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# A Tikhonov-type regularization for equilibrium problems in Hilbert spaces

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ABSTRACT

In this paper we consider a general equilibrium problem in a Hilbert space defined on a closed and convex set. We show a Tikhonov-type regularization method that can be extended for the equilibrium problem. Under mild assumptions we establish the equivalence between the existence of solution of the original problem and the boundedness of the sequence generated by regularized problems.



## 1. Introduction

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$, respectively. Take a nonempty closed and convex set $C \subset \mathcal{H}$ and $f : C \times C \to \mathbb{R}$ is an equilibrium bifunction, i.e., $f(x, x) = 0$ for every $x \in C$.

We consider the following Equilibrium Problem (in short, $EP(f, C)$)

$$EP(f,C)\begin{cases}\text{Find } x^* \in C & \text{such that}\\ f(x^*, y) \geq 0 & \forall y \in C.\end{cases}$$

The set of solutions of $EP(f, C)$ will be denoted by $S(f, C)$.

The Equilibrium problem was first considered and introduced by Ky Fan in [3], but equilibrium problems appeared with this name in the paper of Blum and Oettli in 1994, see [2].

The equilibrium problem is very general in the sense that it includes, among its particular cases, convex minimization problems, variational inequality problems, Nash equilibrium problems, and other applications, see for example [2,6] and their references.

Existence results for solutions to equilibrium problems have been extensively studied, as it can be seen in [2,6,7].

The Tikhonov-type regularization is a well-known method that is widely used in convex optimization and monotone variational inequality studies to handle ill-posed problems. The main idea of the Tikhonov-type regularization method for the monotone variational inequality is that it adds a strongly monotone operator depending on a regularization parameter to obtain a strongly monotone variational inequality. The resulting regularized problem then has a unique solution depending on the regularization parameter. When the cost operator is pseudomonotone, the monotonicity of the regularized problem may fail to hold.

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Regularization methods for equilibrium problems without monotonicity assumptions have been considered in some recent interesting papers, see [8,10,9] and the references therein.

Some other methods have been proposed to solve the equilibrium problems, see for example [1] and the references therein.

In [4], a Tikhonov-type scheme has been proposed for solving finite-dimensional equilibrium problems. The following regularized problem is considered:

$$\text{Find } x^* \in C \text{ such that } f_\lambda(x^*, y) := f(x^*, y) + \lambda g(x^*, y) \geq 0 \quad \forall y \in C, \tag{1.1}$$

where $g$ is a strongly monotone equilibrium bifunction on $C$, $\lambda > 0$ and $g$ satisfies the condition

$$\exists\, \delta > 0, \hat{x} \in C : |g(x, y)| \leq \delta \|x - \hat{x}\| \|y - x\|, \quad \forall x, y \in C. \tag{1.2}$$

However, this condition is quite restrictive (for example it implies $g(\hat{x}, y) = 0$ for all $y \in C$).

In this work, we extend the scheme (1.1) for solving equilibrium problems in Hilbert spaces. Also, we replace condition (1.2) by the following

$$\limsup_{\|y\| \to \infty} \frac{|g(x, y)|}{\|y - x\|} < +\infty \quad \forall x \in C, \tag{1.3}$$

by presenting a practical way to construct strongly monotone bifunctions $g$ satisfying (1.3).

The paper is organized as follows. In Section 2, we recall concepts and basic results that will be important for the development of the work. In Section 3, we present a Tikhonov-type regularization method for equilibrium problems and we show our main results.

## 2. Preliminaries

In this section, we recall definitions and known results that will be important in our subsequent analysis.

**Definition 2.1.** A bifunction $\psi : C \times C \to \mathbb{R}$ is said to be:

(i) strongly monotone on $C$ with modulus $\beta > 0$ if

$$\psi(x, y) + \psi(y, x) \leq -\beta \|x - y\|^2 \quad \forall x, y \in C.$$

(ii) monotone on $C$ if

$$\psi(x, y) + \psi(y, x) \leq 0 \quad \forall x, y \in C.$$

(iii) pseudomonotone on $C$ if

$$\forall x, y \in C : \psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq 0.$$

Clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

**Definition 2.2.** A function $\psi(\cdot, y) : C \to \mathbb{R}$ is said to be weakly upper semicontinuous at the point $x \in C$, when for any sequence $\{x^k\} \subset C$ such that $x^k \rightharpoonup x$, we get that

$$\limsup_{k \to \infty} \psi(x^k, y) \leq \psi(x, y).$$

**Definition 2.3.** A function $\psi(x, \cdot) : C \to \mathbb{R}$ is said to be weakly lower semicontinuous at the point $y \in C$, when for any sequence $\{y^k\} \subset C$ such that $y^k \rightharpoonup y$, we get that

$$\liminf_{k \to \infty} \psi(x, y^k) \geq \psi(x, y).$$

**Remark 2.1.** In [3], it has been established that if $\psi(., y)$ is weakly upper semicontinuous for all $y \in C$, $\psi(x, .)$ is convex for all $x \in C$ and $C$ is weakly compact, then $S(\psi, C)$ is nonempty.

Throughout the paper we assume that $C \subset \mathcal{H}$ is unbounded.

The following technical lemmas will be useful for the existence of solutions of the regularized problems, when $f$ is pseudomonotone.

**Lemma 2.1.** *Let $K$, $C$ be closed and convex subsets of $\mathcal{H}$. Consider a convex function $h : \mathcal{H} \to \mathbb{R}$ and $f(x, .) : C \to \mathbb{R}$ convex for all $x \in C$.*

(i) *If $\bar{x}$ minimizes $h$ on $C \cap K$ and it belongs to the interior of $K$, then $\bar{x}$ minimizes $h$ on $C$.*
(ii) *If $\bar{x}$ solves $EP(f, C \cap K)$ and it belongs to the interior of $K$, then $\bar{x}$ solves $EP(f, C)$.*

**Proof.** See [5, Proposition 3.3]. □

## 3. A Tikhonov-type regularization for the equilibrium problem

Initially, we present our basic assumptions for a bifunction $\psi : C \times C \to \mathbb{R}$:

H1. $\psi(., y) : C \to \mathbb{R}$ is weakly upper semicontinuous for all $y \in C$.
H2. $\psi(x, .) : C \to \mathbb{R}$ is convex and weakly lower semicontinuous for all $x \in C$.
H3. $\psi : C \times C \to \mathbb{R}$ is strongly monotone with modulus $\beta > 0$.
H4. $\limsup_{\|y\|\to\infty} \frac{|\psi(x,y)|}{\|y-x\|} < +\infty \; \forall x \in C$.
H5. For any sequence $\{x^n\} \subset C$ with $\lim_{n\to\infty} \|x^n\| = +\infty$, there exists $u \in C$ and $n_0 \in \mathbb{N}$ such that $\psi(x^n, u) \le 0 \; \forall n \ge n_0$.

Now, we define our regularization procedure for the problem $EP(f, C)$. For this, we consider $\lambda > 0$ and we take a strongly monotone equilibrium bifunction $g : C \times C \to \mathbb{R}$ with modulus $\beta > 0$. Following [4], we define the regularized bifunction $f_\lambda : C \times C \to \mathbb{R}$ by:

$$f_\lambda(x, y) = f(x, y) + \lambda g(x, y). \tag{3.4}$$

Note also that $f_\lambda(x, x) = 0 \; \forall\, x \in C$.

Next, we show that the $EP(f_\lambda, C)$ has a unique solution, when $f$ is monotone. First, we recall an important result.

**Theorem 3.1.** *Assume that $\psi$ is pseudomonotone, satisfying* H1, H2 *and* H5, *then $S(\psi, C)$ is nonempty.*

**Proof.** See [6, Theorem 4.3]. □

**Theorem 3.2.** *Assume that $f$ is monotone and satisfies* H1–H2 *and $g$ satisfies* H1–H3 . *Then for any $\lambda > 0$, $EP(f_\lambda, C)$ has a unique solution.*

**Proof.** It follows easily from (3.4) that $f_\lambda$ satisfies H1–H2. Moreover, as $f$ is monotone and $g$ satisfies H3, we have that $f_\lambda$ is strongly monotone, in particular, $f_\lambda$ is pseudomonotone. To apply the Theorem 3.1 we must show that $f_\lambda$ satisfies the assumption H5. For this, consider a sequence $\{x^n\} \subset C$ with $\lim_{n\to\infty} \|x^n\| = +\infty$. We will show that there exists $u \in C$ and $n_0 \in \mathbb{N}$ such that $f_\lambda(x^n, u) \le 0 \;\; \forall n \ge n_0$. Since $C$ is not empty, take $u \in C$.

Note that:

$$f_\lambda(x^n, u) = f(x^n, u) + \lambda g(x^n, u). \tag{3.5}$$

By using the fact that $f$ is monotone and $g$ satisfies H3, it follows from (3.5) that:

$$f_\lambda(x^n, u) \le -[f(u, x^n) + \lambda g(u, x^n)] - \lambda\beta\|x^n - u\|^2. \tag{3.6}$$

For each $x \in C$, define $h_x : C \to \mathbb{R}$ as $h_x(y) = h(x, y) := f(x, y) + \lambda g(x, y)$. Note that $h(x, .)$ is convex and weakly lower semicontinuous. Thus, (3.6) results in:

$$f_\lambda(x^n, u) \le -h(u, x^n) - \lambda\beta\|x^n - u\|^2. \tag{3.7}$$

Take any element $(\bar{x}, \bar{v})$ in the graph of the subdifferential $\partial h_u$ which is nonempty, since $\partial h_u$ is monotone maximal. By the definition of subdifferential, we have:

$$\langle \bar{v}, x^n - \bar{x}\rangle \le h_u(x^n) - h_u(\bar{x}) = h(u, x^n) - h(u, \bar{x}). \tag{3.8}$$

(3.8) is equivalent to

$$\begin{aligned} -h(u, x^n) &\le \langle \bar{v}, \bar{x} - x^n\rangle - h(u, \bar{x}) \\ &\le \|\bar{v}\|\|\bar{x} - x^n\| - h(u, \bar{x}) \\ &\le \|\bar{v}\|\|\bar{x} - u\| + \|\bar{v}\|\|u - x^n\| - h(u, \bar{x}). \end{aligned} \tag{3.9}$$

Replacing (3.7) and (3.9), we obtain:

$$\begin{aligned} f_\lambda(x^n, u) &\le \|\bar{v}\|\|\bar{x} - u\| + \|\bar{v}\|\|u - x^n\| - h(u, \bar{x}) - \lambda\beta\|x^n - u\|^2 \\ &= \|x^n - u\|\left[\|\bar{v}\| - \lambda\beta\|x^n - u\|\right] + \|\bar{v}\|\|\bar{x} - u\| - h(u, \bar{x}). \end{aligned} \tag{3.10}$$

Since $\|x^n\| \to +\infty$, so that $\lim_{n\to\infty} \|x^n - u\| = +\infty$; so it follows from (3.10) that $\lim_{n\to\infty} f_\lambda(x^n, u) = -\infty$, because $\lambda\beta > 0$. Therefore, for $n$ large enough, it follows that $f_\lambda(x^n, u) \le 0$. With this, we find that $f_\lambda$ satisfies all the assumptions of Theorem 3.1, hence, $S(f_\lambda, C)$ is nonempty.

We now show that the solution is unique. Suppose that $\tilde{x}$ and $\hat{x}$ are solutions of $EP(f_\lambda, C)$. It follows from (3.4) that:

$$0 \le f_\lambda(\tilde{x}, \hat{x}) = f(\tilde{x}, \hat{x}) + \lambda g(\tilde{x}, \hat{x}). \tag{3.11}$$

$$0 \le f_\lambda(\hat{x}, \tilde{x}) = f(\hat{x}, \tilde{x}) + \lambda g(\hat{x}, \tilde{x}). \tag{3.12}$$

Adding (3.11) with (3.12) and using the fact that $f$ is monotone and $g$ satisfies H3, we obtain:

$$0 \leq f(\tilde{x}, \hat{x}) + f(\hat{x}, \tilde{x}) + \lambda[g(\tilde{x}, \hat{x}) + g(\hat{x}, \tilde{x})] \leq -\lambda\beta\|\tilde{x} - \hat{x}\|^2 \leq 0. \tag{3.13}$$

It follows from (3.13) that $\lambda\beta\|\tilde{x} - \hat{x}\| = 0$, hence, $\hat{x} = \tilde{x}$, because $\lambda\beta > 0$. □

We consider a sequence of positive regularization parameter $\{\lambda_k\}$ and construct a sequence of solutions $\left\{x^k\right\} := \{x(\lambda_k)\} \subset C$, of the problem $EP(f_{\lambda_k}, C)$, with $f_{\lambda_k} : C \times C \to \mathbb{R}$ defined by:

$$f_{\lambda_k}(x, y) = f(x, y) + \lambda_k g(x, y). \tag{3.14}$$

We denote by $S(f_{\lambda_k}, C)$ the solution set of the problem EP($f_{\lambda_k}, C$).

Below we will show our main result for the monotone case.

**Theorem 3.3.** *Suppose that $f$ is monotone and satisfies* H1–H2 *and $g$ satisfies* H1–H4 *. If $\left\{x^k\right\}$ is a sequence of solutions of the problems EP($f_{\lambda_k}, C$) and $\lambda_k \to 0$, then the following statements are equivalent:*

(i) *The sequence $\left\{x^k\right\}$ is bounded.*
(ii) *$S(f, C)$ is nonempty.*

**Proof.** From Theorem 3.2 with $\lambda = \lambda_k$, the sequence $\left\{x^k\right\}$ is well-defined.

First, assume that $\left\{x^k\right\}$ is bounded. Then, we have that there exists a subsequence $\left\{x^{k_j}\right\} \subset \left\{x^k\right\}$ that weakly converges to some $\bar{x} \in C$, i.e., $x^{k_j} \rightharpoonup \bar{x}$, with $f_{\lambda_{k_j}}(x^{k_j}, y) \geq 0 \;\; \forall y \in C$. Thus, we have:

$$0 \leq f_{\lambda_{k_j}}(x^{k_j}, y) = f(x^{k_j}, y) + \lambda_{k_j} g(x^{k_j}, y). \tag{3.15}$$

Passing to the limit in (3.15) we obtain:

$$\begin{aligned} 0 &\leq \limsup_{j\to\infty}[f(x^{k_j}, y) + \lambda_{k_j} g(x^{k_j}, y)] \\ &\leq \limsup_{j\to\infty} f(x^{k_j}, y) + \limsup_{j\to\infty} \lambda_{k_j} g(x^{k_j}, y). \end{aligned} \tag{3.16}$$

Since $\lambda_{k_j} \to 0$, by using H1, it follows from (3.16) that

$$f(\bar{x}, y) \geq 0 \quad \forall y \in C.$$

Therefore $S(f, C) \neq \emptyset$.

Take now $x^k \in S(f_{\lambda_k}, C)$ and $\bar{x} \in S(f, C)$. Thus we have:

$$0 \leq f_{\lambda_k}(x^k, \bar{x}) = f(x^k, \bar{x}) + \lambda_k g(x^k, \bar{x}) \quad \text{and} \tag{3.17}$$

$$0 \leq f(\bar{x}, x^k). \tag{3.18}$$

Adding (3.17) with (3.18) and using the monotonicity of $f$, we obtain:

$$0 \leq f(\bar{x}, x^k) + f(x^k, \bar{x}) + \lambda_k g(x^k, \bar{x}) \leq \lambda_k g(x^k, \bar{x}). \tag{3.19}$$

On the other hand, we know that $g$ is strongly monotone with modulus $\beta > 0$. Thus (3.19) implies that:

$$\begin{aligned} g(\bar{x}, x^k) \leq -\beta\|x^k - \bar{x}\|^2 &\Rightarrow \frac{g(\bar{x}, x^k)}{\|x^k - \bar{x}\|} \leq -\beta\|x^k - \bar{x}\| \\ &\Rightarrow -\frac{g(\bar{x}, x^k)}{\|x^k - \bar{x}\|} \geq \beta\|x^k - \bar{x}\| \\ &\Rightarrow \frac{|g(\bar{x}, x^k)|}{\|x^k - \bar{x}\|} \geq \beta\|x^k - \bar{x}\|. \end{aligned} \tag{3.20}$$

Suppose that $\left\{x^k\right\}$ is not bounded. In this case there exists $\left\{x^{k_j}\right\} \subset \left\{x^k\right\}$ such that $\lim_{j\to\infty}\|x^{k_j}\| = +\infty$. It follows from (3.20) that

$$\limsup_{j\to+\infty} \frac{|g(\bar{x}, x^{k_j})|}{\|x^{k_j} - \bar{x}\|} \geq \limsup_{j\to+\infty} \beta\|x^{k_j} - \bar{x}\| = +\infty. \tag{3.21}$$

Note that (3.21) contradicts H4. We conclude, therefore, that the sequence $\left\{x^k\right\}$ is bounded. □

### 3.1. Nonmonotone case

In the case of $f$ monotone, the perturbed problem $EP(f_\lambda, C)$ is strongly monotone. So $EP(f_\lambda, C)$ has a unique solution. When $f$ is pseudomonotone, the regularized problem may be not strongly monotone, or even non pseudomonotone.

From now on, $C_n$ will be the intersection of $C$ with the ball $B(0, n)$ with radius $n$ centered at 0. Note that $C_n$ is weakly closed and also bounded, because it is contained in the $B(0, n)$, hence it is weakly compact, see for example, [11].

**Lemma 3.1.** *Assume that $f$ and $g$ satisfy* H1–H2. *Then for each $\lambda > 0$, there exists $\bar{x} \in C_n$ such that $f_\lambda(\bar{x}, y) \geq 0$, for all $y \in C_n$.*

**Proof.** It is an immediate consequence of Ky Fan's result (see Remark 2.1) applied to $f_\lambda$. □

In the following theorem, we show that $S(f_\lambda, C)$ is nonempty when $f$ is nonmonotone.

**Theorem 3.4.** *Assume that $f$ satisfies* H1, H2, H5 *and $g$ satisfies* H1–H4 . *Then for any $\lambda > 0$, $S(f_\lambda, C)$ is nonempty.*

**Proof.** It is immediate that $f_\lambda$ satisfies H1–H2. Take $\lambda > 0$, $\{x^n\} \subset C$ with $\lim_{n\to\infty} \|x^n\| = +\infty$ and $u \in C$. Thus, we have:

$$\begin{aligned} f_\lambda(x^n, u) &= f(x^n, u) + \lambda g(x^n, u) \\ &\leq f(x^n, u) - \lambda g(u, x^n) - \lambda\beta\|x^n - u\|^2 \\ &= f(x^n, u) + \lambda\|x^n - u\| \left[ \frac{-g(u, x^n)}{\|x^n - u\|} - \beta\|x^n - u\| \right] \\ &\leq f(x^n, u) + \lambda\|x^n - u\| \left[ \frac{|g(u, x^n)|}{\|x^n - u\|} - \beta\|x^n - u\| \right]. \end{aligned} \tag{3.22}$$

Taking the limit in (3.22) with $n \to \infty$ and using H4 and H5, we have that there exists $n_0 \in \mathbb{N}$ such that $f(x^n, u) \leq 0$ and $\limsup_{n\to\infty} \frac{|g(u,x^n)|}{\|x^n-u\|} < +\infty$; so $f_\lambda(x^n, u) \leq 0 \;\; \forall n \geq n_0$. Therefore, for each $\lambda > 0$, we have that $f_\lambda$ satisfies H5.

From Lemma 3.1, we see that there exists $x^n \in C_n$ such that $f_\lambda(x^n, y) \geq 0$ for all $y \in C_n$, hence $x^n$ solves $EP(f_\lambda, C_n)$.

Let us now analyze two cases:

(i) There exists $n \in \mathbb{N}$ such that $\|x^n\| < n$. In this case $x^n \in int(B(0, n))$, and from Lemma 3.1 $x^n$ solves $EP(f_\lambda, C_n) = EP(f_\lambda, C \cap B(0, n))$. From the item (ii) of Lemma 2.1, it follows that $x^n$ solves $EP(f_\lambda, C)$.

(ii) $\|x^n\| \to +\infty$. In this case H5 ensures the existence of $u \in C$ and $n_0 \in \mathbb{N}$ such that

$$f_\lambda(x^n, u) \leq 0, \quad \forall n \geq n_0. \tag{3.23}$$

Take $\bar{n} \geq n_0$ such that $\|u\| < \bar{n}$. Then $u \in C \cap B(0, \bar{n}) = C_{\bar{n}}$ and since $x^{\bar{n}}$ solves $EP(f_\lambda, C_{\bar{n}})$, it follows that

$$f_\lambda(x^{\bar{n}}, u) \geq 0. \tag{3.24}$$

Comparing (3.23) and (3.24), we obtain

$$f_\lambda(x^{\bar{n}}, u) = 0. \tag{3.25}$$

From (3.25), we have

$$f_\lambda(x^{\bar{n}}, u) = 0 \leq f_\lambda(x^{\bar{n}}, y), \quad \forall y \in C_{\bar{n}}. \tag{3.26}$$

Now consider the convex function $f_\lambda^{\bar{n}} : \mathcal{H} \to \mathbb{R}$, defined as $f_\lambda^{\bar{n}}(y) = f_\lambda(x^{\bar{n}}, y)$. Since $u \in C \cap B(0, \bar{n}) = C_{\bar{n}}$, $u$ minimizes $f_\lambda^{\bar{n}}$ on $C_{\bar{n}}$, from (3.26). Since $\|u\| < \bar{n}$, $u \in intB(0, \bar{n})$. It follows from item (i) of Lemma 2.1 that $u$ minimizes $f_\lambda^{\bar{n}}$ on C. It follows from (3.25) that:

$$\begin{aligned} 0 = f_\lambda(x^{\bar{n}}, u) &= f_\lambda^{\bar{n}}(u) \\ &\leq f_\lambda^{\bar{n}}(y) \\ &= f_\lambda(x^{\bar{n}}, y), \quad \forall y \in C. \end{aligned} \tag{3.27}$$

From (3.27), we conclude that $x^{\bar{n}}$ solves $EP(f_\lambda, C)$. □

In the next theorem, we establish our main result for the nonmonotone case.

**Theorem 3.5.** *Suppose that $f$ is pseudomonotone and satisfies* H1, H2, H5, *and $g$ satisfies* H1–H4 . *If $\left\{x^k\right\}$ is a sequence of solutions from* (3.14) *and $\lambda_k \to 0$, then the following statements are equivalent:*

(i) *The sequence $\left\{x^k\right\}$ is bounded.*
(ii) *$S(f, C)$ is nonempty.*

**Proof.** From Theorem 3.4 with $\lambda = \lambda_k$, the sequence $\{x^k\}$ is well-defined. The boundedness of $\{x^k\}$ implies that there must exist at least one subsequence $\{x^{k_j}\} \subset \{x^k\}$ weakly converging to some $\bar{x} \in C$.

Since $f(.,y)$ and $g(.,y)$ satisfy H1 and using $\lambda_k \to 0$, we obtain

$$\begin{aligned} 0 \le \limsup_{j\to\infty} f_{\lambda_{k_j}}(x^{k_j}, y) &= \limsup_{j\to\infty} [f(x^{k_j}, y) + \lambda_{k_j} g(x^{k_j}, y)] \\ &\le \limsup_{j\to\infty} f(x^{k_j}, y) + \limsup_{j\to\infty} [\lambda_{k_j} g(x^{k_j}, y)] \\ &\le f(\bar{x}, y), \quad \forall y \in C. \end{aligned} \tag{3.28}$$

From (3.28), we have that $S(f, C) \neq \emptyset$.

Now, take $\bar{x} \in S(f, C)$. We now show that the sequence $\{x^k\}$ is bounded. Since $x^k \in S(f_{\lambda_k}, C)$, we have:

$$f(\bar{x}, x^k) \ge 0 \quad \text{and} \quad f_{\lambda_k}(x^k, \bar{x}) \ge 0. \tag{3.29}$$

From (3.29) we need to make $f(x^k, \bar{x}) \le 0$, because f is pseudomonotone.

Note that:

$$\begin{aligned} 0 \le f_{\lambda_k}(x^k, \bar{x}) &= f(x^k, \bar{x}) + \lambda_k g(x^k, \bar{x}) \\ &\le \lambda_k g(x^k, \bar{x}). \end{aligned} \tag{3.30}$$

From (3.30) we obtain that $g(x^k, \bar{x}) \ge 0$. By using H3, it results

$$g(x^k, \bar{x}) + g(\bar{x}, x^k) \le -\beta \|x^k - \bar{x}\|^2. \tag{3.31}$$

Since $g(x^k, \bar{x}) \ge 0$, it follows from (3.31) that $g(\bar{x}, x^k) \le -\beta\|x^k - \bar{x}\|^2$.

The conclusion that $\{x^k\}$ is bounded is obtained using the same argument of the proof of the second part of Theorem 3.3. □

**Remark 3.1.** In [4] it has been proposed a regularization method like (3.14) in $\mathbb{R}^n$, where it is assumed that the bifunction $g$ satisfies the following condition:

$$\exists\, \delta > 0, \quad \hat{x} \in C : |g(x, y)| \le \delta \|x - \hat{x}\| \|y - x\|, \quad \forall x, y \in C. \tag{3.32}$$

Note that, (3.32) implies H4. Next, we present an example where the bifunction $g$ satisfies H4, but does not satisfy (3.32).

**Example 3.1.** We consider $g : C \times C \to \mathbb{R}$, $g(x, y) = x(y - x) + \ln(\frac{x}{y})$, where $C := \{x \in \mathbb{R} : x \ge 1\}$. In fact, we have that

$$\limsup_{|y|\to\infty} \frac{|g(x,y)|}{|y-x|} = \limsup_{|y|\to\infty} \left[ \frac{x(y-x)}{|y-x|} + \frac{\ln(x)}{|y-x|} - \frac{\ln(y)}{|y-x|} \right] = x,$$

hence, $g$ satisfies H4.

On the other hand, assume that $g$ verifies (3.32) for some $\delta > 0$ and $\hat{x} \ge 1$. We get that $|g(\hat{x}, y)| = 0,\ \forall\, y \in C$, but

$$\limsup_{|y|\to\infty} |g(\hat{x}, y)| = \limsup_{|y|\to\infty} \left| (y - \hat{x}) \left[ \hat{x} + \ln\left(\frac{\hat{x}}{y}\right) / (y - \hat{x}) \right] \right| = +\infty,$$

which is a contradiction.

**Remark 3.2.** Now, we present a class of strongly monotone bifunctions $g$ satisfying H4. To get our aim, let us consider a strongly monotone operator $T : C \subset \mathcal{H} \to \mathcal{H}$ and a function $\phi : C \to \mathbb{R}$ such that

$$\limsup_{\|y\|\to+\infty} \frac{|\phi(y) - \phi(x)|}{\|y - x\|} < +\infty.$$

Then, define $g(x, y) := \langle T(x), y - x\rangle + \phi(x) - \phi(y)$. Indeed, our conclusion follows from the triangular inequality and Cauchy–Schwarz Inequality.

## 4. Conclusion

In this work, we extended a Tikhonov-type regularization for infinite-dimensional Equilibrium problems. Under mild assumptions we established the equivalence between the existence of solutions of the original problem and the boundedness of the sequence generated by regularized problems.

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