$$v(t) := \frac{\varphi(t,0,\xi)-\mu(t)}{\Gamma(t)} \quad \text{for all } t \in \mathbb{I}.$$

In other words, $\varphi(t,0,\xi)-\mu(t) = \Gamma(t)v(t)$ holds. Note that $v(t)$ is a vector with $\sqrt{v(t)^T M(t,\mu(t))v(t)} = 1$, and thus, $\frac{1}{\sqrt{\lambda_+}} \leqslant \|v(t)\| \leqslant \frac{1}{\sqrt{\lambda_-}}$ holds by (4.2).

Part 3. We show that $\Gamma(t)$ decreases exponentially, and we first calculate the temporal derivative of Γ^2; note also that $M(t,x) = M(t,x)^T$. We obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^2(t) &= 2\big(\varphi(t,0,\xi)-\mu(t)\big)^T M\big(t,\mu(t)\big)\big(f\big(t,\varphi(t,0,\xi)\big)-f\big(t,\mu(t)\big)\big)\\
&\quad + \big(\varphi(t,0,\xi)-\mu(t)\big)^T M'\big(t,\mu(t)\big)\big(\varphi(t,0,\xi)-\mu(t)\big)\\
&= 2\Gamma(t)v(t)^T M\big(t,\mu(t)\big)\big(f\big(t,\mu(t)+\Gamma(t)v(t)\big)-f\big(t,\mu(t)\big)\big)\\
&\quad + \Gamma^2(t)v(t)^T M'\big(t,\mu(t)\big)v(t).
\end{aligned}$$

We have $\|\Gamma(0)v(0)\| \leqslant \delta$. Since $2\delta \leqslant \tilde{\delta}$, there is a maximal $\theta \in (0,T]$ such that

$$\big\|\Gamma(t)v(t)\big\| = \big\|\varphi(t,0,\xi)-\mu(t)\big\| \leqslant \tilde{\delta}$$

for all $t \in [0,\theta]$. We will later show that $\theta = T$.

Now let $t \in [0,\theta]$ and use $L_M(t,\mu(t)) \leqslant -\nu$, the invariance of G and the mean value theorem. Then we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Gamma^2(t) &= 2\Gamma^2(t)v(t)^T M\big(t,\mu(t)\big)\left(\int_0^1 D_x f\big(t,\mu(t)+\lambda\Gamma(t)v(t)\big)\,\mathrm{d}\lambda\right)v(t) + \Gamma^2(t)v(t)^T M'\big(t,\mu(t)\big)v(t)\\
&\leqslant 2\Gamma^2(t)\Bigg(\underbrace{v(t)^T\left(M\big(t,\mu(t)\big)D_x f\big(t,\mu(t)\big)+\frac{1}{2}M'\big(t,\mu(t)\big)\right)v(t)}_{=L_M(t,\mu(t);v(t))}\\
&\quad + v(t)^T M\big(t,\mu(t)\big)\left(\int_0^1 \big(D_x f\big(t,\mu(t)+\lambda\Gamma(t)v(t)\big)-D_x f\big(t,\mu(t)\big)\big)\,\mathrm{d}\lambda\right)v(t)\Bigg)\\
&\leqslant -2\nu\Gamma^2(t) + 2\frac{\Gamma^2(t)\lambda_+}{\lambda_-}\frac{k\lambda_-}{\lambda_+}\nu = -2(1-k)\nu\Gamma^2(t).
\end{aligned}$$

The last inequality follows from (4.2), (4.3) and (4.4). Thus, we obtain with $(1-k)\nu = \gamma$ that

$$\Gamma(t) \leqslant \Gamma(0)e^{-\gamma t} \quad \text{for all } t \in [0,\theta]. \tag{4.5}$$

The inequality (4.5) shows in particular that $\Gamma(t) \leqslant \Gamma(0) \leqslant \sqrt{\lambda_+}\|\xi - x_0\| \leqslant \frac{1}{2}\tilde{\delta}\sqrt{\lambda_-}$ holds, and thus, we have $\|\Gamma(t)v(t)\| \leqslant \frac{1}{2}\tilde{\delta}$ for all $t \in [0,\theta]$. If $\theta < T$, then this contradicts the maximality of θ. Thus, $\theta = T$ and (4.5) holds for all $t \in \mathbb{I}$.

In particular for $t = T$, we obtain from (4.5) that

$$\Gamma(T) \leqslant \Gamma(0)e^{-\gamma T}, \tag{4.6}$$

which means that with $M(0,x) = M(T,x) = \mathbb{1}$, we arrive at

$$\big\|\varphi(T,0,\xi)-\mu(T)\big\| \leqslant \|\xi - x_0\|e^{-\gamma T}. \qquad \square$$

The following corollary deals with the rate of attraction between any two solutions in G when $G(0)$ is convex. In this case, the local result (4.1) for solutions in a neighborhood of x_0 can be extended to a global result for the distance between any two solutions starting in $G(0)$.

Corollary 4.3. *Consider the differential equation* (2.1)*, let $G \subset \mathbb{I} \times \mathbb{R}^n$ be a nonempty, connected, compact and invariant nonautonomous set and M be a Riemannian metric such that $M(0,x) = M(T,x) = \mathbb{1}$ for all $x \in \mathbb{R}^n$. We assume that $G(0)$ is convex and that there exists a $\nu > 0$ such that*

$$L_M(t,x) \leqslant -\nu \quad \textit{for all } (t,x) \in G.$$

Then we have

$$\left\|\varphi(T,0,x) - \varphi(T,0,y)\right\| \leqslant e^{-\nu T}\|x-y\| \quad \textit{for all } x, y \in G(0).$$

Proof. Let $\gamma < \nu$ and $x, y \in G(0)$ be chosen arbitrarily. Theorem 4.2 implies the existence of a $\delta > 0$ such that for all $x_0 \in G(0)$, we have

$$\left\|\varphi(T,0,x_0) - \varphi(T,0,\xi)\right\| \leqslant e^{-\gamma T}\|x_0-\xi\| \quad \text{for all } \xi \in B_\delta(x_0).$$

For given $x, y \in G(0)$, there exist $\kappa < \delta$ and $m \in \mathbb{N}$ such that $m\kappa = \|y-x\|$ and hence $x + m\kappa\frac{y-x}{\|y-x\|} = y$. Since $G(0)$ is convex,

$$x + i\kappa\frac{y-x}{\|y-x\|} \in G(0) \quad \text{for all } i \in \{0,\dots,m\}.$$

With $x_0 = x + i\kappa\frac{y-x}{\|y-x\|}$ and $\xi = x + (i+1)\kappa\frac{y-x}{\|y-x\|}$ for $i \in \{0,\dots,m-1\}$, repeated application of the above estimate implies

$$\left\|\varphi(T,0,x) - \varphi(T,0,y)\right\| \leqslant e^{-\gamma T}\|x-y\|.$$

Since $\gamma < \nu$ was chosen arbitrarily, the assertion follows. □

Example 4.4. Consider again the planar differential equation which was discussed in Example 3.4.

Part 1. Using the Euclidean metric, i.e., $M(t,(x,y)) \equiv \mathbb{1}$, we calculate $L_{\mathbb{1}}(t,(x,y))$ for this example. First observe that

$$L_{\mathbb{1}}\big(t,(x,y);w\big) = w^T D_{(x,y)} f\big(t,(x,y)\big) w = w^T\begin{pmatrix} -1-2cxy & -3cy^2-cx^2 \\ 3cx^2+cy^2 & -1+2cxy\end{pmatrix} w.$$

Then by [13, Corollary to Lemma 6.2], we have

$$\begin{aligned} L_{\mathbb{1}}\big(t,(x,y)\big) &= \max_{w\in\mathbb{R}^2, \|w\|=1} L_{\mathbb{1}}\big(t,(x,y);w\big) \\ &= \frac{1}{2}\Big(-1-2cxy-1+2cxy+\sqrt{\big(-3cy^2-cx^2+3cx^2+cy^2\big)^2+\big(-1-2cxy-(-1+2cxy)\big)^2}\Big) \\ &= \frac{1}{2}\Big(-2+\sqrt{4c^2\big(x^2-y^2\big)^2+16c^2x^2y^2}\Big) = -1+c\big(x^2+y^2\big). \end{aligned}$$

This means that $L_{\mathbb{1}}(t,(x,y)) < 0$ for $x^2+y^2 < \frac{1}{c}$. Since for any $R \geqslant 0$ the set

$$\big\{(t,x,y) \in \mathbb{I}\times\mathbb{R}^2 \,\big|\, x^2+y^2 \leqslant R^2\big\}$$

is positively invariant due to $\dot r = -r$, the invariant set

$$G = \big\{\big(t,\varphi\big(t,0,(x,y)\big)\big) \,\big|\, t\in\mathbb{I},\ x^2+y^2 \leqslant R^2\big\}$$

with $R < \frac{1}{\sqrt{c}}$ satisfies the assumptions of Theorem 4.2 and thus G is an area of exponential attraction.

Part 2. Using a different Riemannian metric, we seek to find a larger area of exponential attraction. Since Borg's criterion considers the worst direction in each point, we get a better estimate if we follow two solutions over a longer time interval. In order to improve the estimate, we fix $T = 1$ and define

$$V\big(t,(x,y)\big) = 2t(t-1)\big(x^2+y^2+0.3\big)$$

and show that

$$M(t,r) := \exp\big(2V(t,r)\big)\mathbb{1}$$

and $G = \{(t,(x,y))\colon \sqrt{x^2+y^2} \leqslant 0.54\exp(-t)\}$ fulfil the conditions of Theorem 4.2. Since $V(0,(x,y)) = V(1,(x,y)) = 0$, we have $M(0,(x,y)) = M(1,(x,y)) = \mathbb{1}$. An easy calculation shows that

$$L_M\big(t,(x,y)\big) = L_{\mathbb{1}}\big(t,(x,y)\big) + V'\big(t,(x,y)\big).$$

Since

$$V'(t,r) = 2(2t-1)\big(r^2+0.3\big) - 4t(t-1)r^2$$

we have

$$L_M(t,r) = -1 + cr^2 + 2(2t-1)\big(r^2+0.3\big) - 4t(t-1)r^2.$$

We can check that

$$L_M\big(t,(x,y)\big) < 0 \quad \text{for } \big(t,(x,y)\big) \in G' := \Big\{\big(t,\varphi\big(t,0,(\xi,\eta)\big)\big) \;\Big|\; t \in \mathbb{I},\ \sqrt{\xi^2+\eta^2} \leqslant 0.54\Big\},$$

using $\|\varphi(t,0,(x,y))\| \leqslant 0.54e^{-t}$ for all $(t,\varphi(t,0,(x,y))) \in G'$. Thus we have enlarged the radius of the set $G(0)$ to $G'(0)$ from $\sqrt{1/c} \approx 0.371$ to 0.54. The bound for the domain of attraction is 0.6116, cf. Example 3.4.

5. Necessity

In this section, the necessity of the conditions of Theorem 4.2 is shown, which means that we construct a Riemannian metric M in a given area of attraction.

Theorem 5.1. *Consider the differential equation* (2.1) *and a compact nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}^n$ which is an area of exponential attraction. Let $-\nu < 0$ be the maximal rate of exponential attraction of all solutions in G (see Remark* 3.3(iii)*). Then for every $\delta > 0$, there exists a Riemannian metric $M : G \to \mathbb{R}^{n\times n}$ in the sense of Definition* 4.1 *with $M(0,x) = M(T,x) = \mathbb{1}$ for all $x \in \mathbb{R}^n$ such that*

$$L_M(t,x) \leqslant -\nu + \delta \quad \textit{for all } (t,x) \in G.$$

Proof. The proof is divided into three parts. In Part 1 we construct M along a certain solution, and in Parts 2 and 3 we extend the construction to G using a partition of unity.

Part 1. Fix $\xi \in G(0)$, and consider the solution $\mu(t) = \varphi(t,0,\xi)$ on $\mathbb{I}$. The invariance of G implies that $(t,\mu(t)) \in G$ for all $t \in \mathbb{I}$. We consider the variational equation along this solution given by

$$\dot{y} = D_x f\big(t,\mu(t)\big)y. \tag{5.1}$$

Denote by $\Phi : \mathbb{I} \to \mathbb{R}^{n\times n}$ the fundamental matrix solution of (5.1) with the initial condition $\Phi(0) = \mathbb{1}$. Since $C := \Phi(T)$ is non-singular, there exists a matrix $B \in \mathbb{C}^{n\times n}$ such that $\Phi(T) = C = \exp(BT)$ (see, e.g., [9, Theorem 2.47]). We define

$$P(t) := \Phi(t)e^{-Bt} \quad \text{for all } t \in \mathbb{I}.$$

Obviously, we have $P(0) = \mathbb{1}$ and $P(T) = \Phi(T)\exp(-BT) = \mathbb{1}$. Since the rate of exponential attraction of $\mu(t)$ is bounded from above by $-\nu$, we have $\operatorname{Re}\alpha \leqslant -\nu$ for all eigenvalues α of B. Indeed, let $\alpha_r + i\alpha_c$ be an eigenvalue of B, i.e. there is an eigenvector $v \in \mathbb{C}^n$ with $\|v\| = 1$ such that $Bv = (\alpha_r + i\alpha_c)v$. Hence,

$$Cv = e^{BT}v = e^{(\alpha_r+i\alpha_c)T}v.$$

Note that $C \in \mathbb{R}^{n\times n}$. Hence,

$$v^*C^TCv = \|Cv\|^2 = e^{2\alpha_r T}\|v\|^2 = e^{2\alpha_r T}.$$

Note that $C = DF_T(\mu(0))$, where F_T is the time-T map. By Proposition 2.5 we have

$$\alpha_r \leqslant \frac{1}{2T}\ln\lambda_{max} = -\nu,$$

where λ_{max} denotes the maximal eigenvalue of C^TC.

Moreover, there exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $S^{-1}BS = A$ is in the special Jordan Normal Form, where the numbers 1 on the side diagonal are replaced by $\varepsilon := 2\delta$. More precisely, S is obtained by $S = S_1S_2$, where S_1 is the matrix containing the (generalised) eigenvectors in the columns and

$$S_2 = \operatorname{diag}\big(1,\varepsilon,\varepsilon^2,\ldots,\varepsilon^{m_1-1},1,\varepsilon,\varepsilon^2,\ldots,\varepsilon^{m_2-1},\ldots,1,\varepsilon,\varepsilon^2,\ldots,\varepsilon^{m_c-1}\big),$$

where m_j is the dimension of the jth eigenspace. With $N_\xi(t) = P^{-1}(t)^*(S^{-1})^*S^{-1}P^{-1}(t)$ for $t \in \mathbb{I}$, we now define the Riemannian metric independent of x by

$$M_\xi(t,x) = M_\xi(t) = \frac{1}{2}\big(N_\xi(t) + \overline{N_\xi(t)}\big) \quad \text{for all } t \in \mathbb{I}.$$

Note that because of the form of $N_\xi(t) = Z(t)^* Z(t)$ for $Z(t) := S^{-1}P^{-1}(t)$, the matrix $N_\xi(t)$ is Hermitian, i.e., $N_\xi(t) = N_\xi(t)^*$. Note also that $v^T N_\xi(t)^* v = (v^T N_\xi(t)^* v)^T = v^T \overline{N_\xi(t)} v$, and thus,

$$v^T M_\xi(t) v = \frac{1}{2}\big(v^T N_\xi(t) v + v^T N_\xi(t)^* v\big) = v^T N_\xi(t) v \quad \text{for all } v \in \mathbb{R}^n. \tag{5.2}$$

Hence, $M_\xi(t)$ is a real, symmetric and positive definite matrix, since $Z(t) = S^{-1}P^{-1}(t)$ is non-singular.

We will now show that $L_{M_\xi}(t, \mu(t); v) \leqslant (-\nu+\varepsilon) v^T M_\xi(t, \mu(t)) v$ for all $v \in \mathbb{R}^n$. First, we have for the derivative of $N_\xi(t)$ that

$$\dot N_\xi(t) = \dot P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) + P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} \dot P^{-1}(t).$$

By using $\frac{\mathrm{d}}{\mathrm{d}t}(P^{-1}(t)P(t)) = 0$, we obtain $\dot P^{-1}(t) = -P^{-1}(t)\dot P(t) P^{-1}(t)$. In addition, since $t \mapsto P(t)e^{Bt}$ is a solution of (5.1), we have $\dot P(t) = D_x f(t, \mu(t))P(t) - P(t)B$. Altogether, we get

$$\dot P^{-1}(t) = -P^{-1}(t) D_x f\big(t, \mu(t)\big) + B P^{-1}(t).$$

Hence,

$$\begin{aligned}\dot N_\xi(t) = {} & -D_x f\big(t, \mu(t)\big)^T P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) + P^{-1}(t)^* B^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) \\ & - P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) D_x f\big(t, \mu(t)\big) + P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} B P^{-1}(t)\end{aligned}$$

and

$$\begin{aligned}2\dot M_\xi(t) = {} & -D_x f\big(t, \mu(t)\big)^T P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) - D_x f\big(t, \mu(t)\big)^T P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1}}\,\overline{P^{-1}(t)} \\ & + P^{-1}(t)^* B^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) + P^{-1}(t)^T B^T \big(S^{-1}\big)^T \overline{S^{-1}}\,\overline{P^{-1}(t)} \\ & - P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) D_x f\big(t, \mu(t)\big) - P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1}}\,\overline{P^{-1}(t)} D_x f\big(t, \mu(t)\big) \\ & + P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} B P^{-1}(t) + P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1} B P^{-1}(t)}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}& 4\left(M_\xi(t) D_x f\big(t, \mu(t)\big) + \frac{1}{2}\dot M_\xi(t)\right) \\ & \quad = P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) D_x f\big(t, \mu(t)\big) + P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1} P^{-1}(t)} D_x f\big(t, \mu(t)\big) \\ & \qquad - D_x f\big(t, \mu(t)\big)^T P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) - D_x f\big(t, \mu(t)\big)^T P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1}}\,\overline{P^{-1}(t)} \\ & \qquad + P^{-1}(t)^* B^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) + P^{-1}(t)^T B^T \big(S^{-1}\big)^T \overline{S^{-1} P^{-1}(t)} \\ & \qquad + P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} B P^{-1}(t) + P^{-1}(t)^T \big(S^{-1}\big)^T \overline{S^{-1} B P^{-1}(t)}.\end{aligned}$$

Furthermore, we have for $v \in \mathbb{R}^n$, using $v^T Z v = (v^T Z v)^T = v^T Z^T v$ repeatedly, that

$$\begin{aligned}& v^T\left(M_\xi(t) D_x f\big(t, \mu(t)\big) + \frac{1}{2}\dot M_\xi(t)\right)v \\ & \quad = \frac{1}{2} v^T \big(P^{-1}(t)^* B^* \big(S^{-1}\big)^* S^{-1} P^{-1}(t) + P^{-1}(t)^* \big(S^{-1}\big)^* S^{-1} B P^{-1}(t)\big) v \\ & \quad = v^T P^{-1}(t)^* \big(S^{-1}\big)^* \left(\frac{1}{2}\big(S^* B^* \big(S^{-1}\big)^* + S^{-1} B S\big)\right) S^{-1} P^{-1}(t) v \\ & \quad = w^* \left(\frac{1}{2}\big(A^* + A\big)\right) w,\end{aligned}$$

where $w := S^{-1}P^{-1}(t)v \in \mathbb{C}^n$ and $A = S^{-1}BS$ was defined above as the special Jordan Normal Form of B. The matrix A is block diagonal with $A = \operatorname{blockdiag}(M_1, \ldots, M_c)$, where the usual Jordan block is replaced by

$$M_j := \begin{pmatrix} \alpha_j & \varepsilon & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon \\ & & & \alpha_j \end{pmatrix} \quad \text{for all } j \in \{1, \ldots, c\}.$$

Here, α_j are the (complex) eigenvalues of B. Thus,

$$\frac{1}{2}\big(A^*+A\big)=\begin{pmatrix} Z_1 & & 0\\ & \ddots & \\ 0 & & Z_c\end{pmatrix},$$

where

$$Z_j:=\begin{pmatrix} \lambda_j & \frac{1}{2}\varepsilon & 0 & & \\ \frac{1}{2}\varepsilon & \lambda_j & \frac{1}{2}\varepsilon & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0\\ & \ddots & \frac{1}{2}\varepsilon & \lambda_j & \frac{1}{2}\varepsilon\\ & & 0 & \frac{1}{2}\varepsilon & \lambda_j\end{pmatrix} \quad \text{and} \quad \lambda_j:=\operatorname{Re}\alpha_j \quad \text{for all } j\in\{1,\dots,c\}.$$

Next, $w^*(\frac{1}{2}(A^*+A))w$ is calculated. Let m_j denote the size of Z_j by, i.e., $Z_j\in\mathbb{R}^{m_j\times m_j}$, and consider $w\in\mathbb{C}^{m_j}$. Then the expression

$$\begin{aligned} w^*Z_jw&=\lambda_j\big(|w_1|^2+|w_2|^2+\cdots+|w_{m_j}|^2\big)\\ &\quad+\frac{\varepsilon}{2}(\overline{w_1}w_2+\overline{w_2}w_1+\overline{w_2}w_3+\overline{w_3}w_2+\cdots+\overline{w_{m_j-1}}w_{m_j}+\overline{w_{m_j}}w_{m_j-1})\end{aligned}$$

is a real number. Note that the Cauchy–Schwarz inequality implies $\mathbb{R}\ni\overline{w_i}w_{i+1}+\overline{w_{i+1}}w_i\leqslant|w_i|^2+|w_{i+1}|^2$, which yields that

$$\begin{aligned} w^*Z_jw&\leqslant\lambda_j\big(|w_1|^2+|w_2|^2+\cdots+|w_{m_j}|^2\big)\\ &\quad+\frac{\varepsilon}{2}\big(|w_1|^2+|w_2|^2+|w_2|^2+|w_3|^2+\cdots+|w_{m_j-1}|^2+|w_{m_j}|^2\big)\\ &\leqslant(\lambda_j+\varepsilon)\big(|w_1|^2+|w_2|^2+\cdots+|w_{m_j}|^2\big).\end{aligned}$$

In addition, note that $\|w\|^2=w^*w=v^TP^{-1}(t)^*(S^{-1})^*S^{-1}P^{-1}(t)v=v^TN_\xi(t)v=v^TM_\xi(t)v$ by (5.2), so we get altogether with $\operatorname{Re}\alpha_j\leqslant-\nu$ for all $j\in\{1,\dots,c\}$ that

$$L_{M_\xi}\big(t,\mu(t);v\big)=w^*\bigg(\frac{1}{2}\big(A^*+A\big)\bigg)w\leqslant\max_{1\leqslant j\leqslant c}(\operatorname{Re}\alpha_j+\varepsilon)\|w\|^2\leqslant(-\nu+\varepsilon)v^TM_\xi(t)v. \tag{5.3}$$

Moreover,

$$L_{M_\xi}\big(t,\mu(t)\big)=\max_{v\in\mathbb{R}^n,\,v^TM_\xi(t)v=1}L_{M_\xi}\big(t,\mu(t);v\big)\leqslant-\nu+\varepsilon. \tag{5.4}$$

Part 2. Since $L_{M_\xi}(t,x)$ is continuous with respect to (t,x), the representation (5.4) implies that there is an open neighborhood U_ξ of $\{(t,\mu(t))\colon t\in\mathbb{I}\}$ in $\mathbb{I}\times\mathbb{R}^n$ such that

$$L_{M_\xi}(t,x)\leqslant-\nu+\frac{\varepsilon}{2}=-\nu+\delta\quad\text{for all }(t,x)\in U_\xi. \tag{5.5}$$

Note that this implies

$$L_{M_\xi}(t,x;v)\leqslant(-\nu+\delta)v^TM_\xi(t)v\quad\text{for all }v\in\mathbb{R}^n\text{ and }(t,x)\in U_\xi. \tag{5.6}$$

Indeed, assume that $L_{M_\xi}(t,x;v)>(-\nu+\delta)v^TM_\xi(t)v$ for some $v\in\mathbb{R}^n$ and $(t,x)\in G$. Since $v\neq0$, we can define $w:=v/\sqrt{v^TM_\xi(t)v}$ such that we have $w^TM_\xi(t)w=1$. Then

$$L_{M_\xi}(t,x)=\max_{\tilde w\in\mathbb{R}^n,\,\tilde w^TM_\xi(t)\tilde w=1}L_{M_\xi}(t,x;\tilde w)\geqslant L_{M_\xi}(t,x;w)=\frac{L_{M_\xi}(t,x;v)}{v^TM_\xi(t)v}>-\nu+\delta,$$

which is a contradiction to (5.5).

Note that M_ξ as well as U_ξ only depends on the solution μ, or in other words, on the point $\xi=\mu(0)\in G(0)$, but $U_\xi\subset\mathbb{I}\times\mathbb{R}^n$. Moreover, we have $\bigcup_{\xi\in G(0)}U_\xi\supset G$, and since G is a compact set, there exist finitely many points $\xi_1,\dots,\xi_N$ such that $\bigcup_{i=1}^N U_{\xi_i}\supset G$. In particular, we have

$$\bigcup_{i=1}^N U_{\xi_i}(0)\supset G(0).$$

Next we choose a partition of unity subordinate to $U_{\xi_i}(0)$, i.e., for $i \in \{1, \dots, N\}$, we choose C^∞-functions $\tilde{g}_i : G(0) \to [0,1]$ such that $\operatorname{supp} \tilde{g}_i \subset U_{\xi_i}(0)$ and $\sum_{i=1}^N \tilde{g}_i(x) = 1$ for all $x \in G(0)$. We define g_i by a prolongation of $\tilde{g}_i$ in a constant way along solutions, i.e., $g_i : G \to [0,1]$ is the C^∞-function defined by $g_i(t,x) = \tilde{g}_i(\varphi(0,t,x))$.

Part 3. Define the Riemannian metric $M : G \to \mathbb{R}^{n\times n}$ by

$$M(t,x) = \sum_{i=1}^N g_i(t,x) M_{\xi_i}(t,x) \quad \text{for all } (t,x) \in G.$$

Obviously, $M(t,x)$ is a symmetric, positive definite matrix for all $(t,x) \in G$. Moreover, we have $M(0,x) = \sum_{i=1}^N g_i(0,x) \times M_{\xi_i}(0,x) = \sum_{i=1}^N g_i(0,x)\mathbb{1} = \mathbb{1}$ and also $M(T,x) = \mathbb{1}$. We now show that $L_M(t,x) \leqslant -\nu + \delta$. Note that we have $g_i'(t,x) \equiv 0$ for the orbital derivative, since g_i is constant along solutions, and thus, the orbital derivative of the product $g_i(t,x)M_{\xi_i}(t,x)$ reads as

$$\bigl(g_i(t,x)M_{\xi_i}(t,x)\bigr)' = g_i'(t,x)M_{\xi_i}(t,x) + g_i(t,x)M_{\xi_i}'(t,x) = g_i(t,x)M_{\xi_i}'(t,x).$$

Thus,

$$L_M(t,x;v) = \sum_{i=1}^N g_i(t,x) v^T \left(M_{\xi_i}(t,x) D_x f(t,x) + \frac{1}{2} M_{\xi_i}'(t,x) \right) v = \sum_{i=1}^N g_i(t,x) L_{M_{\xi_i}}(t,x;v).$$

Since $\operatorname{supp} \tilde{g}_i \subset U_{\xi_i}(0)$ implies $\operatorname{supp} g_i \subset U_{\xi_i}$ and $L_{M_{\xi_i}}(t,x;v) \leqslant (-\nu+\delta) v^T M_{\xi_i}(t,x) v$ by (5.6) for all $(t,x) \in U_{\xi_i}$, we have

$$L_M(t,x;v) \leqslant (-\nu+\delta) \sum_{i=1}^N g_i(t,x) v^T M_{\xi_i}(t,x) v.$$

Finally, since $v^T M(t,x) v = \sum_{i=1}^N g_i(t,x) v^T M_{\xi_i}(t,x) v$, we have

$$L_M(t,x) = \max_{v \in \mathbb{R}^n,\, v^T M(t,x) v = 1} L_M(t,x;v) \leqslant -\nu + \delta \quad \text{for all } (t,x) \in G.$$

This finishes the proof of this theorem. □

The sufficient and necessary conditions of Theorem 4.2 and Theorem 5.1 are summarised in the following corollary.

Corollary 5.2. *Consider the nonautonomous differential equation* (2.1)*. A nonempty, connected, compact and invariant nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}^n$ is an area of exponential attraction if and only if there exist a $-\nu < 0$ and a Riemannian metric $M : G \to \mathbb{R}^{n\times n}$ in the sense of Definition* 4.1 *with $M(0,x) = M(T,x) = \mathbb{1}$ for all $x \in \mathbb{R}^n$ such that*

$$L_M(t,x) \leqslant -\nu \quad \textit{for all } (t,x) \in G.$$

6. Areas and domains of attraction

The results of the preceding sections enable us to prove the following inclusion of area and domains of attraction.

Theorem 6.1. *Consider the nonautonomous differential equation* (2.1)*, and let the nonempty, compact nonautonomous set $G \subset \mathbb{I} \times \mathbb{R}^n$ be an area of exponential attraction, such that $G(0)$ is convex. Let $\mu : \mathbb{I} \to \mathbb{R}^n$ be a solution which lies in G. Then the domain of attraction of μ, denoted by G_μ, satisfies*

$$G_\mu \supset G.$$

Proof. Theorem 5.1 implies the existence of $\gamma > 0$ and a Riemannian metric M with $M(0,x) = M(T,x) = \mathbb{1}$ for all $x \in \mathbb{R}^n$ and $L_M(t,x) \leqslant -\gamma < 0$ for all $(t,x) \in G$. We can now apply Corollary 4.3 and obtain

$$\bigl\|\varphi(T,0,x) - \varphi(T,0,y)\bigr\| \leqslant e^{-\gamma T} \|x - y\| \quad \text{for all } x, y \in G(0).$$

For $x = \mu(0)$ and $y \in G(0)$, this shows that $y \in G_\mu(0)$. Since both G and G_μ are connected and invariant, the assertion of the theorem is proved. □

Note, however, that it is essential that the area of exponential attraction is convex. The following example shows that non-convex areas of exponential attraction are not subsets of the domains of attractions of points within in general.

Example 6.2. For some parameter $a < 0$, we consider the system given in polar coordinates

$$\dot r = ar, \qquad \dot\phi = \begin{cases} \frac{4}{\pi}\phi - 1, & \phi \in [\frac{\pi}{4}, \frac{\pi}{2}),\\ 1, & \phi \in [\frac{\pi}{2}, 5\frac{\pi}{4}),\\ 6 - \frac{4}{\pi}\phi, & \phi \in [5\frac{\pi}{4}, 3\frac{\pi}{2}),\\ 0, & \phi \in [3\frac{\pi}{2}, 2\pi) \cup [0, \frac{\pi}{4}), \end{cases}$$

on the time interval $\mathbb{I} := [0, \frac{\pi}{2}]$, and let $\varphi : \mathbb{I} \times \mathbb{I} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the induced dynamical system on the plane. Note that the right-hand side is not continuously differentiable, but one can easily smoothen this example.

Then it is easy to see that the compact and connected nonautonomous set

$$G := \bigl\{\bigl(t, \varphi\bigl(t, 0, (x, y)\bigr)\bigr) \in \mathbb{I} \times \mathbb{R}^2\colon\ (x, y) \in \overline{B_1(0)} \text{ with } x \leqslant 0 \text{ or } y \leqslant 0\bigr\}$$

is an area of exponential attraction (note that the radial components of adjacent solutions are contracted; in addition, the distance of the angular components of adjacent solutions are contracted for the angles $\phi \in (5\frac{\pi}{4}, 3\frac{\pi}{2})$, they stay constant elsewhere in G). We compare now the time evolutions of $\xi_1 := (1, 0)$ and $\xi_2 := (0, 1)$, which are given by

$$\varphi\left(\frac{\pi}{2}, 0, \xi_1\right) = \bigl(e^{a\frac{\pi}{2}}, 0\bigr) \quad \text{and} \quad \varphi\left(\frac{\pi}{2}, 0, \xi_2\right) = \bigl(-e^{a\frac{\pi}{2}}, 0\bigr).$$

This implies that for all parameters $a \in (-\frac{1}{\pi}\ln 2, 0)$, we have

$$\|\xi_1 - \xi_2\| = \sqrt{2} < 2e^{a\frac{\pi}{2}} = \left\|\varphi\left(\frac{\pi}{2}, 0, \xi_1\right) - \varphi\left(\frac{\pi}{2}, 0, \xi_2\right)\right\|.$$

Hence, the domain of attraction of the solution starting in ξ_1 or ξ_2 does not contain G for such values of a.

We consider an example to illustrate the concepts introduced in this paper and to relate them to the infinite-time case.

Example 6.3. We consider the nonlinear example

$$\dot x = x(x-1) =: f(x). \tag{6.1}$$

In case of an infinite time interval, this system has an attractive equilibrium $x_0 = 0$ and a repulsive equilibrium $x_1 = 1$, and the domain of attraction of x_0 (in $\mathbb{R}$) is given by $(-\infty, 1)$. Moreover, the maximal area of exponential attraction consisting of all exponentially attracting solutions (in $\mathbb{R}$) is $(-\infty, 1)$. The domain of attraction of every solution within the maximal area of exponential attraction is also $(-\infty, 1)$.

Now we consider (6.1) on a finite time interval $\mathbb{I} := [0, T]$ for some $T > 0$. We will show that

1. The maximal area of exponential attraction G is defined by

   $$G(0) = \bigl(-\infty, b(T)\bigr) \subsetneq (-\infty, 1),$$

   cf. (6.3) for the definition of $b(T)$. We have $\lim_{T\to\infty} b(T) = 1$.
2. The domain of attraction of a solution $\varphi(t, 0, \xi)$ with $\xi \in G(0)$ is G_ξ, defined by

   $$G_\xi(0) = \bigl(-\infty, c_\xi(T)\bigr) \supsetneq (-\infty, 1) \quad \text{for } \xi < 0,$$

   cf. (6.5) for the definition of $c_\xi(T)$. We have $\lim_{T\to\infty} c_\xi(T) = 1$.

Part 1. Maximal area of exponential attraction. First we want to determine the maximal area of exponential attraction. Note that the time-T map is given by

$$F_T(x) = \frac{x}{x - (x-1)e^T} \quad \text{for all } x \in \left(-\infty, \frac{e^T}{e^T - 1}\right). \tag{6.2}$$

For larger x, the solutions blow up in a time shorter than T, and thus, the time-T map is not defined for such points. Motivated by Proposition 2.5, we calculate

$$DF_T(x) = \frac{e^T}{(x - (x-1)e^T)^2}$$

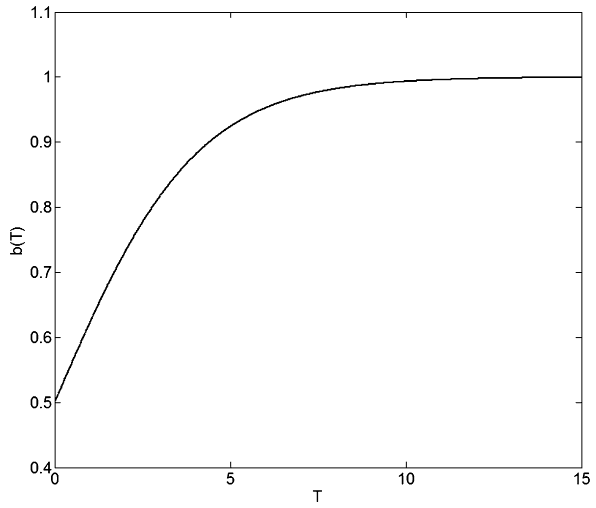

Fig. 2. The function b describing the boundary of the maximal area of exponential attraction, cf. (6.3).

and obtain

$$\begin{aligned} DF_T(x)^T DF_T(x) < 1 \quad &\Leftrightarrow \quad e^{2T} < \big(x - (x-1)e^T\big)^4 \\ &\Leftrightarrow \quad e^T < \big(x\big(1-e^T\big) + e^T\big)^2 \\ &\Leftrightarrow \quad x \notin \left[\frac{e^T - e^{T/2}}{e^T - 1}, \frac{e^T + e^{T/2}}{e^T - 1}\right]. \end{aligned}$$

As remarked above, the calculation is not valid for $x \geqslant e^T/(e^T - 1)$, which means that the 0-fibre of the maximal area of exponential attraction is given as

$$G(0) = \left\{x \in \mathbb{R}\colon\ x < b(T) := \frac{e^T - e^{T/2}}{e^T - 1}\right\}. \tag{6.3}$$

Note that $\lim_{T\to 0} b(T) = \frac{1}{2}$ and $\lim_{T\to\infty} b(T) = 1$. The function $b(T)$ is plotted in Fig. 2.

The rate of exponential attraction $-\nu_\xi$ for the solution $\varphi(t, 0, \xi)$ is given by $\frac{1}{2T}\ln\lambda$, where $\lambda = DF_T(\xi)^2$, i.e.,

$$-\nu_\xi := 1 - \frac{2}{T}\ln\big(\xi - (\xi - 1)e^T\big). \tag{6.4}$$

For example, for $\xi = 0$ we obtain $-\nu_0 = -1$.

Part 2. Domain of attraction. Now we consider a solution $\mu(\cdot) = \varphi(\cdot, 0, \xi)$ lying in G and thus satisfying $\xi < \frac{e^T}{e^T-1}$. We seek to calculate its domain of attraction G_μ. For $x < \frac{e^T}{e^T-1}$ we have $x \in G_\mu(0) \setminus \{\xi\}$ if and only if

$$\begin{aligned} &\big|F_T(x) - F_T(\xi)\big| < |x - \xi| \\ &\quad\Leftrightarrow \quad \left|\frac{x}{x - (x-1)e^T} - \frac{\xi}{\xi - (\xi-1)e^T}\right| < |x - \xi|, \quad \text{cf. (6.2)} \\ &\quad\Leftrightarrow \quad \big|x\big(\xi - (\xi-1)e^T\big) - \xi\big(x - (x-1)e^T\big)\big| < |x-\xi|\big(x - (x-1)e^T\big)\big(\xi - (\xi-1)e^T\big) \\ &\quad\Leftrightarrow \quad e^T < \big(x - (x-1)e^T\big)\big(\xi - (\xi-1)e^T\big) \\ &\quad\Leftrightarrow \quad x < 1 - \frac{\xi}{\xi - e^T(\xi - 1)}. \end{aligned}$$

We thus define

$$c_\xi(T) := 1 - \frac{\xi}{\xi - e^T(\xi - 1)} \tag{6.5}$$

and obtain $G_\xi(0) = (-\infty, c_\xi(T))$. Note that for fixed ξ, $\lim_{T\to\infty} c_\xi(T) = 1$. For $\xi = 0$, we have $c_0(T) = 1$; here the domain of attraction is the same as in the infinite-time case.

Note that for $\xi < 0$ we have $c_\xi(T) > 1$. This means that for negative ξ, the domain of attraction contains also points larger than 1, see also Fig. 3. The reason for this is that in the interval from 0 to T the solutions starting in $\xi < 0$ approach 0 faster than the solutions starting in points near, but larger than 1 tend away from 1.

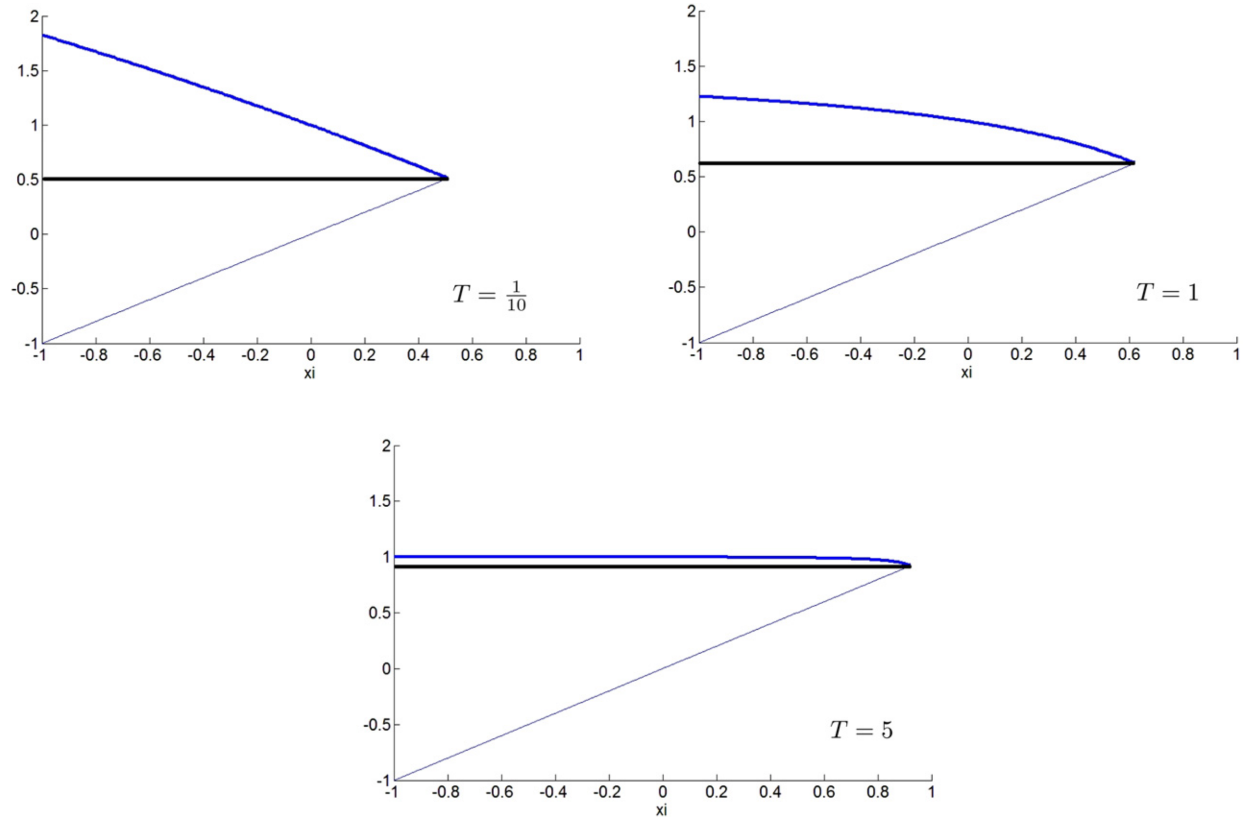

Fig. 3. The upper boundary of the domain of attraction for ξ between -1 and $b(T)$ is illustrated as blue curve, the thin black line represents $x = \xi$ and the thick black line is the upper boundary of the maximal area of exponential attraction. The three figures correspond to different lengths of the finite time interval: $T = 0.1$, $T = 1$ and $T = 5$. The maximal area of exponential attraction is always a subset of the domain of attraction (since the first is convex). For negative ξ, the domain of attraction contains even points bigger than 1. For $T \to \infty$ both the area of attraction and the domains of attraction converge towards $(-\infty, 1)$ which is the corresponding set in the infinite-time case. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

In a final example, we consider a nonautonomous population model and calculate the area of exponential attraction.

Example 6.4. The exponential growth given by $\dot{x} = rx$ for $r > 0$ is unrealistic in real-world applications. Hence, the logistic equation has been introduced to model bounded growth

$$\dot{x} = rx\left(1 - \frac{x}{K}\right),$$

where the positive parameters r and K denote the (maximal) rate of population growth and the carrying capacity, i.e. the maximal population reached by growth from below, respectively.

Since the assumption that both parameters are constant is too restrictive in applications, the nonautonomous logistic equation

$$\dot{x} = r(t)x\left(1 - \frac{x}{K(t)}\right)$$

with time-varying parameters $r(t)$ and $K(t)$ has been considered, for example in applications to biological population models [16], specifically to bacterial populations [3]. We rename $p(t) = r(t)$ and $l(t) = \frac{r(t)}{K(t)}$ and consider the following more general situation, where $p(t)$ and $l(t)$ are not restricted to positive values.

Proposition 6.5. *We consider the nonautonomous logistic equation*

$$\dot{x} = x\big(p(t) - l(t)x\big) \tag{6.6}$$

with continuous functions $p(t)$ *and* $l(t)$ *on* $\mathbb{I} = [0, T]$. *We define* $P := \int_0^T p(t)\,\mathrm{d}t$ *and* $L := \int_0^T \exp(\int_0^t p(s)\,\mathrm{d}s) l(t)\,\mathrm{d}t$.

Then the area of exponential attraction G is defined by

- $G(0) = (-\infty, -\frac{e^{P/2}+1}{L}) \cup (\frac{e^{P/2}-1}{L}, \infty)$ *if* $L > 0$.
- $G(0) = (-\infty, \frac{e^{P/2}-1}{L}) \cup (-\frac{e^{P/2}+1}{L}, \infty)$ *if* $L < 0$.
- $G = \mathbb{I} \times \mathbb{R}$ *if* $L = 0$ *and* $P < 0$.
- $G = \emptyset$ *if* $L = 0$ *and* $P \geqslant 0$.

Proof. The time-T map is given by

$$F_T(x) = \varphi(T, 0, x) = \frac{xe^P}{1 + xL}.$$

We have

$$DF_T(x) = \frac{e^P}{(1 + xL)^2}.$$

By Proposition 2.5 $\mu(t) = \varphi(t, 0, x)$ is exponentially attractive on $\mathbb{I}$ if only if $DF_T(x) < 1$, i.e.

$$e^{P/2} < |1 + xL|,$$

from which the proposition follows. $\square$

Note added in proof

After final submission of this article, the authors became aware of the submitted paper [20], which contains an excellent discussion of domains of attractions of linearized finite-time processes.

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