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Notes on subspace-hypercyclic operators

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ABSTRACT

A bounded linear operator T on a Banach space X is called subspace-hypercyclic for a subspace M if $orb(T, x) \cap M$ is dense in M for a vector $x \in M$. In this case we show that $p(T)$ has a relatively dense range for every real or complex polynomial p, which in turn answers a question posed in Madore and Martinez-Avendano (2011) [8]. As a consequence, the algebraic structure of the set of subspace-hypercyclic vectors can be described.

1. Introduction

Let X be a separable infinite dimensional Banach space. A bounded linear operator $T : X \to X$ is called hypercyclic if there exists a vector $x \in X$ whose orbit $orb(T, x) = \{T^n x : n \geq 1\}$ is dense in X. For example, if B is the backward shift on $\ell^2(\mathbb{N})$, then any multiple λB, $|\lambda| > 1$, is hypercyclic [1]. This phenomenon appears in separable spaces, and it is connected with the invariant subspace problem, dynamical systems, and approximation theory. For references on hypercyclic operators see [2,3].

In the last decade, the concept of hypercyclicity has been generalized in various directions. Considering density in the weak topology instead of the norm topology, we can discuss weak hypercyclicity [4,5]; a bounded linear operator $T : X \to X$ is called weakly hypercyclic if the $orb(T, x)$ is weakly dense in X for some $x \in X$. Quantifying the frequency with which an orbit meets any nonempty open set, we can study the frequently hypercyclic operator [6,7]; a bounded linear operator $T : X \to X$ is frequently hypercyclic if there exists a vector $x \in X$ such that for every nonempty open subset U of X, the set of integers n such that $T^n x$ belongs to U has positive lower density.

Considering the density of the orbit in a nontrivial subspace instead of whole space, the hypercyclicity on subspaces has been recently discussed by Madore and Martinez-Avendano [8]:

Definition 1.1. A bounded linear operator $T : X \to X$ is called subspace-hypercyclic for a subspace M of X if there exists a vector $x \in X$ such that $orb(T, x) \cap M$ is dense in M. The vector x is then called a subspace-hypercyclic vector for T.

It is easy to see that T is subspace-hypercyclic for M if and only if it is so also for $\overline{M}$.

The following examples provide some subspace-hypercyclic operators (not necessarily hypercyclic):

(i) Let T be a hypercyclic operator on a Banach space X and let I be the identity operator on X. Then, the operator $T \oplus I$ is subspace-hypercyclic for the subspace $M := X \oplus \{0\}$ of $X \oplus X$; whereas it cannot be hypercyclic on $X \oplus X$, since the identity operator is not hypercyclic on X.

(ii) Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Let T_φ denote the analytic Toeplitz operator on $H^2(\mathbb{D})$ defined by $T_\varphi(f) = f\varphi$, and C_φ denote the composition operator on $H^2(\mathbb{D})$ defined by $C_\varphi(f) = f \circ \varphi$. Now if φ is a non-identity inner function with $\varphi(0) = 0$, then

$$T_\varphi^* C_\varphi = C_\varphi B,$$

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where B is the backward shift operator on $H^2(\mathbb{D})$. Since λB is hypercyclic on $H^2(\mathbb{D})$ for $|\lambda| > 1$, it can easily be checked that λT_φ^* is subspace-hypercyclic for the subspace $ran\big(C_\varphi\big)$ if $|\lambda| > 1$ (see Examples 2.3 and 2.4 in [8]). On the other hand, observe that λT_φ^* is hypercyclic, by the characterization given by Godefroy and Shapiro [9].

(iii) Let H be a separable Hilbert space, $B(H)$ be the algebra of bounded linear operators on H, and $T \in B(H)$. Since $B(H)$ is not separable with the norm operator topology, the left multiplication operator $L_T : B(H) \to B(H)$ defined by $L_T(S) = TS$ is not hypercyclic with this topology while $L_T : B_0(H) \to B_0(H)$ is hypercyclic with the norm operator topology [10]. Here, $B_0(H)$ is the algebra of compact operators on H. Thus there exists an operator $S \in B_0(H)$ such that $orb(L_T, S)$ is dense in $B_0(H)$ with the operator norm topology, i.e.,

$$\overline{orb(L_T, S) \cap B_0(H)} = \overline{orb(L_T, S)} = B_0(H).$$

So L_T is subspace-hypercyclic for the subspace $B_0(H)$ of $B(H)$.

Besides, there are also examples of hypercyclic operators which are subspace-hypercyclic on proper subspaces. For example, let M be either the subspace of ℓ^2 consisting of all sequences with zeros on the even entries or the subspace consisting of all sequences with zeros on the first m entries. Then λB is subspace-hypercyclic for M where B is the usual backward shift on ℓ^2 and λ is any scalar with $|\lambda| > 1$.

Subspace-hypercyclic operators share many properties with hypercyclic operators. We state some of them in the following theorem which has been proved in [8]:

Theorem 1.2. *Let $T : X \to X$ be a bounded linear operator which is subspace-hypercyclic for a subspace M of X. Then the following statements hold:*

(i) *the spectrum of T intersects the unit circle,*
(ii) *$ker(\lambda - T^*)^n \subseteq M^\perp$ for any scalar λ and integer $n \geq 0$,*
(iii) *X and M are infinite dimensional.*

Our main goal in the present paper is to show that if a bounded linear operator T acting on a Banach space X is subspace-hypercyclic for some subspace M of X and p is a real or complex polynomial, then $ker(p(T^*)) \subseteq M^\perp$, which provides an affirmative answer to the question (v) of [8]. We prove as a consequence, extending a result on hypercyclic operators [11–13], that, at least in some cases, every subspace-hypercyclic operator T for a subspace M of X has a dense linear manifold of M consisting entirely, except for zero, of vectors that are subspace-hypercyclic for T.

2. The main results

In what follows, X will be a separable infinite dimensional Banach space and $L(X)$ will denote the algebra of bounded linear operators on X. Since subspace-hypercyclicity for a subspace and its closure are equivalent, all subspaces of X may be assumed to be closed. If $T \in L(X)$, the orbit of a subspace M of X under T is defined by

$$orb(T, M) = \bigcup_{n \geq 0} T^n M.$$

Proposition 2.1. *Let X be a real or complex Banach space, $T \in L(X)$ be subspace-hypercyclic for a nontrivial subspace M of X, and $M_T := orb(T, M)$; then for any scalar λ,*

$$\big[(\lambda - T)(M_T)\big]^\perp \subseteq M_T^\perp.$$

Proof. Let $x \in M$ be a subspace-hypercyclic vector for T. Assume that Λ is a nonzero linear functional that annihilates the subset $(\lambda - T)(M_T)$. Since M_T is an invariant set under T,

$$\Lambda(T^n x) = \lambda^n \Lambda(x)$$

for every x in M_T and for every positive integer n. If Λ is not zero on M_T, considering the above relation, the definition of M_T and the fact that M_T contains the nontrivial subspace M, we see that Λ is not zero on M. Thus, $\Lambda(M)$ is the whole complex plane $\mathbb{C}$ (or the real line $\mathbb{R}$) and the same is true for $\Lambda(M_T)$. Since x is a subspace-hypercyclic vector for T, $\overline{orb(T, x) \cap M} = M$ and so $M \subseteq \overline{orb(T, x)}$. Hence,

$$M_T = \bigcup_{n=0}^{+\infty} T^n M \subseteq \overline{orb(T, x)},$$

and so the closure of $\{\Lambda(T^n x) : n \geq 0\}$ contains $\Lambda(M_T) = \mathbb{C}$ (or $\mathbb{R}$) while the closure of $\{\lambda^n \Lambda(x) : n \geq 0\}$ does not. This contradiction shows that Λ annihilates the subset M_T and the proof is completed. $\square$

It is worth mentioning that the above proposition may be restated: if T is subspace-hypercyclic on M, $M_T = orb(T, M)$ and $[M_T]$ denotes the linear span of M_T, then,

$$[M_T] \subseteq \overline{(\lambda - T)([M_T])}. \tag{1}$$

This is the key point for proving the next theorem.

Theorem 2.2. *Let X be a complex Banach space, $T \in L(X)$ be subspace-hypercyclic for a nontrivial subspace M of X, and $M_T := orb(T, M)$. Then for every nonzero complex polynomial p,*

$$[M_T] \subseteq \overline{p(T)([M_T])}. \tag{2}$$

Proof. We may express p as a product of linear factors;

$$p(z) = \lambda_0(\lambda_1 - z)(\lambda_2 - z) \cdots (\lambda_n - z)$$

where λ_i is a complex numbers for $0 \le i \le n$. Then $p(T)$ is a scalar times a product of factors of the form $\lambda_i - T$, that is,

$$p(T) = \lambda_0(\lambda_1 - T)(\lambda_2 - T) \cdots (\lambda_n - T).$$

By the last proposition, each of the factors $\lambda_i - T$ satisfies the relation (1) for M_T and $p(T)$ is written as a product of such operators. Hence (2) holds and this completes the proof. □

Corollary 2.3. *Let X be a complex Banach space, $T \in L(X)$ be subspace-hypercyclic for a nontrivial subspace M. Then for every nonzero complex polynomial p, $p(T)$ has a dense range in M, or equivalently, $ker(p(T^*)) \subseteq M^{\perp}$.*

Given a real Banach space X, we denote by $\tilde{X}$ the complexification of X. That is, $\tilde{X}$ is the product space $X \times X$ endowed with the complex scalar product given by

$$(a + ib)(x, y) := (ax - by, ay + bx) \quad (x, y \in X, a, b \in \mathbb{R}).$$

Furthermore, given $T \in L(X)$, we denote its complexification by $\tilde{T} \in L(\tilde{X})$. That is,

$$\tilde{T}(x, y) := (Tx, Ty) \quad (x, y \in X).$$

Lemma 2.4. *Let X be a real Banach space, and $T \in L(X)$. If T is subspace-hypercyclic for a subspace M of X, then $\tilde{T} \in L(\tilde{X})$ is also subspace-hypercyclic for the subspace $\tilde{M} = M \times \{0\}$ of $\tilde{X}$.*

Proof. It is enough to note that if $x \in M$ is a subspace-hypercyclic vector for T then $(x, 0) \in \tilde{M}$ is a subspace-hypercyclic vector for $\tilde{T}$. □

Theorem 2.5. *For a real Banach space, the conclusions of Theorem 2.2 and Corollary 2.3 remain true.*

Proof. Let X be a real Banach space, and $T \in L(X)$ be a subspace-hypercyclic operator for a subspace M of X. Pick a real nonzero polynomial p, and put $M_T := orb(T, M)$, and $\tilde{M_T} := orb(\tilde{T}, \tilde{M})$. Consider the complex polynomial $\tilde{p}$ with the same coefficients as p. Now by Lemma 2.4, $\tilde{T}$ is subspace-hypercyclic for $\tilde{M}$ and applying Theorem 2.2, we see that the relation (2) holds for $\tilde{T}$, $\tilde{M_T}$, and $\tilde{p}$. Hence the same is true for T, M_T, p and hence Theorem 2.2 and Corollary 2.3 remain true for real Banach spaces. □

Herrero and Bourdon proved an important result on the algebraic structure of the set of hypercyclic vectors:

Proposition 2.6. *If x is any hypercyclic vector for T, then*

$$\{p(T)x : p \text{ is a polynomial } \} \setminus \{0\},$$

is a dense set of hypercyclic vectors. In particular, any hypercyclic operator admits a dense invariant subspace consisting, except for zero, of hypercyclic vectors.

For the complex case, the above theorem is due to Herrero [13] and Bourdon [12]. The real case was pointed out by Bes [11].

Applying Corollary 2.3 and Theorem 2.5, we can extend the above theorem to subspace-hypercyclic operators in the following way:

Theorem 2.7. *Let $T \in L(X)$ and M be a nontrivial subspace of X such that*

$$M_{n,T} := \bigcup_{k=0}^{+\infty} T^{kn}(M)$$

is a subspace of X for some $n \in \mathbb{N}$. If T is subspace-hypercyclic for M, then there is a dense invariant linear manifold of X consisting entirely, except for zero, of vectors that are subspace-hypercyclic for T with respect to $M_{n,T}$.

Proof. Let $x \in M$ be a subspace-hypercyclic vector for T and

$$\Delta = \{p(T)x : p \text{ is a polynomial}\};$$

then Δ is a nontrivial subspace of X. We show that every nonzero element of Δ is a subspace-hypercyclic vector for T with respect to $M_{n,T}$. Let $M_n = T^n(M)$; then

$$M_{n,T} = \bigcup_{k=0}^{+\infty} T^k(M_n).$$

Since $x \in M$ is a subspace-hypercyclic vector for T,

$$M = \overline{orb(T,x) \cap M}$$

and so

$$M_n = T^n(M) \subseteq \overline{orb(T, T^n x) \cap M_n} = \overline{orb(T,x) \cap M_n}.$$

Moreover,

$$M_{n,T} = \bigcup_{k=0}^{+\infty} T^k(M_n) \subseteq \overline{orb(T,x) \cap M_{n,T}}.$$

Hence T is subspace-hypercyclic for both M_n and $\overline{M_{n,T}}$. Let $p(T)x$ be an arbitrary nonzero element of Δ; then Theorem 2.2 and the fact that $M_{n,T}$ is a T-invariant subspace of X imply that

$$\overline{M_{n,T}} = \overline{P(T)(M_{n,T})}.$$

Thus

$$\begin{aligned}\overline{M_{n,T}} &= p(T)(\overline{M_{n,T}}) = p(T)\left(\overline{orb(T,x) \cap M_{n,T}}\right) \\ &\subseteq \overline{p(T)(orb(T,x)) \cap p(T)(\overline{M_{n,T}})} \\ &\subseteq \overline{orb(T, p(T)x) \cap \overline{M_{n,T}}},\end{aligned}$$

and hence $p(T)x$ is a subspace-hypercyclic vector for T with respect to $\overline{M_{n,T}}$. Finally, it is easy to see that Δ is T-invariant and dense in M. Now the proof is completed. □

To the best of our knowledge, all examples of subspace-hypercyclic operators T for nontrivial subspaces M given in [8] are either of kind (i) or kind (ii) in the following corollary which forces the set $M_{n,T}$ in Theorem 2.7 to be a subspace.

Corollary 2.8. *Let $T \in L(X)$ be subspace-hypercyclic for the nontrivial subspace M of X and $M_{n,T}$ be as in the previous theorem.*

(i) *If $T^n(M) \subseteq M$ for some integer $n \geq 0$, then $M_{n,T} = M$ and so there is a T-invariant linear manifold Δ of X which is dense in M and every nonzero vector of Δ is subspace-hypercyclic for T with respect to M.*
(ii) *If $M \subseteq T^n(M)$ for some integer $n \geq 0$, then $M_{n,T}$ is a subspace of X and so there is a T-invariant linear manifold Δ of X which is dense in M and every nonzero vector of Δ is subspace-hypercyclic for T with respect to $M_{n,T}$.*
(iii) *If $T^n(M) = X$ for some integer $n \geq 0$, then $M_{n,T} = X$, and so there is a dense T-invariant linear manifold Δ of X such that every nonzero vector of Δ is hypercyclic for T.*

Example 2.9. Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and consider the operator $T := \lambda B$ where B is the backward shift on ℓ^2.

(i) Let M be the subspace of $X := \ell^2$ consisting of all sequences with zeros on the even entries; that is,

$$M := \left\{(a_n) \in X : a_{2k} = 0 \text{ for all } k \geq 1\right\}.$$

Then by the previous theorem there is a T-invariant linear manifold of X which is dense in M consisting entirely, except for zero, of vectors that are subspace-hypercyclic for T with respect to M.

(ii) Let $m \in \mathbb{N}$ and M be the subspace of X consisting of all sequences with zeros on the first m entries; that is,

$$M := \left\{(a_n) \in X : a_n = 0 \text{ for } n < m\right\}.$$

Then $M \subseteq T(M), T^m(M) = X$ and so by the previous theorem there is a dense T-invariant linear manifold of X consisting entirely, except for zero, of vectors that are hypercyclic for T.

The following proposition is implicitly proven in [8]. It shows that bounded linear operators on a Banach space with a sufficiently large supply of eigenvectors are not subspace-hypercyclic. Here we state it as an application of Proposition 2.1.

Proposition 2.10. *Let $T \in L(X)$, and let there exists a nonempty subset $A \subseteq \mathbb{C}$ such that the vector space spanned by the kernels $ker(\lambda - T^*)$ with $\lambda \in A$, i.e.,*

$$span\left(\bigcup_{\lambda \in A} ker(\lambda - T^*)\right),$$

is dense in X. Then T is not subspace-hypercyclic.

Proof. Applying Proposition 2.1, we see that $ker(\lambda - T^*) \subseteq M^{\perp}$ for every scalar $\lambda \in A$. Thus

$$span\left(\bigcup_{\lambda \in A} ker(\lambda - T^*)\right) \subseteq M^{\perp}$$

and by density of the vector space spanned by the kernels, $M^{\perp} = X$ and hence $M = \{0\}$. □

As in the case of hypercyclicity, multiplication operators on Hardy spaces can never be subspace-hypercyclic. This is explicitly proven in [8]; here we state it as an application of the above proposition:

Example 2.11. Let φ be a bounded holomorphic function on the unit disk $\mathbb{D}$; then M_{φ}, the operator of "multiplication by φ", defined on the usual Hardy space $H^2(\mathbb{D})$ by $(M_{\varphi}f)(z) = \varphi(z)f(z)$ is clearly a bounded linear operator on $H^2(\mathbb{D})$. We want to prove that M_{φ} is not subspace-hypercyclic on $H^2(\mathbb{D})$. First suppose that φ is a non-constant holomorphic function and let $k_z \in H^2(\mathbb{D})$ be the reproducing kernel for $z \in \mathbb{D}$, i.e., $k_z(f) = f(z)$ for every $f \in H^2(\mathbb{D})$. Then, it is well known that $M_{\varphi}^*(k_z) = \overline{\varphi(z)}k_z$, i.e., $k_z \in ker(\overline{\varphi(z)} - M_{\varphi}^*)$. Since, the reproducing kernels have a dense linear span in $H^2(\mathbb{D})$ (see Theorem 4.42 in [3]), the vector space spanned by the kernels $ker(\lambda - T^*)$ with $\lambda \in rang(\varphi)$ is dense in $H^2(\mathbb{D})$. Hence by the previous proposition, M_{φ} cannot be subspace-hypercyclic on $H^2(\mathbb{D})$. Now suppose that φ is a constant function; then M_{φ} is a scalar multiple of the identity operator on $H^2(\mathbb{D})$. So for some scalar λ, the orbit of M_{φ} under any vector f is of the form $\{\lambda^n f : n \geq 0\}$ which is not dense in any nontrivial subspace of $H^2(\mathbb{D})$. Hence M_{φ} is not subspace-hypercyclic in this case and the proof is completed.

The paper is ended with several questions on subspace-hypercyclic operators.

Bourdon and Feldman [14] proved that any orbit that is somewhere dense is (everywhere) dense. A similar question for subspace-hypercyclicity can be posed as follows:

Question 1. Let M be a nontrivial subspace of a Banach space X and $x \in M$. Does $orb(T, x) \cap M$ being somewhere dense in M imply that it is everywhere dense in M?

Ansari [15] showed that powers of hypercyclic operators are hypercyclic with the same hypercyclic vectors. A similar question for subspace-hypercyclicity can be given:

Question 2. Let M be a nontrivial subspace of a Banach space X and T be subspace-hypercyclic for M; is T^n also subspace-hypercyclic for M for every integer $n > 1$?

From the definition of subspace-hypercyclicity, we find that if an operator is subspace-hypercyclic for a subspace M, it must have a dense orbit in M. Hence, it is natural to ask the following question:

Question 3. Let $T \in L(X)$, M be an infinite dimensional subspace and $x \in M$. Does the density of $orb(T, x)$ in M, i.e., $M \subseteq \overline{orb(T, x)}$, imply that T is subspace-hypercyclic for M?

As far as we know, all the subspace-hypercyclic operators T for the nontrivial subspace M satisfy either $T^n(M) \subseteq M$ or $M \subseteq T^n(M)$ for some integer $n \geq 0$. Now, we are interested in proposing the following question:

Question 4. Does there exist a subspace-hypercyclic operator T for a nontrivial subspace M such that we have neither $T^n(M) \subseteq M$ nor $M \subseteq T^n(M)$ for any integer $n \geq 0$?

In view of Theorem 2.7 and Corollary 2.8, it would be interesting to answer the following question:

Question 5. If T is subspace-hypercyclic for a nontrivial subspace M, must there be a dense linear manifold consisting of subspace-hypercyclic vectors for M?

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