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Finite rank sums of products of Toeplitz and Hankel operators

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ABSTRACT

On the Dirichlet space of the unit disk, we consider operators which are finite sums of Toeplitz products, Hankel products or products of Toeplitz and Hankel operators. We then give characterizations of when such operators have finite rank on the Dirichlet space.

1. Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$. The Sobolev space $\mathscr{S}$ is the completion of the space of all smooth functions f on D for which

$$\left|\int_D f\,dA\right|^2 + \int_D \left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2\right) dA < \infty$$

where dA denotes the normalized Lebesgue measure on D. The space $\mathscr{S}$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_D f dA \int_D \bar{g}\, dA + \int_D \left(\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}}\overline{\frac{\partial g}{\partial \bar{z}}}\right) dA.$$

The Dirichlet space $\mathscr{D}$ is the closed subspace of $\mathscr{S}$ consisting of all holomorphic functions $f \in \mathscr{S}$ with $f(0) = 0$. Let Q denote the orthogonal projection from $\mathscr{S}$ onto $\mathscr{D}$. Put

$$\mathscr{L}^{1,\infty} = \left\{u \in \mathscr{S} : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in L^\infty(D)\right\}$$

where the derivatives are taken in the sense of distributions and $L^p(D) = L^p(D, dA)$ denotes the usual Lebesgue space on D. By Sobolev's embedding theorem [1, Theorem 5.4], each function in $\mathscr{L}^{1,\infty}$ can be extended to a continuous function on the closed unit disk $\bar{D}$. We will use the same notation between each function in $\mathscr{L}^{1,\infty}$ and its continuous extension.

Given $u \in \mathscr{L}^{1,\infty}$, the *Toeplitz operator* T_u and the *(little) Hankel operator* H_u with symbol u are defined on $\mathscr{D}$ respectively by

$$T_u f = Q(uf), \qquad H_u f = QJ(uf)$$

for functions $f \in \mathscr{D}$. Here J is the flip operator defined by $Jf(z) = f(\bar{z})$. Then, it is easy to see that T_u and H_u are bounded linear operators on $\mathscr{D}$.

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In several recent papers, some algebraic properties of Toeplitz and Hankel operators on the Dirichlet space have been studied. The (semi-)commuting Toeplitz operators have been characterized by the author with harmonic symbols in [2]. Also, results in [2] were later extended in [3] to the class of operators which are finite sums of products of two Toeplitz operators with harmonic symbols. Later, the results in [2] have been extended in [4] to more general symbols satisfying a certain absolute continuity condition and in [5] to general symbols in $\mathscr{L}^{1,\infty}$. At the same time, the product problem of when the product T_uT_v equals another Toeplitz operator has been studied. Also, finite rank (semi-)commutators of two Toeplitz operators have been characterized in [6] and commuting Hankel operators have been studied as in [5] or [7].

Very recently, a more general class of operators including (semi-)commutators and products of two Toeplitz operators or two Hankel operators has been considered. More explicitly, the author and Zhu [8] considered operators L of the form

$$L = \sum_{j=1}^{N} A_jB_j,$$

where each of A_j and B_j is a Toeplitz operator or a Hankel operator. Then, they gave characterizations of when such operators are equal to 0 on $\mathscr{D}$.

Motivated by these results, in this paper we study more generally the problem of characterizing when an operator of the form L has finite rank on the Dirichlet space. The corresponding problems on the Hardy space or Bergman space have been studied in [9–12] or [13].

In Section 2, we collect some preliminary results which will be used in our characterizations. In Section 3, we consider operators of the form L where A_j, B_j are all Toeplitz operators and then give a characterization for the operators to have finite rank in terms of certain boundary conditions and harmonicity of a function induced by the symbols; see Theorem 3.5. In addition, we study the finite rank product problem of when a product of several Toeplitz operators has finite rank. Our result shows that a product of several Toeplitz operators can have finite rank only in an obvious case; see Theorem 3.10. In Section 4, we study the corresponding problem in case when A_j, B_j are all Hankel operators and obtain a characterization; see Theorem 4.2. Specifically, in the case of rank 0, our results not only give complete different characterizations from results in [8] but also recover several known results mentioned above concerning (semi-)commutators and products of Toeplitz operators or Hankel operators. In Section 5, we consider operators which are sums of any two of the form T_uH_v or H_uT_v and characterize such operators to have finite rank; see Theorem 5.4 and Corollaries 5.5 and 5.6.

2. Preliminaries

Each point evaluation is easily verified to be a bounded linear functional on $\mathscr{D}$. Hence, for each $a \in D$, there exists a unique $r_a \in \mathscr{D}$ such that $f(a) = \langle f, r_a\rangle$ for every $f \in \mathscr{D}$. It is known that the function r_a is given by

$$r_a(z) = \log\left(\frac{1}{1-\bar{a}z}\right), \quad z \in D.$$

Using the explicit formula for r_a, one can see that Q can be represented by

$$Q\psi(a) = \int_D \frac{a}{1-a\bar{w}}\frac{\partial\psi}{\partial w}(w)\,dA(w), \quad a \in D \tag{1}$$

for functions $\psi \in \mathscr{S}$. Specifically, for a function $\psi \in \mathscr{S}$ with series expansion

$$\psi(re^{i\theta}) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}$$

we see from a simple calculation using (1) that

$$Q\psi(z) = \sum_{j=1}^{\infty} a_j z^j, \quad z = re^{i\theta} \in D. \tag{2}$$

The Bergman space L_a^2 is the closed subspace of $L^2(D)$ consisting of all holomorphic functions. Let P be the Bergman projection which is the orthogonal projection from $L^2(D)$ onto L_a^2. It is known that P can be represented by

$$P\psi(a) = \int_D \psi(w)\overline{k_a(w)}\,dA(w), \quad a \in D$$

where k_a denotes the Bergman kernel for L_a^2 given by

$$k_a(w) = \frac{1}{(1-\bar{a}w)^2}, \quad w \in D.$$

For each $a \in D$, we put

$$\rho_a(z) = \frac{z}{1-\bar{a}z}, \quad z \in D.$$

Since $\rho_a' = k_a$ and $\rho_a(0) = 0$, we have

$$\langle \psi, \rho_a \rangle = \int_D \frac{\partial \psi}{\partial w} \overline{\rho_a'} \, dA = P\left(\frac{\partial \psi}{\partial w}\right)(a)$$

for every $\psi \in \mathscr{S}$. In particular, we have $\langle \psi, \rho_a \rangle = \psi'$ for $\psi \in \mathscr{D}$. See [2] or Chapter 4 of [14] for details and related facts.

3. Sums of Toeplitz products

In this section, we consider operators which are finite sums of products of two Toeplitz operators and characterize such operators to have finite rank.

We first recall the notion of Lipschitz spaces. Let $0 < \alpha < 1$. Given a complex function f on D, we say $f \in \Lambda_\alpha$ if

$$\sup \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty$$

where the supremum is taken over all $z, w \in D$ with $z \neq w$. It is known that for a function f holomorphic on D, $f \in \Lambda_\alpha$ if and only if $(1 - |z|^2)^{1-\alpha} f'(z)$ is bounded on D and each holomorphic function in Λ_α is continuous up to the boundary ∂D. Also, note that each function in $\mathscr{L}^{1,\infty}$ belongs to Λ_α and the projection P maps Λ_α into Λ_α; see [5,15] or Chapter 7 of [16] for details.

Lemma 3.1. *Let $u, v \in \mathscr{L}^{1,\infty}$ and fix a point $z \in D$. Define a function Φ on D given by $\Phi(a) = T_u T_v \rho_a(z)$ for $a \in D$. Then Φ is bounded on D.*

Proof. Fix $z \in D$ and recall $\rho_a' = k_a$ for $a \in D$. Using (1), we first have $T_v \rho_a = \psi_{1,a} + \psi_{2,a}$ where

$$\psi_{1,a}(w) = \int_D \frac{w}{1 - w\bar{\zeta}} \frac{\partial v}{\partial \zeta}(\zeta) \rho_a(\zeta) \, dA(\zeta),$$

$$\psi_{2,a}(w) = \int_D \frac{w}{1 - w\bar{\zeta}} v(\zeta) k_a(\zeta) \, dA(\zeta)$$

for every $a, w \in D$. Hence $\Phi = F_1 + F_2 + G_1 + G_2$ where

$$F_j(a) = z \int_D \frac{\frac{\partial u}{\partial w}(w) \psi_{j,a}(w)}{1 - z\bar{w}} \, dA(w),$$

$$G_j(a) = z \int_D \frac{u(w) \psi_{j,a}'(w)}{1 - z\bar{w}} \, dA(w)$$

for $j = 1, 2$ and $a \in D$. Now we show $F_j, G_j \in L^\infty(D)$ for each $j = 1, 2$. Since $u, v \in \mathscr{L}^{1,\infty}$, we see by Lemma 3.10 of [14]

$$\begin{aligned} |F_1(a)| &= \left| z \int_D \int_D \frac{\frac{\partial u}{\partial w}(w) w \frac{\partial v}{\partial \zeta}(\zeta) \zeta}{(1 - z\bar{w})(1 - w\bar{\zeta})(1 - \bar{a}\zeta)} \, dA(\zeta) \, dA(w) \right| \\ &\leq C \int_D \int_D \frac{1}{|1 - \bar{a}\zeta| \, |1 - w\bar{\zeta}|} \, dA(w) \, dA(\zeta) \\ &\leq C \int_D \frac{1}{|1 - \bar{a}\zeta|} \, dA(\zeta) \\ &\leq C \end{aligned}$$

for some constants C independent of $a \in D$. Hence $F_1 \in L^\infty(D)$. Also, by an application of Fubini's theorem, we obtain

$$F_2(a) = \int_D v\bar{h} k_a \, dA = \overline{P(\bar{v}h)(a)}$$

where

$$h(\zeta) = \bar{z} \int_D \frac{\overline{w \frac{\partial u}{\partial w}(w)}}{(1 - \bar{z}w)(1 - \zeta\bar{w})} \, dA(w), \quad \zeta \in D.$$

On the other hand, using again Lemma 3.10 of [14], we can see $(1 - |\zeta|^2)^{1-\alpha} h'(\zeta)$ is bounded on D and hence $h \in \Lambda_\alpha$ for all $\alpha \in (0, 1)$. Since $v \in \Lambda_\alpha$, we have $P[\bar{v}h] \in \Lambda_\alpha$ for all $\alpha \in (0, 1)$ and thus $F_2 \in L^\infty(D)$. Also, by a similar argument, we have

$$G_1(a) = \int_D \frac{\frac{\partial v}{\partial \zeta}(\zeta) \zeta}{1 - \zeta\bar{a}} \overline{P[k](\zeta)} \, dA(\zeta)$$

where

$$k(w) = \frac{\bar{z}}{1 - \bar{z}w}\bar{u}(w), \quad w \in D.$$

Then similarly we can see $P[k] \in \Lambda_\alpha$ for all $\alpha \in (0, 1)$ and hence $P[k] \in L^\infty(D)$. Now, by Lemma 3.10 of [14] again, we see $G_1 \in L^\infty(D)$. Finally, using a similar argument, we see

$$G_2(a) = \int_D v\overline{P[k]}k_a \, dA = \overline{P[\bar{v}P(k)](a)}.$$

Also, using $\bar{v}P[k] \in \Lambda_\alpha$ for all $\alpha \in (0, 1)$, we see $G_2 \in L^\infty(D)$. Therefore Φ is bounded on D. The proof is complete. □

Given a Hilbert space K with an inner product $(,)$ and $a, b \in K$, we let $a \otimes b$ be the rank one operator on K defined by $[a \otimes b]x = (x, b)a$ for $x \in K$. Recall that each finite rank operator S on K can be written in the form

$$S = \sum_{j=1}^{N} a_j \otimes b_j$$

for some $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ in K. In case S is an operator which is a finite sum of products of two Toeplitz operators on $\mathscr{D}$, we will need some more information on the functions b_j. To do this, we first need the following lemma which is taken from Lemma 2.4 of [17].

Lemma 3.2. *Let $\{g_j\}_{j=1}^N$ be a linearly independent set in $\mathscr{D}$. Then there exist points $z_1, \ldots, z_N \in D$ such that the matrix*

$$\begin{pmatrix} g_1(z_1) & \cdots & g_1(z_N) \\ \vdots & & \vdots \\ g_N(z_1) & \cdots & g_N(z_N) \end{pmatrix}$$

is invertible.

Proposition 3.3. *Let $u_j, v_j \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$ and $x_j, y_j \in \mathscr{D}$ for $j = 1, \ldots, N$. Suppose $\{x_j\}_{j=1}^N$ is linearly independent and*

$$\sum_{j=1}^{M} T_{u_j}T_{v_j} = \sum_{j=1}^{N} x_j \otimes y_j$$

holds. Then y_j' is bounded on D for all $j = 1, \ldots, N$.

Proof. Since $\langle \rho_a, y_j \rangle = \overline{y_j'(a)}$ for each j, we first have

$$\sum_{j=1}^{N} [x_j \otimes y_j]\rho_a = \sum_{j=1}^{N} \langle \rho_a, y_j \rangle x_j = \sum_{j=1}^{N} x_j\overline{y_j'(a)}$$

for every $a \in D$. Thus, letting

$$F_z(a) := \sum_{j=1}^{M} \overline{T_{u_j}T_{v_j}\rho_a(z)}, \quad z, a \in D,$$

we have

$$F_z(a) = \sum_{j=1}^{N} \overline{x_j(z)}y_j'(a), \quad z, a \in D$$

and hence

$$\begin{pmatrix} F_{z_1}(a) \\ \vdots \\ F_{z_N}(a) \end{pmatrix} = \begin{pmatrix} \overline{x_1(z_1)} & \cdots & \overline{x_N(z_1)} \\ \vdots & & \vdots \\ \overline{x_1(z_N)} & \cdots & \overline{x_N(z_N)} \end{pmatrix} \begin{pmatrix} y_1'(a) \\ \vdots \\ y_N'(a) \end{pmatrix}$$

for all points $a \in D$ and $z_1, \ldots, z_N \in D$. Now, since the functions $x_1, \ldots, x_N$ are linearly independent, Lemma 3.2 shows that the $N \times N$ matrix in the above displayed equation is invertible for some points $z_1, \ldots, z_N \in D$. Thus, each y_j' is a linear combination of functions $F_{z_1}(a), \ldots, F_{z_N}(a)$. On the other hand, by Lemma 3.1 F_z is bounded for each $z \in D$. Thus each y_j' is bounded on D. The proof is compete. □

In the course of our proofs, we will use some known results on the Hardy space. Thus we need to introduce the well known Hardy space.

For $1 \le p \le \infty$, we let $L^p(\partial D) = L^p(\partial D, \sigma)$ denote the usual Lebesgue space on ∂D where σ is the normalized Lebesgue measure on ∂D. For $1 \le p < \infty$, the Hardy space H^p consists of all holomorphic functions f on D such that

$$\sup_{0\le r<1} \int_{\partial D} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty.$$

It is well known that H^p can be identified with a closed subspace of $L^p(\partial D)$ via boundary values. We use the same notation to denote a function in H^p and its boundary value in $L^p(\partial D)$.

We let $C : L^2(\partial D) \to H^2$ be the orthogonal projection from $L^2(\partial D)$ onto H^2. Given $\psi \in L^\infty(\partial D)$, we let $t_\psi : H^2 \to H^2$ be the (*Hardy space*) Toeplitz operator with symbol ψ defined by $t_\psi f = C(\psi f)$ for functions $f \in H^2$. Then, t_ψ is a bounded linear operator on H^2. See Chapter 10 of [14] for more information.

Recall that each function in $\mathscr{L}^{1,\infty}$ can be extended to a continuous function (with the same notation) to $\bar{D}$. We sometimes call the restriction to ∂D of u the boundary function of u. In particular, when we write t_u with $u \in \mathscr{L}^{1,\infty}$, the symbol of t_u should be understood to be the boundary function of u.

The following connection between Toeplitz operators on the Dirichlet and Hardy spaces has been known for any $u \in \mathscr{L}^{1,\infty}$:

$$T_u f = z t_u \left(\frac{1}{z} f \right), \quad f \in \mathscr{D}; \tag{3}$$

see Lemma 3.2 of [6].

For $\psi \in L^\infty(\partial D)$, we recall that ψ admits a unique decomposition $\psi = f + g$ where $f, \bar{g}$ are analytic, all negative Fourier coefficients of f and nonnegative Fourier coefficients of g vanish. In this case, we call f the analytic part and g the co-analytic part of ψ. Also, we then use the same notation between the (co-)analytic part on ∂D and its (anti-)holomorphic extension to D via Poisson extension. Moreover, we have $f, \bar{g} \in H^p$ for all p. For details, see Chapter 9 of [14] for example.

The following lemma taken from Corollary 4.3 of [10] will be useful in our characterization of finite rank sums of products of two Toeplitz operators.

Lemma 3.4. *Let $\varphi_j, \psi_j, \rho \in L^\infty(\partial D)$ for $j = 1, \ldots, M$ and $\alpha_j, \beta_j \in H^2$ for $j = 1, \ldots, N$. Let ϵ_j, δ_j be the analytic parts of $\varphi_j, \overline{\psi_j}$ respectively. Then*

$$\sum_{j=1}^{M} t_{\varphi_j} t_{\psi_j} = t_\rho + \sum_{j=1}^{N} \alpha_j \otimes \beta_j$$

holds on H^2 if and only if the following two conditions hold.

(a) $\sum_{j=1}^{M} \varphi_j \psi_j = \rho$ *on ∂D.*
(b) $\sum_{j=1}^{M} \epsilon_j \overline{\delta_j} - (1 - |z|^2) \sum_{j=1}^{N} \alpha_j \overline{\beta_j}$ *is harmonic on D.*

Given two functions f, g holomorphic on D, put $u = f + \bar{g}$ and suppose

$$\sup_{0<r<1} \int_{\partial D} |u(r\zeta)|^2 \, d\sigma(\zeta) < \infty. \tag{4}$$

Then, we see $f, g \in H^2$ from the boundedness of the Riesz projection C, since $f = Cu$ and $\bar{g} = (Id - C)u$ on ∂D. In the following, we will use

$$\langle \varphi, \psi \rangle_2 = \int_{\partial D} \varphi \bar{\psi} \, d\sigma$$

for functions $\varphi, \psi \in L^2(\partial D)$.

Now we are ready to prove the main result of this section which characterizes finite rank sums of products of two Toeplitz operators.

Theorem 3.5. *Let $u_j, v_j, \tau \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$ and f_j, k_j be the analytic parts of the boundary functions of $u_j, \overline{v_j}$ respectively. Let $x_j, y_j \in \mathscr{D}$ for $j = 1, \ldots, N$. Then*

$$\sum_{j=1}^{M} T_{u_j} T_{v_j} = T_\tau + \sum_{j=1}^{N} x_j \otimes y_j \tag{5}$$

holds on $\mathscr{D}$ if and only if the following two conditions hold.

(a) $\sum_{j=1}^{M} u_j v_j = \tau$ *on ∂D.*
(b) $\sum_{j=1}^{M} f_j \overline{k_j} - (1 - |z|^2) \sum_{j=1}^{N} \frac{x_j}{z} \overline{y_j'}$ *is harmonic on D.*

Proof. For simplicity, we put

$$T = \sum_{j=1}^{M} T_{u_j} T_{v_j} - T_\tau, \qquad S = \sum_{j=1}^{M} t_{u_j} t_{v_j} - t_\tau .$$

First suppose the set $\{x_j\}_{j=1}^N$ is linearly independent. Assume (5) holds. Using (3), we have

$$S\left(\frac{h}{z}\right) = \frac{1}{z} T(h) = \sum_{j=1}^{K} \langle h, y_j \rangle \frac{x_j}{z}$$

for every $h \in \mathscr{D}$. Since $z^{-1}\mathscr{D}$ is dense in H^2, it follows that

$$S\psi = \sum_{j=1}^{N} \langle z\psi, y_j \rangle \frac{x_j}{z}$$

for every $\psi \in H^2$. Note $y_j' \in H^2$ for each j by Proposition 3.3. On the other hand, a simple calculation shows that $\langle zz^n, y_j \rangle = \langle z^n, y_j' \rangle_2$ for each j and integer $n \geq 0$. Hence $\langle z\psi, y_j \rangle = \langle \psi, y_j' \rangle_2$ for each j and $\psi \in H^2$. It follows that

$$S\psi = \sum_{j=1}^{N} \langle \psi, y_j' \rangle_2 \frac{x_j}{z} = \sum_{j=1}^{N} \left[\frac{x_j}{z} \otimes y_j' \right] \psi$$

for every $\psi \in H^2$ and thus

$$S = \sum_{j=1}^{N} \frac{x_j}{z} \otimes y_j' \quad \text{on } H^2. \tag{6}$$

Now, by Lemma 3.4, we have (a) and (b).

To prove the converse, assume (a) and (b) hold. First, we show $y_j' \in H^2$ for each j. To do this, we will use a similar argument as in Proposition 3.3. By (b), we can write

$$\sum_{j=1}^{M} f_j \overline{k_j} - (1 - |z|^2) \sum_{j=1}^{N} \frac{x_j}{z} \overline{y_j'} = F + \overline{G}$$

for some functions F, G holomorphic on D. Since $y_j' \in L_a^2$, we see $(1 - |z|^2) y_j'$ is bounded on D by Theorem 4.14 of [14]. Note $z^{-1} x_j \in H^2$ for each j. Also, since $f_j, k_j \in H^p$ for all $p \geq 1$, we see $f_j \overline{k_j}$ satisfies (4) for each j. Thus $F + \overline{G}$ also satisfies (4) and hence $G \in H^2$. On the other hand, by the complexification lemma (see [18, Lemma 2]), we have

$$\sum_{j=1}^{M} f_j(z) \overline{k_j(a)} - \frac{1 - z\bar{a}}{z} \sum_{j=1}^{N} x_j(z) \overline{y_j'(a)} = F(z) + \overline{G(a)}$$

for every $z, a \in D$. Letting

$$A_z(a) = \frac{\bar{z}}{1 - \bar{z}a} \left[\sum_{j=1}^{M} \overline{f_j(z)} k_j(a) - \overline{F(z)} - G(a) \right]$$

for $a, z \in D$, we have

$$\begin{pmatrix} \overline{x_1(z_1)} & \cdots & \overline{x_N(z_1)} \\ \vdots & & \vdots \\ \overline{x_1(z_N)} & \cdots & \overline{x_N(z_N)} \end{pmatrix} \begin{pmatrix} y_1'(a) \\ \vdots \\ y_N'(a) \end{pmatrix} = \begin{pmatrix} A_{z_1}(a) \\ \vdots \\ A_{z_N}(a) \end{pmatrix}$$

for all points $a \in D$ and $z_1, \ldots, z_N \in D$. Now, since $x_1, \ldots, x_N$ are linearly independent, as in the proof of Proposition 3.3, each y_j' is a linear combination of $A_{z_1}(a), \ldots, A_{z_N}(a)$ for some points $z_1, \ldots, z_N \in D$. Note $A_z \in H^2$ for each $z \in D$ because $G, k_j \in H^2$ for each j as observed above. Thus $y_j' \in H^2$ for each j. Now, (a) and (b), together with Lemma 3.4, imply that (10) holds. Since $\langle z\psi, y_j \rangle = \langle \psi, y_j' \rangle_2$ for each j and $\psi \in H^2$ as observed above, we have

$$S\psi = \sum_{j=1}^{N} \langle \psi, y_j' \rangle_2 \frac{x_j}{z} = \frac{1}{z} \sum_{j=1}^{N} \langle z\psi, y_j \rangle x_j$$

for every $\psi \in H^2$. Since $Th = zS(z^{-1}h)$ for $h \in \mathscr{D}$ by (3), we obtain

$$Th = \sum_{j=1}^{N} \langle h, y_j \rangle x_j = \sum_{j=1}^{N} [x_j \otimes y_j] h$$

for every $h \in \mathscr{D}$; thus (5) holds on $\mathscr{D}$.

Now suppose the set $X = \{x_j\}_{j=1}^{N}$ is linearly dependent and choose a maximal subset Y of linearly independent elements in X so that each function in $X \setminus Y$ is a linear combination of functions in Y. Without loss of generality, we may assume $Y = \{x_1, \ldots, x_K\}$ for some $K < N$. For each $x_i \in X \setminus Y$, write $x_i = \sum_{j=1}^{K} a_{ij} x_j$ for some constants a_{ij}. Putting $Y_j = y_j + \sum_{i=K+1}^{N} \overline{a_{ij}} y_i$ for $j = 1, \ldots, K$, we note

$$\sum_{j=1}^{N} x_j \otimes y_j = \sum_{j=1}^{K} x_j \otimes Y_j.$$

Thus, by the result what we have proved in the previous case, we see (5) holds if and only if (a) holds and $\sum_{j=1}^{M} f_j \overline{k_j} - (1 - |z|^2) \sum_{j=1}^{K} \frac{x_j}{z} \overline{Y_j'}$ is harmonic. But, since

$$\sum_{j=1}^{K} \frac{x_j}{z} \overline{Y_j'} = \sum_{j=1}^{K} \frac{x_j}{z} \left[\overline{y_j'} + \sum_{i=K+1}^{N} a_{ij} \overline{y_i'} \right] = \sum_{j=1}^{N} \frac{x_j}{z} \overline{y_j'},$$

we have the desired result. The proof is complete. $\square$

Now we obtain several immediate consequences of Theorem 3.5. First, we have a characterization for finite rank sums of semi-commutators as in the next corollary. Given Toeplitz operators T_u and T_v, we let $[T_u, T_v) = T_u T_v - T_{uv}$ denote the semi-commutator.

Corollary 3.6. *Let $u_j, v_j \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$ and f_j, k_j be the analytic parts of the boundary functions of $u_j, \overline{v_j}$ respectively. Let $x_j, y_j \in \mathscr{D}$ for $j = 1, \ldots, N$. Then*

$$\sum_{j=1}^{M} [T_{u_j}, T_{v_j}) = \sum_{j=1}^{N} x_j \otimes y_j$$

holds on $\mathscr{D}$ if and only if

$$\sum_{j=1}^{M} f_j \overline{k_j} - (1 - |z|^2) \sum_{j=1}^{N} \frac{x_j}{z} \overline{y_j'} \quad \text{is harmonic on } D.$$

To obtain more concrete descriptions, we have the following characterization of harmonicity of functions which are finite sums of products of holomorphic and antiholomorphic functions. The following is taken from Theorem 3.3 of [9].

Lemma 3.7. *Let $f_1, \ldots, f_N$ and $g_1, \ldots, g_N$ be holomorphic functions on D. Then $\sum_{j=1}^{N} f_j \overline{g_j}$ is harmonic on D if and only if*

$$\sum_{j=1}^{N} [f_j - f_j(0)] [\overline{g_j - g_j(0)}] = 0 \quad \text{on } D.$$

Taking $x_j = y_j = 0$ for all j in Theorem 3.5 and using Lemma 3.7, we obtain the following consequence which extends Theorem 1.1 of [3] where harmonic symbols u_j, v_j and certain special τ have been considered.

Corollary 3.8. *Let $u_j, v_j, \tau \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$. Let f_j, k_j be the analytic parts of the boundary functions of $u_j, \overline{v_j}$ respectively. Then the following conditions are equivalent.*

(a) $\sum_{j=1}^{M} T_{u_j} T_{v_j} = T_\tau$.
(b) $\sum_{j=1}^{M} u_j v_j = \tau$ *on* ∂D *and* $\sum_{j=1}^{M} [f_j - f_j(0)] [\overline{k_j - k_j(0)}] = 0$ *on* D.

Also, as another application, taking $x_j = y_j = 0$ in Corollary 3.6 and using Lemma 3.7, we characterize zero sums of semi-commutators which extends Corollary 3.7 of [8] where a complete different argument was used for the case $M = 1$.

Corollary 3.9. *Let $u_j, v_j, \tau \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$. Let f_j, k_j be the analytic parts of the boundary functions of $u_j, \overline{v_j}$ respectively. Then $\sum_{j=1}^{M}[T_{u_j}, T_{v_j}) = 0$ if and only if*

$$\sum_{j=1}^{M}[f_j - f_j(0)][\overline{k_j - k_j(0)}] = 0 \quad \text{on } D.$$

Let us consider some examples concerning Theorem 3.5. Note $T_z T_{\bar{z}} = T_1 - z \otimes z$ by a straightforward computation; see [8, Lemma 2.1] for example. More generally, for any positive integer $N \geq 2$, since

$$|z|^{2N} + (1 - |z|^2)\sum_{j=1}^{N-1}|z|^{2j} = 1,$$

we have

$$T_{z^N} T_{\overline{z^N}} = T_1 - \sum_{j=2}^{N} z^j \otimes \frac{z^j}{j}$$

by Theorem 3.5. For $u \in \mathscr{L}^{1,\infty}$, note $T_u = 0$ if and only if $u = 0$ on ∂D; see Proposition 3.1 of [8]. It follows that

$$\sum_{j=1}^{M} T_{\varphi_j z^N} T_{\overline{\psi_j z^N}} = T_M - M\sum_{j=2}^{N} z^j \otimes \frac{z^j}{j}$$

for any integer $M \geq 1$ and $\varphi_j, \psi_j \in \mathscr{L}^{1,\infty}$ with $\varphi_j = \psi_j = 1$ on ∂D.

In conjunction with Theorem 3.5, we consider the finite rank product problem of when a product of several Toeplitz operators has finite rank on $\mathscr{D}$. In the case of rank zero, the following theorem was proved in Theorem 3.10 of [8].

Theorem 3.10. *Let $u_j \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, N$. Then the following statements are all equivalent.*

(a) *$T_{u_1} \cdots T_{u_N}$ has finite rank on $\mathscr{D}$.*
(b) *$T_{u_j} = 0$ for some j.*
(c) *$u_j = 0$ on ∂D for some j.*

Proof. By Lemma 3.3 of [6] (or Proposition 5.1), we see that (a) holds if and only if $t_{u_1} \cdots t_{u_N}$ has finite rank on H^2, which is equivalent to $u_j = 0$ on ∂D for some j by Corollary 4.1 of [19]. Now, the result follows from the fact that for $u \in \mathscr{L}^{1,\infty}$, $T_u = 0$ if and only if $u = 0$ on ∂D; see Proposition 3.1 of [8]. This completes the proof. □

4. Sums of Hankel products

In this section, we consider operators which are finite sums of products of two Hankel operators and characterize such operators to have finite rank.

We first prove the following proposition showing that a product of two Hankel operators can be written as a semi-commutator of two Toeplitz operators. Such a result is well known in the Hardy space setting; see [20, Lemma B4.4.3] for example. In what follows, we put $\hat{u}(z) = Ju(z) = u(\bar{z})$ for $z \in D$ for simplicity.

Proposition 4.1. *Let $u, v \in \mathscr{L}^{1,\infty}$. Then*

$$H_u H_v = T_{\hat{u}v} - T_{\widehat{zu}} T_{zv}$$

holds on $\mathscr{D}$. In other words, $H_u H_v = [T_{-\widehat{zu}}, T_{zv})$ holds.

Proof. By Proposition 5.1 of [8], we first note $H_{\bar{z}\hat{u}} H_{\bar{z}v} = T_{uv} - T_u T_v$ holds. Replacing u, v with $\bar{z}\hat{u}, zv$ respectively, we obtain

$$H_{|z|^2 u} H_{|z|^2 v} = T_{|z|^2 \hat{u} v} - T_{\widehat{zu}} T_{zv}. \tag{7}$$

On the other hand, for $\varphi, \psi \in \mathscr{L}^{1,\infty}$ such that $\varphi = \psi$ on ∂D, we note $T_\varphi = T_\psi$ and $H_\varphi = H_\psi$; see Propositions 3.1 and 4.1 of [8]. Hence $H_{|z|^2 u} H_{|z|^2 v} = H_u H_v$ and $T_{|z|^2 \hat{u} v} = T_{\hat{u}v}$ holds. Now, (7) yields the desired results. The proof is complete. □

Given a function $\psi = f + g \in L^\infty(\partial D)$ where f is the analytic part and g is the co-analytic part of ψ, we note that the Poisson extension of the analytic part of $\widehat{z\psi}$ is $z^{-1}\widehat{g}$. Also, $z^{-1}\overline{g}$ is the Poisson extension of the analytic part of $\overline{z\psi}$.

The following is the main result of this section.

Theorem 4.2. *Let $u_j, v_j, \tau \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$ and g_j, h_j be the co-analytic parts of the boundary functions of u_j, v_j respectively. Let $x_j, y_j \in \mathscr{D}$ for $j = 1, \ldots, N$. Then*

$$\sum_{j=1}^{M} H_{u_j} H_{v_j} = T_\tau + \sum_{j=1}^{N} x_j \otimes y_j \tag{8}$$

holds on $\mathscr{D}$ if and only if the following two conditions hold.

(a) $\tau = 0$ *on* ∂D.
(b) $\frac{1}{|z|^2}\sum_{j=1}^{M}\widehat{g_j}h_j + (1-|z|^2)\sum_{j=1}^{N}\frac{x_j}{z}\overline{y_j'}$ *is harmonic on* D.

Proof. By Proposition 4.1, we see (8) holds if and only if

$$\sum_{j=1}^{M} T_{\widehat{\overline{z}u_j}}T_{zv_j} = T_{-\tau+\sum_{j=1}^{M}\widehat{\overline{z}u_j}zv_j} - \sum_{j=1}^{N} x_j \otimes y_j$$

holds. Note $z^{-1}\widehat{g_j}$ and $z^{-1}\overline{h_j}$ are Poisson extensions of the analytic parts of $\widehat{\overline{z}u_j}$ and $\overline{zv_j}$ respectively. Now, the result follows from Theorem 3.5. The proof is complete. □

For $u \in \mathscr{L}^{1,\infty}$, recall that $T_u = 0$ if and only if $u = 0$ on ∂D. So Theorem 4.2 shows that $\sum_{j=1}^{M} H_{u_j}H_{v_j}$ is a finite rank perturbation of a Toeplitz operator only when it is a finite rank operator.

As a consequence, taking $x_j = y_j = 0$ in Theorem 4.2, we obtain a characterization on when a sum of products of two Hankel operators equals a Toeplitz operator. The corresponding result is stated as Theorem 4.2 in [8] in case $\tau = 0$ below.

Corollary 4.3. *Let $u_j, v_j, \tau \in \mathscr{L}^{1,\infty}$ for $j = 1, \ldots, M$ and g_j, h_j be co-analytic parts of boundary functions of u_j, v_j respectively. Then the following two statements are equivalent.*

(a) $\sum_{j=1}^{M} H_{u_j}H_{v_j} = T_\tau$.
(b) $\tau = 0$ *on* ∂D *and* $\sum_{j=1}^{M}[\widehat{g_j} - \widehat{g_j}'(0)z][h_j - \widehat{h_j}'(0)\bar{z}] = 0$ *on* D.

Proof. By Theorem 4.2, we see (a) holds if and only if $\tau = 0$ on ∂D and

$$\frac{1}{|z|^2}\sum_{j=1}^{M}\widehat{g_j}h_j = \sum_{j=1}^{M}\frac{\widehat{g_j}}{z}\frac{h_j}{\bar{z}}$$

is harmonic on D. Since $\frac{\widehat{g_j}}{z}(0) = \widehat{g_j}'(0)$ and $\frac{h_j}{\bar{z}}(0) = \widehat{h_j}'(0)$ for each j, the result follows from Lemma 3.7. The proof is complete. □

Note that if the co-analytic part of the boundary function of $u \in \mathscr{L}^{1,\infty}$ has the form of $a\bar{z}$ for some constant a, we see $H_u = 0$ by Proposition 4.1 of [8]. Thus, as a consequence of Corollary 4.3, we have the following characterization which extends Corollary 4.4 of [8] where a completely different argument has been used.

Corollary 4.4. *Let $u, v, \varphi, \psi, \tau \in \mathscr{L}^{1,\infty}$. Then $H_uH_v - H_\varphi H_\psi = T_\tau$ on $\mathscr{D}$ if and only if $\tau = 0$ on ∂D and one of the following two conditions holds.*

(a) H_u *or* H_v *is* 0, *and* H_φ *or* H_ψ *is* 0.
(b) $H_\varphi = \beta H_u$ *and* $H_v = \beta H_\psi$ *for some constant* β.

Proof. Let g, h, f, k be co-analytic parts of the boundary functions of u, v, φ, ψ respectively. By Corollary 4.3, $H_uH_v - H_\varphi H_\psi = T_\tau$ holds if and only if $\tau = 0$ on ∂D and

$$[\widehat{g} - \widehat{g}'(0)z][h - \widehat{h}'(0)\bar{z}] = [\widehat{f} - \widehat{f}'(0)z][k - \widehat{k}'(0)\bar{z}] \tag{9}$$

holds on D. If $\widehat{g} = \widehat{g}'(0)z$ on D, then $H_u = 0$, and $\widehat{f} = \widehat{f}'(0)z$ or $k = \widehat{k}'(0)\bar{z}$, which implies $H_\varphi = 0$ or $H_\psi = 0$ by the remark just before this corollary, so (a) holds. Similarly, if $k = \widehat{k}'(0)\bar{z}$ on D, we see $H_\psi = 0$, and $H_u = 0$ or $H_v = 0$. So (a) holds. Now we assume neither $\widehat{g} - \widehat{g}'(0)z$ nor $k - \widehat{k}'(0)\bar{z}$ is identically 0 on D. By (9), we have

$$\frac{\widehat{f} - \widehat{f}'(0)z}{\widehat{g} - \widehat{g}'(0)z} = \frac{h - \widehat{h}'(0)\bar{z}}{k - \widehat{k}'(0)\bar{z}}$$

at all points of D except the countable set consisting of zeros of $[\widehat{g} - \widehat{g}'(0)z][k - \widehat{k}'(0)\bar{z}]$. Thus the above must be equal to a constant β on D. So $\widehat{f} - \widehat{f}'(0)z = \beta[\widehat{g} - \widehat{g}'(0)z]$ and $h - \widehat{h}'(0)\bar{z} = \beta[k - \widehat{k}'(0)\bar{z}]$ on D. Hence $H_{\varphi-\beta u} = 0$ and $H_{v-\beta\psi} = 0$, so (b) holds. The converse implication is clear because $T_\tau = 0$ if and only if $\tau = 0$ on ∂D. The proof is complete. □

Corollary 4.4 implies that $H_uH_v = 0$ if and only if $H_u = 0$ or $H_v = 0$, which is already noticed in Corollary 4.3 of [8]. In view of Theorem 3.10, one might ask whether this property can be extended to products of arbitrary many Hankel operators. The answer is *no*. Using (2), one can easily check that $H_{\bar{z}^2}H_{\bar{z}^3}H_{\bar{z}^2} = 0$ but none of them equals zero.

Let us consider some examples concerning Theorem 4.2. For any integer $N \geq 2$ and $k \geq 0$, we note

$$\frac{1}{|z|^2}z^{k+N+1}\overline{z^{N+1}} + (1-|z|^2)\sum_{j=1}^{N-1} z^{k+j}\overline{z^j} = z^k.$$

Thus, by Theorem 4.2, we have

$$H_{\overline{z^{k+N+1}}}H_{\overline{z^{N+1}}} = -\sum_{j=2}^{N} z^{k+j} \otimes \frac{z^j}{j}$$

and also

$$\sum_{k=0}^{M} H_{\overline{z^{k+N+1}}}H_{\overline{z^{N+1}}} = -\sum_{k=0}^{M}\sum_{j=2}^{N} z^{j+k} \otimes \frac{z^j}{j} = -\sum_{j=2}^{N} \frac{z^j(1-z^{M+1})}{1-z} \otimes \frac{z^j}{j}$$

for every $M \geq 0$ and $N \geq 2$.

5. Products of Toeplitz and Hankel operators

In this section, we characterize operators which are sums of any two of the form T_uH_v or H_uT_v. We first need some notations. The notation $\mathscr{P}$ denotes the set of all nonzero analytic polynomials. Also, given a function $u \in \mathscr{L}^{1,\infty}$, we say $u \in \mathscr{A}$ if there exists $p \in \mathscr{P}$ such that pu^* is analytic on ∂D where u^* is the boundary function of u. Also, for a bounded operator S on K where $K = H^2$ or $\mathscr{D}$, we write $S \in \mathscr{F}(K)$ if S has finite rank on K.

The following proposition will be useful when we apply the Hardy space results to derive some results on the Dirichlet space. The idea comes from Lemma 3.3 of [6] where sums of products of Toeplitz operators have been considered.

Proposition 5.1. *Let T be a bounded operator on $\mathscr{D}$ and S be a bounded operator on H^2. Suppose $Tf = zS(z^{-1}f)$ for every $f \in \mathscr{D}$. Then $T \in \mathscr{F}(\mathscr{D})$ if and only if $S \in \mathscr{F}(H^2)$.*

Proof. Let A, B be ranges of T, S respectively. Note that $z^{-1}\mathscr{D}$ is in H^2. Since $Tf = zS(z^{-1}f)$ for every $f \in \mathscr{D}$, we have $A \subset zB$, which implies that the rank of T is less than or equal to the rank of S. Also, since $S(z^{-1}\mathscr{D}) = z^{-1}A$ and $z^{-1}\mathscr{D}$ is dense in H^2, we see that the rank of S equals the rank of T by the boundedness of S. This completes the proof. □

Given $\psi \in L^\infty(\partial D)$, the (*Hardy space*) Hankel operator h_ψ with symbol ψ is a bounded linear operator on H^2 defined by $h_\psi f = CJ(\psi f)$ for functions $f \in H^2$. See Chapter 4 of [21] for details. As before, the symbol of h_u with $u \in \mathscr{L}^{1,\infty}$ should be understood to be the boundary function of u.

The following lemma gives a useful connection between Hankel operators on the Dirichlet space and Hardy space.

Lemma 5.2. *For $v \in \mathscr{L}^{1,\infty}$, we have $H_vf = zh_{vz^2}(z^{-1}f)$ for every $f \in \mathscr{D}$.*

Proof. Let V be the Poisson integral of the boundary function of v and write

$$V(z) = \sum_{j<0} a_j\overline{z^{-j}} + \sum_{j\geq 0} a_jz^j, \quad z \in D$$

for the series expansion of V. Using (1), one can see that

$$Q(z^k\bar{z}^\ell) = \begin{cases} z^{k-\ell}, & \text{if } k \geq \ell+1, \\ 0, & \text{if } k < \ell+1 \end{cases} \tag{10}$$

for all nonnegative integers k and ℓ. Then, for any integer $\ell \geq 1$, since

$$J[Vz^\ell](re^{i\theta}) = \sum_{j<0} a_jr^{-j+\ell}e^{i(-j-\ell)\theta} + \sum_{j\geq 0} a_jr^{j+\ell}e^{i(-j-\ell)\theta}$$

for $re^{i\theta} \in D$, it follows from (10) that

$$\begin{aligned} H_V[z^\ell] &= \sum_{j<0} a_jQ(z^{-j}\bar{z}^\ell) + \sum_{j\geq 0} a_jQ(\bar{z}^{j+\ell}) \\ &= \sum_{j\geq 1} a_{-j-\ell}z^j. \end{aligned}$$

On the other hand, since $v(e^{i\theta}) = \sum_{j=-\infty}^{\infty} a_je^{ij\theta}$ on ∂D, one can also check

$$h_{vz^2}[z^{\ell-1}](z) = C\left[\sum_{j=-\infty}^{\infty} a_je^{-i(j+\ell+1)\theta}\right] = z^{-1}\sum_{j\geq 1} a_{-j-\ell}z^j$$

for every $z \in D$ and integers $\ell \geq 1$. Thus $H_Vf = zh_{vz^2}(z^{-1}f)$ for every $f \in \mathscr{D}$. Now the result follows from the fact that $H_v = H_V$ on $\mathscr{D}$; see Theorem 2 of [5]. The proof is complete. □

A well known Kronecker theorem [21] says that for $\psi \in L^\infty(\partial D)$, $h_\psi \in \mathscr{F}(H^2)$ if and only if $p\psi$ is analytic on ∂D for some $p \in \mathscr{P}$. Using this result, we characterize finite rank Hankel operators on the Dirichlet space as shown in the next proposition.

Proposition 5.3. *Let $u \in \mathscr{L}^{1,\infty}$. Then $H_u \in \mathscr{F}(\mathscr{D})$ if and only if $u \in \mathscr{A}$.*

Proof. By Proposition 5.1 and Lemma 5.2, $H_u \in \mathscr{F}(\mathscr{D})$ if and only if $h_{z^2u} \in \mathscr{F}(H^2)$, which is in turn equivalent to $z^2u \in \mathscr{A}$ by Kronecker's theorem. Now the result follows from the fact that $z^2u \in \mathscr{A}$ if and only if $u \in \mathscr{A}$. The proof is complete. □

Now, we give a characterization of when a sum of two operators of the form H_uT_v is a finite rank perturbation of a Hankel operator. But, we were not able to give a characterization for general finite sums of the products.

Theorem 5.4. *Let $u, v, \varphi, \psi, \tau \in \mathscr{L}^{1,\infty}$. Then $H_uT_v + H_\varphi T_\psi - H_\tau \in \mathscr{F}(\mathscr{D})$ if and only if one of the following statements holds.*

(a) $u, \varphi, \tau \in \mathscr{A}$.
(b) $u, \psi, \varphi\psi - \tau \in \mathscr{A}$.
(c) $v, \varphi, uv - \tau \in \mathscr{A}$.
(d) $v, \psi, uv + \varphi\psi - \tau \in \mathscr{A}$.
(e) *There exist $p_1, p_2, q_1, q_2, r \in \mathscr{P}$ with $p_1q_1 + p_2q_2 = 0$ such that $p_1u + p_2\varphi$, $q_1v + q_2\psi$ and $r[p_2\varphi(q_1v + q_2\psi) + p_1q_1\tau]$ are analytic.*

Proof. Put

$$T = H_uT_v + H_\varphi T_\psi - H_\tau, \qquad S = h_{z^2u}t_v + h_{z^2\varphi}t_\psi - h_{z^2\tau}.$$

By (3) and Lemma 5.2, we have $Tf = zS(z^{-1}f)$ for every $f \in \mathscr{D}$. Thus, by Proposition 5.1, we see $T \in \mathscr{F}(\mathscr{D})$ if and only if $S \in \mathscr{F}(H^2)$. On the other hand, using Theorems 4.2 and 3.1 of [11], we see that $S \in \mathscr{F}(H^2)$ if and only if one of the following conditions holds.

(i) $h_{z^2u}, h_{z^2\varphi}, h_{z^2\tau} \in \mathscr{F}(H^2)$.
(ii) $h_{z^2u}, h_\psi, h_{z^2(\varphi\psi-\tau)} \in \mathscr{F}(H^2)$.
(iii) $h_v, h_{z^2\varphi}, h_{z^2(uv-\tau)} \in \mathscr{F}(H^2)$.
(iv) $h_v, h_\psi, h_{z^2(uv+\varphi\psi-\tau)} \in \mathscr{F}(H^2)$.
(v) There exist $p_1, p_2, q_1, q_2, r \in \mathscr{P}$ with $p_1q_1 + p_2q_2 = 0$ such that $p_1u + p_2\varphi$, $q_1v + q_2\psi$ and $r[p_2\varphi(q_1v + q_2\psi) + p_1q_1\tau]$ are analytic.

For $g \in \mathscr{L}^{1,\infty}$, note that $H_g \in \mathscr{F}(\mathscr{D})$ if and only if $h_{gz^2} \in \mathscr{F}(H^2)$ by Proposition 5.1 and Lemma 5.2. Now, by Kronecker's theorem and Proposition 5.3, we have the desired result. The proof is complete. □

Also, we characterize sums of two operators of the form T_uH_v in the corollary below. In the proof, we will use the following connection between Toeplitz and Hankel operators:

$$T_uH_v + H_{\bar{z}\hat{u}}T_{zv} = H_{\hat{u}v}, \quad u, v \in \mathscr{L}^{1,\infty}; \tag{11}$$

see Proposition 5.2 of [8].

Corollary 5.5. *Let $u, v, \varphi, \psi \in \mathscr{L}^{1,\infty}$. Then $T_uH_v + T_\varphi H_\psi \in \mathscr{F}(\mathscr{D})$ if and only if one of the following statements holds.*

(a) $\hat{u}, \hat{\varphi}, \hat{u}v + \hat{\varphi}\psi \in \mathscr{A}$.
(b) $\hat{u}, \psi, \hat{u}v \in \mathscr{A}$.
(c) $v, \hat{\varphi}, \hat{\varphi}\psi \in \mathscr{A}$.
(d) $v, \psi \in \mathscr{A}$.
(e) *There exist $p_1, p_2, q_1, q_2, r \in \mathscr{P}$ with $p_1q_1 + p_2q_2 = 0$ such that $p_1\bar{z}\hat{u} + p_2\bar{z}\hat{\varphi}$, $q_1zv + q_2z\psi$ and $r[p_2\hat{\varphi}(q_1v + q_2\psi) + p_1q_1(\hat{u}v + \hat{\varphi}\psi)]$ are analytic.*

Proof. By (11), we have

$$-[T_uH_v + T_\varphi H_\psi] = H_{\bar{z}\hat{u}}T_{zv} + H_{\bar{z}\hat{\varphi}}T_{z\psi} - H_{\hat{u}v+\hat{\varphi}\psi}.$$

Note $z\bar{z} = 1$ on ∂D. Also, given $\varphi \in \mathscr{L}^{1,\infty}$, we note $\varphi \in \mathscr{A}$ if and only if $z\varphi \in \mathscr{A}$, which is equivalent to $\bar{z}\varphi \in \mathscr{A}$. Now, Theorem 5.4 gives the desired result. The proof is complete. □

By (11), we have

$$H_uT_v - T_\varphi H_\psi = H_uT_v + H_{\bar{z}\hat{\varphi}}T_{z\psi} - H_{\hat{\varphi}\psi}$$

for $u, v, \varphi, \psi \in \mathscr{L}^{1,\infty}$. Thus the following characterization is a consequence of Theorem 5.4.

Corollary 5.6. *Let $u, v, \varphi, \psi \in \mathscr{L}^{1,\infty}$. Then $H_uT_v - T_\varphi H_\psi \in \mathscr{F}(\mathscr{D})$ if and only if one of the following statements holds.*

(a) $u, \hat{\varphi}, \hat{\varphi}\psi \in \mathscr{A}$.
(b) $u, \psi \in \mathscr{A}$.

(c) $v, \hat{\varphi}, uv - \hat{\varphi}\psi \in \mathscr{A}$.
(d) $v, \psi, uv \in \mathscr{A}$.
(e) *There exist* $p_1, p_2, q_1, q_2, r \in \mathscr{P}$ *with* $p_1q_1 + p_2q_2 = 0$ *such that* $p_1u + p_2\bar{z}\hat{\varphi}$, $q_1v + q_2z\psi$ *and* $r\hat{\varphi}v$ *are analytic.*

In the cases of $\varphi = v$ and $\psi = u$ in Corollary 5.6, we characterize finite rank commutators of Toeplitz and Hankel operators as shown in the following.

Corollary 5.7. *Let* $u, v \in \mathscr{L}^{1,\infty}$. *Then* $H_uT_v - T_vH_u \in \mathscr{F}(\mathscr{D})$ *if and only if one of the following statements holds.*

(a) $u \in \mathscr{A}$.
(b) $v, \hat{v}, uv - \hat{v}u \in \mathscr{A}$.
(c) *There exist* $p_1, p_2, q_1, q_2, r \in \mathscr{P}$ *with* $p_1q_1 + p_2q_2 = 0$ *such that* $p_1u + p_2\bar{z}\hat{v}$, $q_1v + q_2zu$ *and* $r\hat{v}v$ *are analytic.*

Finally, taking $\varphi = \psi = 0$ or $u = v = 0$ in Corollary 5.6, we characterize finite rank products of Toeplitz and Hankel operators.

Corollary 5.8. *Let* $u, v \in \mathscr{L}^{1,\infty}$. *Then the following statements holds.*

(a) $H_uT_v \in \mathscr{F}(\mathscr{D})$ *if and only if either* $u \in \mathscr{A}$ *or* $v, uv \in \mathscr{A}$.
(b) $T_\varphi H_\psi \in \mathscr{F}(\mathscr{D})$ *if and only if either* $\psi \in \mathscr{A}$ *or* $\hat{\varphi}, \hat{\varphi}\psi \in \mathscr{A}$.

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