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# Nonlinear stability of generalized Benjamin–Bona–Mahony–Burgers shock profiles in several dimensions

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## ABSTRACT

This paper is concerned with nonlinear stability of planar viscous shock profiles of the generalized Benjamin–Bona–Mahony–Burgers equations in two dimensions. Our analysis is motivated by J. Goodman's work (1989) [4] on the nonlinear stability of scalar viscous shock profiles in two dimensions and is based on some new decay estimates on the planar viscous shock profiles of the generalized Benjamin–Bona–Mahony–Burgers equations.

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## 1. Introduction

This paper is concerned with the precise description of the large time behavior of global smooth solutions of the generalized Benjamin–Bona–Mahony–Burgers equations in two dimensions

$$u_t - \Delta u_t + f(u)_x + g(u)_y = \Delta u, \quad (t, x, y) \in \mathbf{R}^+ \times \mathbf{R}^2 \quad (1.1)$$

with prescribed initial data

$$u(0, x, y) = u_0(x, y) \rightarrow u_{\pm} \quad \text{as } x \rightarrow \pm\infty \text{ for } \forall y \in \mathbf{R}. \quad (1.2)$$

Here  $u_- > u_+$  are two given constants and the nonlinear flux functions  $f(u)$  and  $g(u)$  are assumed to be sufficiently smooth and are genuinely nonlinear in the  $x$ -direction, i.e.

$$f''(u) \geq a > 0 \quad (1.3)$$

holds for some constant  $a > 0$  and all  $u \in \mathcal{M} = [u_+, u_-]$ .

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Before stating our main results, we first use several sentences to explain the Benjamin–Bona–Mahony–Burgers equation. In fact, as pointed out in [1], when attempting to describe the propagation of small-amplitude long waves in a nonlinear dispersive media, it is frequently necessary to take account of dissipative mechanisms to accurately reflect real situations. Oftentimes the mechanisms leading to the degradation of the wave are quite complex and not well understood. In such cases one may be forced to rely upon *ad hoc* models of dissipation, cf. [3]. One of the equations that have gained some currency when the need to append dissipation to nonlinearity and dispersion arises in modeling unidirectional propagation of planar waves is

$$u_t + uu_x - \nu u_{xx} - u_{xxt} = 0, \quad (1.4)$$

where  $\nu > 0$  is a fixed constants and  $u = u(t, x)$  is a real-valued function of the two real variables  $t$  and  $x$ , which, in applications, are typically proportional to distance in the direction of propagation and to elapsed time, respectively. The dependent variable may represent a displacement of the underlying medium or a velocity, for example.

When  $\nu = 0$ , (1.4) is the so-called regularized long-wave (called RLW in short in the sequel) equation proposed by Peregrine [10] and Benjamin et al. [2], and is called the Benjamin–Bona–Mahony (called BBM in the sequel for simplicity) equation. This equation features a balance between the nonlinear dispersive effect but takes no account of dissipation. If  $\nu > 0$ , since the dispersive effect of (1.4) is the same as the BBM equation

$$u_t + uu_x - u_{xxt} = 0,$$

while the dissipative effect is the same as the Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0,$$

(1.4) is called the Benjamin–Bona–Mahony–Burgers (called BBM–Burgers in the sequel for simplicity) equation.

The study on the large time behavior of global solutions to the generalized BBM–Burgers equation has been the subject of numerous investigations (a complete literature in this direction is beyond the scope of the paper; the references cited in [1–3,5,9,13,15–20] and the monograph [12] can help the interested readers into the literature) and, to the best of our knowledge, almost all the results are concentrated on the one-dimensional case: For the Cauchy problem of the generalized BBM–Burgers equations

$$\begin{cases} u_t - u_{xxt} - \nu u_{xx} + f(u)_x = 0, & x \in \mathbf{R}, t > 0, \\ u(0, x) = u_0(x) \rightarrow u_{\pm} \text{ as } x \rightarrow \pm\infty, \end{cases} \quad (1.5)$$

when  $u_- = u_+ = 0$ ,  $f(u) = \frac{1}{2}u^2$ , by using energy estimates, a maximum principle, and a transformation of Cole–Hopf type, it is shown in [1] that the unique global smooth solution  $u(t, x)$  of the Cauchy problem (1.5) satisfies the following decay estimate

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}} \int_{-\infty}^{+\infty} u^2(t, x) dx = \frac{4\nu^2(v_+ - 1)^2}{2\pi\sqrt{\nu}} \int_{-\infty}^{+\infty} \frac{\exp(-2u^2)}{\left(1 + \frac{v_+ - 1}{\sqrt{\pi}} \int_u^{+\infty} \exp(-s^2)\right)^2} du, \quad (1.6)$$

where

$$v_+ = \exp\left(-\frac{1}{2\nu} \int_{-\infty}^{+\infty} u_0(x) dx\right).$$

The estimate (1.6) implies that, if  $\int_{\mathbf{R}} u_0(x) dx \neq 0$ , the  $L^2(\mathbf{R})$ -norm decay estimate (1.6) is optimal, i.e. there exists a positive constant  $C > 0$ , which depends only on the initial data, such that

$$\|u(t, x)\|_{L^2(\mathbf{R})} \geq C(1+t)^{-\frac{1}{4}}. \quad (1.7)$$

On the other hand, when  $f(u) = \frac{u^{p+1}}{p+1}$ ,  $u_- = u_+ = 0$ , and  $\int_{\mathbf{R}} u_0(x) dx = 0$ , it is shown in [9] that, the  $L^2$ -norm decay estimate (1.6) can be improved to

$$\|u(t, x)\|_{L^2(\mathbf{R})} \leq C(1+t)^{-\frac{3}{4}}. \quad (1.8)$$

Here  $p \geq 1$  is an integer and  $C$  is some positive constant.

For the case of  $u_- \neq u_+$ , it is believed that the large time behavior of solutions to (1.5) is closely related to the structure of the unique global entropy solution  $u^r(t, x)$  to the following Riemann problem of the corresponding scalar conservation laws

$$\begin{cases} u_t^r + f(u^r)_x = 0, \\ u^r(t, x)|_{t=0} = u_0^r(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \end{cases}$$

When  $f(u)$  satisfies (1.3) and  $u_- > u_+$ , it is expected that the global solutions  $u(t, x)$  of the Cauchy problem (1.5) converges to the corresponding viscous shock profiles  $\phi(\xi) \equiv \phi(x - st)$  (for the higher dimensional case, it is called as *planar viscous shock profiles*) which solve

$$\begin{cases} s\phi''' - \phi'' - s\phi' + f(\phi)' = 0, \\ \phi(\pm\infty) = u_{\pm}, \end{cases} \quad (1.9)$$

and such an expectation is justified mathematically in [15,16] and both the algebraic and the exponential convergence rates of its global solutions toward a suitably shifted viscous shock profiles  $\phi(\xi + x_0)$  are obtained by the space-time weighted method developed by Kawashima and Matsumura in [8].

For the following initial-boundary value problem for the generalized BBM–Burgers equations in the half space

$$\begin{cases} u_t - u_{xxt} - \nu u_{xx} + f(u)_x = 0, & x \in \mathbf{R}^+, t > 0, \\ u(0, x) = u_0(x) \rightarrow u_{\pm} & \text{as } 0 \leq x \rightarrow \pm\infty, \\ u(t, 0) = u_b, & t \geq 0, \end{cases} \quad (1.10)$$

it is shown in [18,17] that to describe its large time behavior, a new kind of nonlinear wave, the so-called *boundary layers*, should be taken into account and the nonlinear stability together with the convergence rates, both algebraic and exponential, for these boundary layers are obtained in [18,17].

All the above results are concentrated on the one dimensional case, for the higher dimensional case, the only result available is [19] which deals with the case  $u_- = u_+ = 0$ . For the case  $u_- \neq u_+$ , to the best of our knowledge, fewer results have been obtained in multi-dimensional case and the main purpose of our present manuscript is devoted to the nonlinear stability of the planar viscous shock profiles  $\phi(x - st)$  to the Cauchy problem (1.1), (1.2).

We now outline the main results and the main difficulties encountered. Note that for the nonlinear stability of one-dimensional generalized BBM–Burgers shock profiles, the existence and monotonicity of such viscous shock profiles is discussed in [15] by the phase-plane analysis method (cf. [14]) and the strictly monotonicity of the viscous shock profiles are enough to yield the nonlinear stability result. But for the two dimensional case, we need to derive the precise information on the decay of the planar viscous shock profiles as  $\xi \rightarrow \pm\infty$  and the relations between  $\phi''(\xi)$ ,  $\phi'(\xi)$ , and  $\phi(\xi) - u_{\pm}$  plays an essential role in the analysis. In fact, we can show that the monotonic planar viscous shock profiles to the two-point boundary value problem (1.9) obtained in [15] satisfies

$$\begin{cases} |u_- - \phi(\xi)| \leq O(1)\eta e^{-C\eta|\xi|}, & \forall \xi \leq 0, \\ |\phi(\xi) - u_+| \leq O(1)\eta e^{-C\eta|\xi|}, & \forall \xi \geq 0, \\ \left| \frac{\partial^k(\phi(\xi) - u_{\pm})}{\partial \xi^k} \right| \leq O(1)\eta^2 \exp(-C\eta|\xi|), & \forall \xi \in \mathbf{R}, \eta = |u_- - u_+|, \\ |\phi''(\xi)| \leq O(1)|\phi'(\xi)|, & \forall \xi \in \mathbf{R}, \\ |\phi(\xi) - u_-| \leq \frac{O(1)|\phi'(\xi)|}{|u_- - u_+|}, & \forall \xi \leq 0, \\ |\phi(\xi) - u_+| \leq \frac{O(1)|\phi'(\xi)|}{|u_- - u_+|}, & \forall \xi \geq 0. \end{cases} \quad (1.11)$$

where  $k > 0$  is an integer and  $O(1)$  and  $C > 0$  are positive constants independent of  $\eta$ .

Based on the decay estimates (1.11), we can show by the elementary energy method that if the initial data of the Cauchy problem (1.1), (1.2) satisfy

$$\sum_{\alpha_1 + \alpha_2 \leq 5} \int_{\mathbf{R}} \int_{\mathbf{R}} (1 + x^2)^{p'} \left| \frac{\partial^{\alpha_1 + \alpha_2}(u_0(x, y) - \phi(x))}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|^2 dx dy \leq D_0 \eta^{p'+1} \quad (1.12)$$

for some  $p' > 1$ , then we can show that the generalized BBM–Burgers planar viscous shock profiles  $\phi(\xi)$  are nonlinear stable provided that  $\eta$  is sufficiently small. Here  $D_0 > 0$  is some positive constant independent of  $\eta$ .

Our analysis is motivated by the method developed by J. Goodman in [4] to discuss the nonlinear stability of scalar viscous shock profiles in two dimensions. Comparing with the case of the viscous scalar conservation laws considered in [4], the main difficulties we encounter here are the following

- The first is, as mentioned above, on the derivation of the precise information on the decay of the planar viscous shock profiles  $\phi(\xi)$  as  $\xi \rightarrow \pm\infty$  and on the relations between  $\phi''(\xi)$ ,  $\phi'(\xi)$ , and  $\phi(\xi) - u_{\pm}$ . Notice that for the planar scalar viscous shock profiles studied in [4], such properties are well-established, cf. [8];
- The second is due to the appearance of the third-order differential operator  $\partial_t \Delta$ . Fortunately, such a difficulty can be overcome by some delicate energy type estimates.

Before concluding this section, we give several remarks:

- In this paper, since we are interested in deducing the nonlinear stability result by employing the elementary energy method, we need to ask that  $\eta = |u_- - u_+|$ , the strength of the planar viscous shock profiles, to be small, we are convinced that the arguments developed by Hoff and Zumbrun in [7,6] seems to be adapted to deduce similar result for general  $\eta$  together with considerable additional details on the decay rates. The study for this case is left for the future.
- We have shown that the compressive planar waves for the generalized BBM–Burgers equations are nonlinear stable, but it seems quite difficult to deduce the nonlinear stability of the planar expansive (rarefaction) waves and such a problem is under our current research.

This manuscript is organized as follows. After stating some notations, we reformulate our problem (1.1), (1.2) by considering the perturbation of the solutions  $u(t, x, y)$  of the Cauchy problem (1.1), (1.2) around the planar viscous shock profiles  $\phi(\xi)$  defined in (1.9) and determine the shift  $\delta(t, y)$  as in (2.26) and (2.32) in Section 2. Also, the decay of the planar viscous shock profiles  $\phi(\xi)$  are proved and the local solvability of the reformulated problem is stated in this section. Finally the proof of our main result is given in Section 3.

*Notations:* Throughout the rest of this paper,  $C$  or  $O(1)$  will be used to denotes a generic positive constant independent of  $\eta$ ,  $t$  and  $\xi$ . Note that all these constants may vary from line to line. If the dependence needs to be explicitly pointed out, the notations  $C_i$  ( $i \in \mathbf{Z}_+$ ) or  $C(\cdot, \cdot)$ , etc. are used.

Let  $\Omega = \mathbf{R}$  or  $\mathbf{R}^2$ .  $H^l(\Omega)$  ( $l \geq 0$ ) and  $L^p(\Omega)$  stand for the  $l$ -th order Sobolev space and the Lebesgue space with the usual  $\|\cdot\|_{H^l(\Omega)}$  norm and  $\|\cdot\|_{L^p(\Omega)}$  norm respectively and  $\|\cdot\|_{L^p}$  ( $1 \leq p \leq +\infty$ ) and  $\|\cdot\|_l$  are used to denote the corresponding  $L^p(\Omega)$  norm and  $H^l(\Omega)$  norm. For simplicity,  $\|f(t)\|_{L^p}$  denotes  $\|f(t, \cdot)\|_{L^p(\Omega)}$  and  $\|f(t)\|_l$  denotes  $\|f(t, \cdot)\|_{H^l(\Omega)}$ . Especially, when  $p = 2$ , we denote  $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$ .

For the weight function  $w(\xi) = \sqrt{1 + \xi^2}$ ,  $L_w^2(\mathbf{R}^2)$  denotes the space of measurable functions  $f(t, \xi, y)$  satisfying  $\sqrt{w(\xi)} f(t, \xi, y) \in L^2(\mathbf{R}^2)$  with the norm  $\|\sqrt{w}f(t)\|^2 = \int_{\mathbf{R}} \int_{\mathbf{R}^2} w(\xi) |f(t, \xi, y)|^2 d\xi dy$ .  $H_w^l(\mathbf{R}^2)$  ( $l \geq 1$ ) denotes the space of functions  $f(t, \xi, y)$  satisfying  $\partial^\alpha f(t, \xi, y) \in L_w^2(\mathbf{R}^2)$  ( $|\alpha| \leq l$ ) with the norm  $\|f(t)\|_{H_w^l}^2 = \sum_{|\alpha|=0}^l \|\sqrt{w} \partial^\alpha f(t)\|^2$ . Here we used the notation that for each multi-index  $\alpha = (\alpha_1 \alpha_2)$ ,  $\partial^\alpha f(t, \xi, y) = \frac{\partial^{\alpha_1 + \alpha_2} f(t, \xi, y)}{\partial \xi^{\alpha_1} \partial y^{\alpha_2}}$ . For  $k \in \mathbf{Z}$  and  $p \geq 1$ ,  $W^{k,p}(\Omega)$  and  $W_w^{k,p}(\Omega)$  can be defined similarly.

Finally, The symbol  $\iiint$  will stand for the triple integral over  $[0, t] \times \mathbf{R}^2$  and  $\int f$  will represent the integral on  $\mathbf{R}^2$  unless otherwise stated.

## 2. Preliminaries and reformulation of the original problem

The main purpose of this section is two-fold: The first is to concentrate on deducing certain decay estimates on the planar viscous shock profiles  $\phi(\xi)$  and the relations between  $\phi''(\xi)$ ,  $\phi'(\xi)$ , and  $\phi(\xi) - u_\pm$ , and the second is to reformulate our original problem and to state the local existence result.

Firstly, we cite the result obtained in [8,15] on the solvability of the two-point boundary problem (1.9). For the case  $s = 0$ , it is proved in [8,15] that

**Lemma 2.1** (The Case  $s = 0$ , Cf. [8,15]). Suppose that the smooth nonlinear function  $f(u)$  and  $u_\pm$  satisfy  $u_+ < u_-$ , (1.3),  $f(u_-) = f(u_+)$ , and the Lax geometric entropy condition  $f'(u_+) < 0 < f'(u_-)$ , then there exists a stationary solution  $\phi(\xi)$ , unique up to a shift, to the two-point boundary problem (1.9) and  $\phi(\xi)$  satisfies  $\phi'(\xi) < 0$ ,  $u_+ < \phi(\xi) < u_-$  for all  $\xi \in \mathbf{R}$  and for any positive integer  $k$

$$\begin{aligned} |u_- - \phi(\xi)| &\leq O(1)\eta e^{-C\eta|\xi|}, \quad \forall \xi \leq 0, \\ |\phi(\xi) - u_+| &\leq O(1)\eta e^{-C\eta|\xi|}, \quad \forall \xi \geq 0, \\ |\partial_\xi^k \phi(\xi)| &\leq O(1)\eta^2 e^{-C\eta|\xi|}, \quad \forall \xi \in \mathbf{R}. \end{aligned} \quad (2.1)$$

Here  $\eta = u_- - u_+$ ,  $O(1)$  and  $C$  are some positive constants independent of  $\eta$ .

For the case  $s \neq 0$ , it is shown in [15] that

**Lemma 2.2** (The Case  $s \neq 0$ , Cf. [15]). Suppose that  $u_+ < u_-$ ,  $s$  and the smooth nonlinear function  $f(u)$  satisfy (1.3), the Rankine–Hugoniot (R–H) condition  $s(u_- - u_+) = f(u_-) - f(u_+)$ , the Lax geometric entropy condition  $f'(u_+) < s < f'(u_-)$ , and

$$\begin{cases} s > 0, \\ 4s^2 - 4sf'(u_-) + 1 \geq 0 \end{cases} \quad (2.2)$$

or

$$\begin{cases} s < 0, \\ 4s^2 - 4sf'(u_+) + 1 \geq 0, \end{cases} \quad (2.3)$$

then there exists a solution  $\phi(\xi)$ , unique up to a shift, to the two-point boundary problem (1.9) and  $\phi(\xi) = \phi(x - st)$  verifies  $\phi'(\xi) < 0$  and  $u_+ < \phi(\xi) < u_-$  for all  $x \in \mathbf{R}$ .

For the corresponding decay estimates on the planar viscous shock profiles  $\phi(\xi)$  for the generalized BBM–Burgers equation obtained in Lemma 2.2, motivated by the analysis in [11], we have the following result

**Lemma 2.3** (Decay Estimates on  $\phi(\xi)$ ). Assume that the conditions listed in Lemma 2.2 hold, then the planar viscous shock profiles  $\phi(\xi)$  for the generalized BBM–Burgers equation obtained in Lemma 2.2 satisfy the estimates (2.1).

**Proof.** We only deal with the case of  $s > 0$  in details since the other case can be treated similarly.

For the monotonic solution  $\phi(\xi)$  obtained above, Lemma 2.1.1 in [14] tells us that

$$\lim_{\xi \rightarrow \pm\infty} \phi''(\xi) = \lim_{\xi \rightarrow \pm\infty} \phi'(\xi) = 0. \quad (2.4)$$

Let  $\varphi(\xi) = \frac{\phi(\xi) - u_-}{u_+ - u_-}$ , we get from (1.9)<sub>1</sub> that

$$\varphi'' - \frac{\varphi'}{s} + F(\varphi) = 0, \quad (2.5)$$

where  $F(\varphi) = \frac{f(u_- + \varphi(u_+ - u_-)) - f(u_-) - s\varphi(u_+ - u_-)}{s(u_+ - u_-)}$ . Then it is easy to see that  $\varphi(\xi) \rightarrow 0$  as  $\xi \rightarrow -\infty$  and  $F(0) = F(1) = 0$ .

From (2.5), we can deduce that

$$\frac{\varphi''}{\varphi} - \frac{\varphi'}{s\varphi} \rightarrow -F'(0) = -\frac{f'(u_-) - s}{s} \quad \text{as } \xi \rightarrow -\infty. \quad (2.6)$$

If we differentiate Eq. (2.5) with respect to  $\xi$  once, we obtain

$$\frac{\varphi'''}{\varphi'} - \frac{\varphi''}{s\varphi'} \rightarrow -F'(0) = -\frac{f'(u_-) - s}{s} \quad \text{as } \xi \rightarrow -\infty. \quad (2.7)$$

Now setting  $\varphi_1(\xi) = \frac{\varphi'(\xi)}{\varphi(\xi)}$ ,  $\varphi_2(\xi) = \frac{\varphi''(\xi)}{\varphi'(\xi)}$ , we can deduce that

$$\varphi_1'(\xi) = \frac{\varphi''(\xi)}{\varphi(\xi)} - \varphi_1^2(\xi), \quad \varphi_2'(\xi) = \frac{\varphi'''(\xi)}{\varphi'(\xi)} - \varphi_2^2(\xi).$$

(2.6) together with (2.7) yield

$$\varphi_1'(\xi) - \frac{\varphi_1(\xi)}{s} + \varphi_1^2(\xi) \rightarrow -F'(0) = -\frac{f'(u_-) - s}{s} \quad \text{as } \xi \rightarrow -\infty, \quad i = 1, 2. \quad (2.8)$$

Then we claim that  $\varphi_i(\xi) \rightarrow \lambda$  as  $\xi \rightarrow -\infty$ , where  $\lambda$  is a solution of the following algebraic equation

$$\lambda^2 - \frac{\lambda}{s} + \frac{f'(u_-) - s}{s} = 0. \quad (2.9)$$

We show first that  $|\varphi_i(\xi)|$  cannot tend to infinity as  $\xi \rightarrow -\infty$ . Indeed, let  $|\varphi_i(\xi)| \rightarrow \infty$  as  $\xi \rightarrow -\infty$ . Then  $\bar{\psi}(\xi) = \varphi_i^{-1}(\xi) \rightarrow 0$  as  $\xi \rightarrow -\infty$ . Dividing (2.8) by  $\varphi_i^2(\xi)$ , we obtain  $\lim_{\xi \rightarrow -\infty} \frac{d\bar{\psi}(\xi)}{d\xi} = 1$ , which is impossible.

Now we show that  $\varphi_i(\xi)$  has a finite limit as  $\xi \rightarrow -\infty$ . Let us assume the contrary. We can then choose strictly decreasing sequences  $\{x_n\}$ ,  $\{y_n\}$ ,  $x_n \rightarrow -\infty$ ,  $y_n \rightarrow -\infty$ ,  $\varphi_i(x_n) \rightarrow \lambda_1$ ,  $\varphi_i(y_n) \rightarrow \lambda_2$ ,  $\lambda_1 < \lambda_2$ . Let  $\lambda_0$  satisfies the inequality  $\lambda_1 < \lambda_0 < \lambda_2$  and not be a solution of Eq. (2.9). For example,

$$\lambda_0^2 - \frac{\lambda_0}{s} + \frac{f'(u_-) - s}{s} > 0.$$

Without loss of generality, we assume  $x_n < y_n < x_{n-1}$  and  $\varphi_i(x_n) < \lambda_0 < \varphi_i(y_n)$  for any  $n \in \mathbf{N}$ . By the continuity of  $\varphi_i(\xi)$  and  $\varphi_i'(\xi)$ , we can choose  $\{z_n\}$ ,  $z_n \rightarrow -\infty$  such that  $x_n < z_n < y_n$ ,  $\varphi_i(z_n) = \lambda_0$  and  $\varphi_i'(z_n) \geq 0$ . Then we obtain

$$\varphi_i'(z_n) - \frac{\varphi_i(z_n)}{s} + \varphi_i^2(z_n) \geq \lambda_0^2 - \frac{\lambda_0}{s} > -\frac{f'(u_-) - s}{s},$$

which contradicts (2.8).

Thus,  $\varphi_i(\xi) \rightarrow \lambda$  as  $\xi \rightarrow -\infty$ , whence

$$\varphi_i'(\xi) \rightarrow -\lambda^2 + \frac{\lambda}{s} - \frac{f'(u_-) - s}{s}.$$

This limit must be equal to zero in view of the boundedness of  $\varphi_i(\xi)$ . This completes the claim above.

From the expressions of  $\varphi_i(\xi)$  ( $i = 1, 2$ ), we see that  $\frac{\phi'(\xi)}{\phi(\xi)-u_-}$ ,  $\frac{\phi''(\xi)}{\phi'(\xi)}$  has a finite limit as  $\xi \rightarrow -\infty$ . Similarly, let  $\varphi(\xi) = \frac{\phi(\xi)-u_+}{u_- - u_+}$ , we can prove that  $\frac{\phi'(\xi)}{\phi(\xi)-u_+}$ ,  $\frac{\phi''(\xi)}{\phi'(\xi)}$  has a finite limit as  $\xi \rightarrow +\infty$ . Therefore, we divide (1.9)<sub>1</sub> by  $\phi'(\xi)$  and get that  $\lim_{\xi \rightarrow \pm\infty} \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)}$  exist and are finite. Noticing that  $\phi'(\xi) < 0$ ,  $\forall \xi \in \mathbf{R}$ ,  $(f(\phi(\xi)) - f(u_{\pm})) - s(\phi(\xi) - u_{\pm}) < 0$ ,  $\forall \xi \in \mathbf{R}$  and both of them are continuous with respect to  $\xi$ , we can thus deduce that

$$\frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} > 0$$

holds for all fixed  $\xi \in \mathbf{R}$  and

$$\lim_{\xi \rightarrow \pm\infty} \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} \geq 0.$$

On the other hand, we have from (2.5) that

$$\begin{cases} \lim_{\xi \rightarrow -\infty} \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} \equiv \lim_{\xi \rightarrow -\infty} \frac{f(\phi(\xi)) - f(u_-) - s(\phi(\xi) - u_-)}{\phi'(\xi)} = 1 - s \lim_{\xi \rightarrow -\infty} \frac{\phi''(\xi)}{\phi'(\xi)}, \\ \lim_{\xi \rightarrow +\infty} \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} \equiv \lim_{\xi \rightarrow +\infty} \frac{f(\phi(\xi)) - f(u_+) - s(\phi(\xi) - u_+)}{\phi'(\xi)} = 1 - s \lim_{\xi \rightarrow +\infty} \frac{\phi''(\xi)}{\phi'(\xi)}. \end{cases}$$

Since  $\frac{1}{s}$  is not a solution of (2.9), we can deduce that

$$\lim_{\xi \rightarrow \pm\infty} \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} > 0.$$

The above analysis implies that there exists a constant  $m > 1$  such that

$$\frac{1}{m} \leq \frac{f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})}{\phi'(\xi)} \leq m, \quad \forall \xi \in \mathbf{R}. \quad (2.10)$$

Namely

$$-\frac{1}{m} [f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})] \leq -\phi'(\xi) \leq -m [f(\phi(\xi)) - f(u_{\pm}) - s(\phi(\xi) - u_{\pm})]. \quad (2.11)$$

Thus for any  $\xi \leq 0$ , we obtain from the assumption (1.3) imposed on  $f(u)$  that

$$\begin{aligned} (u_- - \phi(\xi))' &\geq -\frac{1}{m} (u_- - \phi(\xi)) \left[ \frac{f(u_+) - f(u_-)}{u_+ - u_-} - \frac{f(\phi(\xi)) - f(u_-)}{\phi(\xi) - u_-} \right] \\ &= -\frac{1}{m} (u_- - \phi(\xi)) \int_0^1 [f'(u_- + \theta(u_+ - u_-)) - f'(u_- + \theta(\phi(\xi) - u_-))] d\theta \\ &\geq \frac{1}{m} (u_- - \phi(\xi)) (\phi(\xi) - u_+) \int_0^1 f''(\xi_1) \theta d\theta \\ &\geq \frac{a}{2m} (u_- - \phi(\xi)) (\phi(0) - u_+), \end{aligned} \quad (2.12)$$

where  $\xi_1$  is between  $u_+$  and  $u_-$ .

Consequently

$$\left[ e^{-\frac{a}{2m}(\phi(0)-u_+)\xi} (u_- - \phi(\xi)) \right]' \geq 0. \quad (2.13)$$

Integrating (2.13) with respect to  $\xi$  over  $[\xi, 0]$  yields

$$u_- - \phi(\xi) \leq (u_- - \phi(0)) e^{\frac{a}{2m}(\phi(0)-u_+)\xi}, \quad \forall \xi \leq 0. \quad (2.14)$$

This gives (2.1)<sub>1</sub>. On the other hand, for any  $\xi \leq 0$ , (2.11) implies that

$$|\phi'(\xi)| = (u_- - \phi(\xi))' \leq m [s(\phi(\xi) - u_-) - (f(\phi(\xi)) - f(u_-))], \quad (2.15)$$

and similar to that of (2.12), we can obtain

$$(u_- - \phi(\xi))' \leq \frac{m}{2} \bar{a} (u_- - \phi(\xi)) (u_- - u_+), \quad (2.16)$$

where  $\bar{a} = \max_{u_+ \leq \phi \leq u_-} f''(\phi)$ .

Putting (2.14) and (2.16) together, we finally arrive at

$$|\phi'(\xi)| \leq O(1)\eta^2 e^{-\frac{\alpha}{2m}(\phi(0)-u_+)\xi}, \quad \forall \xi \leq 0. \quad (2.17)$$

By employing similar arguments used above, we can prove that

$$\phi(\xi) - u_+ \leq (\phi(0) - u_+)e^{-\frac{\alpha}{2m}(u_- - \phi(0))\xi}, \quad \forall \xi \geq 0 \quad (2.18)$$

and

$$|\phi'(\xi)| \leq O(1)\eta^2 e^{-\frac{\alpha}{2m}(\phi(0)-u_+)\xi}, \quad \forall \xi \geq 0. \quad (2.19)$$

Thus (2.1) with  $k = 1$  follows immediately from (2.17) and (2.19). Finally the case  $k > 1$  can be obtained by differentiating (1.9)<sub>1</sub> with respect to  $\xi$  and by exploiting the principle of mathematical induction. This completes the proof of Lemma 2.3.  $\square$

To deduce the nonlinear stability of the planar viscous shock profiles  $\phi(\xi)$  constructed above, in addition to the decay estimates (2.1), we need also the relations between  $\phi''(\xi)$ ,  $\phi'(\xi)$ , and  $\phi(\xi) - u_{\pm}$  which are obtained in the following corollary:

**Corollary 2.1.** Assume that the assumptions listed in Lemmas 2.1 and 2.2 are satisfied, we can deduce that

$$\begin{cases} |\phi''(\xi)| \leq O(1)|\phi'(\xi)|, & \forall \xi \in \mathbf{R}, \\ |\phi(\xi) - u_-| \leq \frac{O(1)|\phi'(\xi)|}{|u_- - u_+|}, & \forall \xi \leq 0, \\ |\phi(\xi) - u_+| \leq \frac{O(1)|\phi'(\xi)|}{|u_- - u_+|}, & \forall \xi \geq 0. \end{cases} \quad (2.20)$$

Here  $O(1)$  is some positive constant independent of  $\xi$  and  $\eta = |u_- - u_+|$ .

**Proof.** We only deal with the case  $s > 0$ , the other cases can be treated similarly.

The analysis in Lemma 2.3 implies that

$$\lim_{\xi \rightarrow -\infty} \frac{\phi''(\xi)}{\phi'(\xi)} = \lambda \quad (2.21)$$

where  $\lambda$  is a solution of the algebraic Eq. (2.9) which is bounded by  $\frac{1+2|f'(u_-)|s+2s^2}{|s|}$  and such a bound is independent of  $\eta$ .

On the other hand, the analysis in the proof of Lemma 2.3 together with the assumption (1.3) imply that

$$0 < \ell = \lim_{\xi \rightarrow -\infty} \frac{f(\phi(\xi)) - f(u_-) - s(\phi(\xi) - u_-)}{\phi'(\xi)} = (f'(u_-) - s) \lim_{\xi \rightarrow -\infty} \frac{\phi(\xi) - u_-}{\phi'(\xi)}$$

and

$$|s - f'(u_-)| = \left| \frac{f(u_+) - f(u_-)}{u_- - u_+} - f'(u_-) \right| = \left| \frac{f''(\theta u_- + (1-\theta)u_+)}{2}(u_+ - u_-) \right| \geq \frac{a\eta}{2}, \quad \theta \in (0, 1).$$

Putting the above estimates together, we can easily deduce that (2.20) holds for  $\xi \leq 0$ . The case  $\xi \geq 0$  can be proved similarly and thus the proof of Corollary 2.1 is completed.  $\square$

Now we turn to reformulate our original problem and to deduce the local solvability of the reformulated problem. To this end, following the idea of [4], we first preform some normalizations as in the following:

- **Normalization 1:** Letting  $t^{(1)} = t$ ,  $\xi = x - st$ ,  $y^{(1)} = y$ .
- **Normalization 2:** Letting  $t^{(2)} = t^{(1)}$ ,  $x^{(2)} = \xi$ ,  $y^{(2)} = y^{(1)} + \rho\xi$ .
- **Normalization 3:** Letting  $t^{(3)} = t^{(2)}$ ,  $x^{(3)} = x^{(2)}$ ,  $y^{(3)} = y^{(2)} - \sigma t^{(2)}$ .

Here  $\rho = -\frac{g''(\phi(0))}{f''(\phi(0))}$  and  $\sigma = \frac{g(u_-) - g(u_+)}{u_- - u_+}$ . Notice that after each normalization, the functions  $f(u)$  and  $g(u)$  may change but the convexity of  $f(u)$  does not change.

After the normalizations given above, we have from Corollary 2.1 that

$$\begin{aligned} f(u_-) &= f(u_+), & |f'(\phi)| &= O(1)(u_- - u_+), & g(u_-) &= g(u_+), \\ |g''(\phi)| &= O(1)(u_- - u_+), & |g'(\phi)| &= O(1)(u_- - u_+)^2, & |g(\phi)| &\leq O(1)(u_- - u_+)|\phi'|. \end{aligned} \quad (2.22)$$

Without loss of generality, we suppose that  $g(u_-) = g(u_+) = 0$ . For simplicity, we remain write  $t^{(3)} = t, x^{(3)} = \xi$  and  $y^{(3)} = y$ . Then the Cauchy problem (1.1), (1.2) can be rewritten as

$$\begin{cases} u_t + [su_{\xi\xi\xi} + \rho_1 u_{\xi\xi y} + \rho_2 u_{\xi y y} + \rho_3 u_{y y y}] - (u_{\xi\xi t} + 2\rho u_{\xi y t} + (1 + \rho^2)u_{y y t}) + f(u)_\xi + g(u)_y \\ = u_{\xi\xi} + 2\rho u_{\xi y} + (1 + \rho^2)u_{y y}, \\ u(0, \xi, y) = u_0(\xi, y) \rightarrow u_\pm, \quad \text{as } \xi \rightarrow \pm\infty \text{ for any } y \in \mathbf{R}. \end{cases} \quad (2.23)$$

Here  $\rho_1 = 3\rho s + \sigma, \rho_2 = 2\sigma\rho + 3s\rho^2, \rho_3 = (\rho s + \sigma)(1 + \rho^2)$ .

Since the solution  $u(t, x, y)$  of (2.23) is expected to approach the planar viscous shock profiles  $\phi = \phi(\xi)(\xi = x - st)$ , if we define the perturbation as

$$u(t, \xi, y) = \phi(\xi + \delta(t, y)) + U_\xi(t, \xi, y), \quad (2.24)$$

then we have from (2.23)<sub>1</sub> and (2.24) that

$$\begin{aligned} & U_{\xi t} + [sU_{\xi\xi\xi} + \rho_1 U_{\xi\xi y} + \rho_2 U_{\xi y y} + \rho_3 U_{y y y}] - (U_{\xi\xi t} + 2\rho U_{\xi y t} + (1 + \rho^2)U_{y y t}) \\ & + [\rho_1 \phi''' \delta_y + \rho_2 (\phi''' \delta_y^2 + \phi'' \delta_{y y}) + \rho_3 (\phi''' \delta_y^3 + 3\phi'' \delta_y \delta_{y y})] \\ & - [\phi''' \delta_t + 2\rho (\phi''' \delta_y \delta_t + \phi'' \delta_{y t}) + (1 + \rho^2)(\phi''' \delta_y^2 \delta_t + 2\phi'' \delta_y \delta_{y t} + \phi' \delta_t \delta_{y y})] \\ & + (f(\phi + U_\xi) - f(\phi))_\xi + g(\phi + U_\xi)_y + \phi' [\delta_t - (1 + \rho^2)\delta_{y y} - (1 + \rho^2)\delta_{y y} + \rho_3 \delta_{y y y}] \\ & = U_{\xi\xi\xi} + 2\rho U_{\xi\xi y} + (1 + \rho^2)U_{\xi y y} + 2\rho \phi'' \delta_y + (1 + \rho^2)\phi'' \delta_y^2, \end{aligned} \quad (2.25)$$

where we have used (1.9).

To guarantee that

$$\int (u(t, \xi, y) - \phi(\xi + \delta(t, y))) d\xi = 0,$$

we need to define the shift function  $\delta(t, y)$  with given initial data  $\delta(0, y) = \delta_0(y)$  which satisfies

$$\int (u_0(x, y) - \phi(x + \delta_0(y))) dx = 0.$$

Thus

$$\delta_0(y) = -\eta^{-1} \int (u_0(x, y) - \phi(x)) dx. \quad (2.26)$$

With  $\delta(t, y)$  chosen as above, we can define  $U(t, \xi, y)$  and  $U_0(x, y)$  by

$$U(t, \xi, y) = \int_{-\infty}^{\xi} (u(t, z, y) - \phi(z + \delta(t, y))) dz \quad (2.27)$$

and

$$U_0(x, y) = \int_{-\infty}^x (u_0(z, y) - \phi(z + \delta_0(y))) dz, \quad (2.28)$$

respectively.

Moreover, due to

$$f(\phi + U_\xi) = f(\phi) + f'(\phi)U_\xi + q(\phi, U_\xi)U_\xi^2, \quad g(\phi + U_\xi) = g(\phi) + g'(\phi)U_\xi + r(\phi, U_\xi)U_\xi^2, \quad (2.29)$$

we define the decomposition

$$\begin{aligned} \left( \int g(\phi + U_\xi) d\xi \right)_y &= - \left( \int g''(\phi) \phi' U d\xi \right)_y + \left( \int r(\phi, U_\xi) U_\xi^2 d\xi \right)_y \\ &= \left( \int \alpha \phi' d\xi \right)_y + \left( \int \beta \phi' d\xi \right)_y \\ &= -\eta(\alpha_y + \beta_y), \end{aligned} \quad (2.30)$$

namely

$$\begin{aligned} -g''(\phi) \phi'(\xi + \delta(t, y)) U(t, \xi, y) &= \alpha(t, y) \phi'(\xi + \delta(t, y)) + V(t, \xi, y)_\xi, \\ r(\phi, U_\xi) U_\xi^2(t, \xi, y) &= \beta(t, y) \phi'(\xi + \delta(t, y)) + W(t, \xi, y)_\xi, \end{aligned} \quad (2.31)$$

where  $V, W \in L^2(\mathbf{R}^2)$ .

Then it is easy to see from integrating (2.25) with respect to  $\xi$  that  $\delta(t, y)$  verifies

$$\delta_t - (1 + \rho^2)\delta_{yyt} - (1 + \rho^2)\delta_{yy} + \rho_3\delta_{yyy} + \alpha_y + \beta_y = 0. \quad (2.32)$$

Inserting (2.29) and (2.30) into (2.25) and integrating the resultant equation with respect to  $\xi$  over  $(-\infty, \xi)$  yield

$$\begin{aligned} U_t &+ [sU_{\xi\xi\xi} + \rho_1 U_{\xi\xi y} + \rho_2 U_{\xi yy} + \rho_3 U_{yyy} - (U_{\xi\xi t} + 2\rho U_{\xi yt} + (1 + \rho^2)U_{yyt})] \\ &+ [\rho_1 \phi'' \delta_y + \rho_2 (\phi'' \delta_y^2 + \phi' \delta_{yy}) + \rho_3 (\phi'' \delta_y^3 + 3\phi' \delta_y \delta_{yy})] \\ &- [\phi'' \delta_t + 2\rho (\phi'' \delta_y \delta_t + \phi' \delta_{yt}) + (1 + \rho^2)(\phi'' \delta_y^2 \delta_t + 2\phi' \delta_y \delta_{yt} + \phi' \delta_t \delta_{yy})] \\ &+ f'(\phi)U_\xi + q(\phi, U_\xi)U_\xi^2 + g(\phi)\delta_y + (g'(\phi)U)_y + \alpha\phi' \delta_y + V_y + \beta\phi' \delta_y + W_y \\ &= U_{\xi\xi} + 2\rho U_{\xi y} + (1 + \rho^2)U_{yy} + 2\rho\phi' \delta_y + (1 + \rho^2)\phi' \delta_y^2. \end{aligned} \quad (2.33)$$

Now we give some elementary estimates on  $\alpha$ ,  $\beta$ ,  $V$ , and  $W$  which will be used frequently later. The first two lemmas are borrowed from [4] (Notice that to simplify the notation, in the rest part of this paper, we use  $\psi(\xi + \delta)$  to denote  $|\phi'(\xi + \delta)| = -\phi'(\xi + \delta)$ )

**Lemma 2.4** (Cf. [4]). Let  $l_\xi(t, \xi, y) = m(t, \xi, y)\psi$  and assume that for any fixed  $t, y$ ,  $l(t, \xi, y) \in H^1(-\infty, 0)$ , then

$$\int_{-\infty}^0 l^2 d\xi \leq \int_{-\infty}^0 |\xi| \psi d\xi \int_{-\infty}^0 m^2 \psi d\xi. \quad (2.34)$$

**Lemma 2.5** (Cf. [4]). Let  $l_\xi(t, \xi, y) = rm^2(t, \xi, y)$  and assume that for any fixed  $t, y$ ,  $l(t, \xi, y) \in L^2(-\infty, 0)$ , then

$$\int_{-\infty}^0 l^2 d\xi \leq \sup |r|^2 \int_{-\infty}^0 |\xi| m^2 d\xi \int_{-\infty}^0 m^2 d\xi. \quad (2.35)$$

Based on Lemmas 2.4 and 2.5, we can deduce the following estimates on  $\alpha(t, y)$ ,  $\beta(t, y)$ ,  $V(t, \xi, y)$ , and  $W(t, \xi, y)$ :

**Lemma 2.6.** The following estimates hold

$$\int \alpha^2 dy \leq O(1)\eta \iint U^2 \psi d\xi dy, \quad (2.36)$$

$$\int \beta^2 dy \leq O(1) [\eta^{-2} \sup |U_\xi|^2 + 1] \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \iint U_\xi^2 d\xi dy, \quad (2.37)$$

$$\int \alpha_y^2 dy \leq O(1) (\sup |\delta_y|^2 + \eta) \eta \iint (U^2 \psi + U_y^2) d\xi dy, \quad (2.38)$$

$$\int \beta_y^2 dy \leq O(1) [\sup |\delta_y|^2 + \eta^{-2} \sup |U_\xi|^2 + 1] \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \iint (U_\xi^2 + U_{\xi y}^2) d\xi dy, \quad (2.39)$$

$$\int \alpha_t^2 dy \leq O(1) (\sup |\delta_t|^2 + \eta) \eta \iint (U^2 \psi + U_t^2) d\xi dy, \quad (2.40)$$

$$\int \beta_t^2 dy \leq O(1) [\sup |\delta_t|^2 + \eta^{-2} \sup |U_\xi|^2 + 1] \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \iint (U_{\xi t}^2 + U_\xi^2) d\xi dy, \quad (2.41)$$

$$\begin{aligned} \int \alpha_{yt}^2 dy &\leq O(1) [\sup |\delta_t|^2 + \sup |U|^2 + \sup |\delta_y|^2 + \eta^2] \\ &\times \left[ \iint (U_y^2 + U_t^2 + U_{yt}^2) d\xi dy + \int (\delta_y^2 + \delta_{yt}^2) dy \right], \end{aligned} \quad (2.42)$$

$$\begin{aligned} \int \beta_{yt}^2 dy &\leq O(1) \left[ (\sup |\delta_y|^2 + \eta^{-2} \sup |U_\xi|^2) \sup_{y \in \mathbf{R}} \int U_{\xi y}^2 d\xi + \sup |\delta_y|^2 \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \right] \iint U_{\xi t}^2 d\xi dy \\ &+ [1 + \sup |\delta_t|^2 + \sup |\delta_y|^2 + \sup |\delta_{yt}|^2 + \eta^{-2} \sup |U_\xi|^2] \\ &\times \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \iint (U_\xi^2 + U_{\xi y}^2 + U_{\xi yt}^2) d\xi dy, \end{aligned} \quad (2.43)$$

$$\iint V^2 d\xi dy \leq O(1)\eta^2 \int |\xi| \psi d\xi \iint U^2 \psi d\xi dy, \quad (2.44)$$

$$\iint W^2 d\xi dy \leq O(1) \left( \int |\xi| \psi d\xi + 1 \right) [\eta^{-1} \sup |U_\xi|^2 + 1] \sup_{y \in \mathbf{R}} \int w U_\xi^2 d\xi \iint U_\xi^2 d\xi dy, \quad (2.45)$$

$$\iint V_y^2 d\xi dy \leq O(1) \int |\xi| \psi d\xi [\sup |\delta_y|^2 + \eta^2] \iint (U^2 \psi + U_y^2) d\xi dy, \quad (2.46)$$

$$\iint W_y^2 d\xi dy \leq O(1) [1 + \sup |\delta_y|^2 + \eta^{-1} \sup |U_\xi|^2] \times \sup_{y \in \mathbf{R}} \int w U_\xi^2 d\xi \iint (U_\xi^2 + U_{\xi y}^2) d\xi dy. \quad (2.47)$$

Here  $O(1)$  is some positive constant independent of  $\eta$ .

**Proof.** We only prove (2.36), (2.37) and (2.45), the remainders can be treated similarly.

To show that (2.36) and (2.37) hold true, we first have from (2.31) that

$$\alpha(t, y) = \eta^{-1} \int g''(\phi) \phi' U d\xi, \quad \beta(t, y) = -\eta^{-1} \int r U_\xi^2 d\xi. \quad (2.48)$$

Moreover, (2.22) and (2.29) imply that

$$\begin{aligned} |r(\phi, U_\xi)| &= \frac{1}{2} |g''(\phi + \theta U_\xi)| \\ &\leq \frac{1}{2} |g''(\phi)| + O(1) |U_\xi| \\ &\leq O(1) \eta + O(1) |U_\xi|, \quad \theta \in (0, 1), \end{aligned} \quad (2.49)$$

we can then deduce (2.36) and (2.37) immediately from Lemma 2.3.

For (2.45), we get from (2.31)<sub>2</sub>, Lemmas 2.4 and 2.5 that

$$\begin{aligned} \iint W^2 dy &\leq \eta \int |\xi| \psi d\xi \int \beta^2 dy + \sup_{y \in \mathbf{R}} |r|^2 \sup_{y \in \mathbf{R}} \int |\xi| U_\xi^2 d\xi \iint U_\xi^2 d\xi dy \\ &\leq O(1) \int |\xi| \psi d\xi [\eta^{-1} \sup |U_\xi|^2 + 1] \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \iint U_\xi^2 d\xi dy \\ &\quad + O(1) [\sup |U_\xi|^2 + \eta^2] \sup_{y \in \mathbf{R}} \int |\xi| U_\xi^2 d\xi \iint U_\xi^2 d\xi dy, \end{aligned} \quad (2.50)$$

which gives the desired estimate (2.45). Here we have used (2.37) and (2.49). This completes the proof of Lemma 2.6.  $\square$

Now we continue to show the local existence of solutions  $U(t, \xi, y)$  and  $\delta(t, y)$  to the Cauchy problems (2.33), (2.28), (2.32) and (2.26) respectively. To this end, we define the following solution spaces for  $U(t, \xi, y)$  and  $\delta(t, y)$

$$X_M(t_1, t_2) = \left\{ U(t, \xi, y) \left| \begin{array}{l} U(t, \xi, y) \in C([t_1, t_2]; H^5(\mathbf{R}^2)) \cap C^1([t_1, t_2]; H^4(\mathbf{R}^2)), \\ U_\xi(t, \xi, y) \in C([t_1, t_2]; H_w^5(\mathbf{R}^2)) \cap C^1([t_1, t_2]; H_w^4(\mathbf{R}^2)), \\ \sup_{t \in [t_1, t_2]} \left( \|U(t)\|_5^2 + \|U_\xi(t)\|_{H_w^5}^2 + \int_0^t \|\sqrt{\psi} U(\tau)\|^2 d\tau \right) \leq M \end{array} \right. \right\} \quad (2.51)$$

with the norm

$$\|U\|_X = \sup_{t \in [t_1, t_2]} \left( \|U(t)\|_5^2 + \|U_\xi(t)\|_{H_w^5}^2 + \int_0^t \|\sqrt{\psi} U(\tau)\|^2 d\tau \right) \quad (2.52)$$

and

$$Y_M(t_1, t_2) = \left\{ \delta(t, y) \left| \begin{array}{l} \delta(t, y) \in C([t_1, t_2]; H^5(\mathbf{R})) \cap C^1([t_1, t_2]; H^4(\mathbf{R})), \\ \sup_{t \in [t_1, t_2]} \|\delta(t)\|_5^2 \leq M \end{array} \right. \right\} \quad (2.53)$$

with the norm

$$\|\delta\|_Y = \sup_{t \in [t_1, t_2]} \|\delta(t)\|_5 \quad (2.54)$$

respectively. Then the local existence result of the Cauchy problems (2.33), (2.28), (2.32) and (2.26) can be stated in the following proposition.

**Proposition 2.1** (Local Existence Result). Consider the following Cauchy problems

$$\begin{cases} U_t + [sU_{\xi\xi\xi} + \rho_1 U_{\xi\xi y} + \rho_2 U_{\xi y y} + \rho_3 U_{y y y}] - (U_{\xi\xi t} + 2\rho U_{\xi y t} + (1 + \rho^2)U_{y y t}) \\ + [\rho_1 \phi'' \delta_y + \rho_2 (\phi'' \delta_y^2 + \phi' \delta_{y y}) + \rho_3 (\phi'' \delta_y^3 + 3\phi' \delta_y \delta_{y y})] \\ - [\phi'' \delta_t + 2\rho (\phi'' \delta_y \delta_t + \phi' \delta_{y t}) + (1 + \rho^2)(\phi'' \delta_y^2 \delta_t + 2\phi' \delta_y \delta_{y t} + \phi' \delta_t \delta_{y y})] \\ + f'(\phi)U_{\xi} + q(\phi, U_{\xi})U_{\xi}^2 + g(\phi)\delta_y + (g'(\phi)U)_y + \alpha\phi' \delta_y + V_y + \beta\phi' \delta_y + W_y \\ = U_{\xi\xi} + 2\rho U_{\xi y} + (1 + \rho^2)U_{y y} + 2\rho\phi' \delta_y + (1 + \rho^2)\phi' \delta_y^2, \\ U(t, \xi, y)|_{t=0} = U_0(\xi, y) \end{cases} \quad (2.55)$$

and

$$\begin{cases} \delta_t - (1 + \rho^2)\delta_{y y t} - (1 + \rho^2)\delta_{y y} + \rho_3 \delta_{y y y} + \alpha_y + \beta_y = 0, \\ \delta(t, y)|_{t=0} = \delta_0(y). \end{cases} \quad (2.56)$$

If  $U_0(\xi, y) \in H^5(\mathbf{R}^2)$ ,  $U_{0\xi}(\xi, y) \in H_w^5(\mathbf{R}^2)$ ,  $\delta_0(y) \in H^5(\mathbf{R})$ , then there exists  $t_0 > 0$ , which depends only on  $\|U_0\|_5$ ,  $\|U_{0\xi}\|_{H_w^5}$ ,  $\|\delta_0\|_5$  and is chosen sufficiently small such that there exist a unique solution  $U(t, \xi, y)$  to (2.33), (2.28) in  $X_{2M_1}(0, t_0)$  and a unique solution  $\delta(t, y)$  to (2.32), (2.26) in  $Y_{2M_2}(0, t_0)$ , respectively. Here

$$M_1 = \|U_0\|_5^2 + \|U_{0\xi}\|_{H_w^5}^2, \quad M_2 = \|\delta_0\|_5^2.$$

**Proof.** We construct the following sequence of iterating approximate solutions:

$$U^{(0)}(t, \xi, y) = \delta^{(0)}(t, y) \equiv 0$$

and for  $n \geq 1$ ,  $U^{(n)}(t, \xi, y)$  and  $\delta^{(n)}(t, y)$  are the unique solutions to the following linear Cauchy problems

$$\begin{cases} U_t^{(n)} + [sU_{\xi\xi\xi}^{(n)} + \rho_1 U_{\xi\xi y}^{(n)} + \rho_2 U_{\xi y y}^{(n)} + \rho_3 U_{y y y}^{(n)}] - (U_{\xi\xi t}^{(n)} + 2\rho U_{\xi y t}^{(n)} + (1 + \rho^2)U_{y y t}^{(n)}) \\ + \phi'(\xi + \delta^{(n-1)}) \left( \rho_2 \delta_{y y}^{(n-1)} + 3\rho_3 \delta_y^{(n-1)} \delta_{y y}^{(n-1)} - 2\rho \delta_{y t}^{(n-1)} \right. \\ \left. - (1 + \rho^2) \left( 2\delta_y^{(n-1)} \delta_{y t}^{(n-1)} + \delta_t^{(n-1)} \delta_{y y}^{(n-1)} \right) \right) + \phi''(\xi + \delta^{(n-1)}) \left( \rho_1 \delta_y^{(n-1)} + \rho_2 (\delta_y^{(n-1)})^2 \right. \\ \left. + \rho_3 (\delta_y^{(n-1)})^3 - \delta_t^{(n-1)} - 2\rho \delta_y^{(n-1)} \delta_t^{(n-1)} - (1 + \rho^2) (\delta_y^{(n-1)})^2 \delta_t^{(n-1)} \right) \\ + f'(\phi(\xi + \delta^{(n-1)}))U_{\xi}^{(n)} + q(\phi(\xi + \delta^{(n-1)}), U_{\xi}^{(n-1)}) \left( U_{\xi}^{(n-1)} \right)^2 \\ + g(\phi(\xi + \delta^{(n-1)}))\delta_y^{(n-1)} + (g'(\phi(\xi + \delta^{(n-1)}))U^{(n-1)})_y + (V(U^{(n-1)}, \delta^{(n-1)}))_y \\ + \phi'(\xi + \delta^{(n-1)})\delta_y^{(n-1)} (\alpha(U^{(n-1)}, \delta^{(n-1)}) + \beta(U^{(n-1)}, \delta^{(n-1)})) + (W(U^{(n-1)}, \delta^{(n-1)}))_y \\ = U_{\xi\xi}^{(n)} + 2\rho U_{\xi y}^{(n)} + (1 + \rho^2)U_{y y}^{(n)} + \phi'(\xi + \delta^{(n-1)}) \left( 2\rho \delta_y^{(n-1)} + (1 + \rho^2) (\delta_y^{(n-1)})^2 \right), \\ U^{(n)}(t, \xi, y)|_{t=0} = U_0(\xi, y) \end{cases} \quad (2.57)$$

and

$$\begin{cases} \delta_t^{(n)} - (1 + \rho^2)\delta_{y y t}^{(n)} - (1 + \rho^2)\delta_{y y}^{(n)} + \rho_3 \delta_{y y y}^{(n)} + (\alpha_y + \beta_y)(U^{(n-1)}, \delta^{(n-1)}) = 0, \\ \delta^{(n)}(t, y)|_{t=0} = \delta_0(y). \end{cases} \quad (2.58)$$

By employing the standard method, it is easy to show that there exists a sufficiently small positive constant  $t_0 > 0$ , which depends only on  $\|U_0\|_5$ ,  $\|U_{0\xi}\|_{H_w^5}$ , and  $\|\delta_0\|_5$ , such that the approximate sequences  $\{U^n(t, \xi, y)\}$  and  $\{\delta^n(t, y)\}$  are Cauchy sequences in  $X_{2M_1}(0, t_0)$  and  $Y_{2M_2}(0, t_0)$  respectively. The unique limits  $U(t, \xi, y) \in X_{2M_1}(0, t_0)$  and  $\delta(t, y) \in Y_{2M_2}(0, t_0)$  are then the desired local solutions. This completes the proof of Proposition 2.1.  $\square$

### 3. Nonlinear stability of planar viscous shock profiles $\phi(\xi)$

This section is devoted to deducing the nonlinear stability of the planar generalized BBM–Burgers shock profile  $\phi(\xi)$  and our main results can be summarized as in the following

**Theorem 3.1** (Nonlinear Stability Result). Suppose that the assumptions imposed in Lemmas 2.1 and 2.2 hold,  $U_0(\xi, y) \in H^5(\mathbf{R}^2)$ ,  $U_{0\xi}(\xi, y) \in H_w^5(\mathbf{R}^2)$ , and  $\delta_0(y) \in H^5(\mathbf{R})$ . If we assume further that there exists a positive constant  $d_0 > 0$  independent of  $t, \xi, y, \eta = u_- - u_+ > 0$  such that

$$\|U_0\|_5^2 + \|U_{0\xi}\|_{H_w^5}^2 + \eta\|\delta_0\|_5^2 \leq d_0\eta^{p'} \quad (3.1)$$

holds for some  $p' > 1$ , then there exists a sufficiently small  $\eta_0 > 0$  such that if  $0 < \eta \leq \eta_0$ , the Cauchy problems (2.33), (2.28), (2.32) and (2.26) admit unique global solutions  $U(t, \xi, y)$  and  $\delta(t, y)$  which satisfy

$$\begin{aligned} & \|U(t)\|_5^2 + \|U_\xi(t)\|_{H_w^5}^2 + \eta \|\delta(t)\|_5^2 \\ & + \int_0^t \left( \|\nabla U(\tau)\|_4^2 + \|\nabla U_\xi(\tau)\|_{H_w^4}^2 + \|U_t(\tau)\|_4^2 + \|U_{\xi t}(\tau)\|_4^2 + \eta \|\delta_y(\tau)\|_4^2 + \eta \|\delta_t(\tau)\|_4^2 \right) d\tau \\ & \leq C_0 \left( \|U_0\|_5^2 + \|U_{0\xi}\|_{H_w^5}^2 + \eta \|\delta_0\|_5^2 \right) \end{aligned} \quad (3.2)$$

for some positive constant  $C_0$  independent of  $t, \xi, y$  and any  $t \geq 0$ .

Consequently

$$\lim_{t \rightarrow \infty} \left( \|u(t, \xi, y) - \phi(\xi)\|_{W^{3,\infty}(\mathbb{R}^2)} + \|U(t, \xi, y)\|_{W^{3,\infty}(\mathbb{R}^2)} + \|\delta(t, y)\|_{W^{4,\infty}(\mathbb{R})} \right) = 0. \quad (3.3)$$

**Theorem 3.1** will be proved by the continuation argument which is based on the local existence result and certain energy type *a priori* estimates. The local existence result has been established in Section 2 and consequently we need only to derive certain energy type *a priori* estimates on  $U(t, \xi, y)$  and  $\delta(t, y)$ . To this end, suppose that the local solutions  $U(t, \xi, y)$  and  $\delta(t, y)$  obtained in Proposition 2.1 has been extended to the time step  $t = T$ , then to prove Theorem 3.1, all that we need to do is to deduce certain energy type estimates on  $U(t, \xi, y)$  and  $\delta(t, y)$  based on the following *a priori* assumption

$$N(T) = \sup_{0 \leq t \leq T} \left( \|U(t)\|_5^2 + \|U_\xi(t)\|_{H_w^5}^2 + \eta \|\delta(t)\|_5^2 \right) \leq C_1 \eta^{p'}. \quad (3.4)$$

Here  $p' > 1$  is some positive constant and  $C_1 > 0$  is independent of  $T$  and  $\eta$ .

Let  $\bar{p} = p' - 1 > 0$ , we have from the *a priori* assumption (3.4) and the Sobolev inequality that

$$\begin{cases} \|U(t)\|_{W^{3,\infty}(\mathbb{R}^2)}^2 + \|\sqrt{w}U_\xi(t)\|_{W^{3,\infty}(\mathbb{R}^2)}^2 \leq C_2 \eta^{p'}, \\ \sup_{y \in \mathbb{R}} \left\{ \|U(t, \cdot, y)\|_{H^4(\mathbb{R}_\xi)}^2 + \|U_\xi(t, \cdot, y)\|_{H_w^4(\mathbb{R}_\xi)}^2 \right\} \leq C_2 \eta^{p'}, \\ \|\delta(t)\|_{W^{4,\infty}(\mathbb{R})}^2 \leq C_2 \eta^{\bar{p}} \end{cases} \quad (3.5)$$

hold for all  $0 \leq t \leq T$ , where  $C_2$  is some positive constant independent of  $T$  and  $\eta$ .

Based on the *a priori* assumption (3.4), we can deduce the following *a priori* estimates on  $U(t, \xi, y)$  and  $\delta(t, y)$

**Proposition 3.1** (A Priori Estimates). Suppose that  $U(t, \xi, y), U_\xi(t, \xi, y) \in X_{C_2 \eta^{p'}}([0, T])$ ,  $\delta(t, y) \in Y_{C_2 \eta^{\bar{p}}}([0, T])$ . Then there exists a sufficiently small constant  $\eta_1 > 0$  such that if  $0 < \eta \leq \eta_1$ , the estimate (3.2) holds for all  $t \in [0, T]$  and some positive constant  $C_0 > 0$  independent of  $T$  and  $\eta$ .

**Proposition 3.1** will be proven by the following a series of lemmas. The first one deals with the basic energy estimate on  $U(t, \xi, y)$

**Lemma 3.1.** Under the assumptions given in Proposition 3.1, there exist a constant  $C_3 > 0$ , which is independent of  $T, \xi, y$  and  $\eta$ , and a sufficiently small constant  $\eta_2 > 0$  such that for any  $0 < \eta \leq \eta_2$  and  $t \in [0, T]$ ,  $U(t, \xi, y)$  satisfies

$$\|U(t)\|_1^2 + \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|^2 \right) d\tau \leq C_3 \left( \|U_0\|_1^2 + \eta \int_0^t (\|\delta_y(\tau)\|^2 + \|\delta_t(\tau)\|_1^2) d\tau \right). \quad (3.6)$$

**Proof.** Multiplying (2.33) by  $U(t, \xi, y)$  and integrating the resulting equation with respect to  $\xi, y, t$  over  $\mathbb{R}^2 \times [0, t]$  yield

$$\begin{aligned} & \iint A_1(t, \xi, y) d\xi dy + \int_0^t \iint A_2(\tau, \xi, y) d\xi dy d\tau \\ & + \underbrace{\int_0^t \iint U \phi' [\rho_2 \delta_{yy} + 3\rho_3 \delta_y \delta_{yy} - 2\rho \delta_{yt} - (1 + \rho^2)(2\delta_y \delta_{yt} + \delta_t \delta_{yy})] d\xi dy d\tau}_{I_{1,1}} \\ & + \underbrace{\int_0^t \iint U \phi'' [\rho_1 \delta_y + \rho_2 \delta_y^2 + \rho_3 \delta_y^3 - \delta_t - 2\rho \delta_y \delta_t - (1 + \rho^2) \delta_y^2 \delta_t] d\xi dy d\tau}_{I_{1,2}} \\ & + \underbrace{\int_0^t \iint f'(\phi) U U_\xi d\xi dy d\tau}_{I_{1,3}} + \underbrace{\int_0^t \iint U (q U_\xi^2 + g(\phi) \delta_y + (g'(\phi) U)_y) d\xi dy d\tau}_{I_{1,4}} \end{aligned}$$

$$\begin{aligned}
 & + \underbrace{\int_0^t \iint U \phi' \delta_y (\alpha + \beta) \, d\xi dy d\tau}_{I_{1,5}} - \underbrace{\int_0^t \iint U_y (V + W) \, d\xi dy d\tau}_{I_{1,6}} \\
 & = \iint A_1(0, \xi, y) \, d\xi dy + \underbrace{\int_0^t \iint U (2\rho \phi' \delta_y + (1 + \rho^2) \phi' \delta_y^2) \, d\xi dy d\tau}_{I_{1,7}}.
 \end{aligned} \tag{3.7}$$

Here

$$A_1 = \frac{1}{2} (U^2 + U_\xi^2 + 2\rho U_\xi U_y + (1 + \rho^2) U_y^2), \quad A_2 = U_\xi^2 + 2\rho U_\xi U_y + (1 + \rho^2) U_y^2.$$

Noticing that  $U_\xi^2 + 2\rho U_\xi U_y + (1 + \rho^2) U_y^2$  is a positive definite quadratic form, there exists a positive constant  $0 < b < 1$  such that

$$U_\xi^2 + 2\rho U_\xi U_y + (1 + \rho^2) U_y^2 \geq b (U_\xi^2 + U_y^2),$$

then it is easy to get that

$$\begin{aligned}
 A_1 & \geq \frac{1}{2} (U^2 + b (U_\xi^2 + U_y^2)), \\
 A_2 & \geq b (U_\xi^2 + U_y^2).
 \end{aligned} \tag{3.8}$$

We are now turning to estimate  $I_{1,1}$ – $I_{1,7}$  term by term. Firstly, we have from [Corollary 2.1](#) and some integrations by parts that

$$\begin{aligned}
 |I_{1,1}| & \leq \frac{a}{20} \int_0^t \left\| \sqrt{\psi} U(\tau) \right\|^2 d\tau + \frac{b}{20} \int_0^t \|U_y(\tau)\|^2 d\tau \\
 & \quad + O(1) \eta (\sup |\delta_y|^2 + \sup |\delta_t|^2 + \sup |U|^2 + 1) \int_0^t (\|\delta_y(\tau)\|^2 + \|\delta_{yt}(\tau)\|^2) d\tau.
 \end{aligned} \tag{3.9}$$

For the terms  $I_{1,2}$  and  $I_{1,3}$ , it is easy to see that

$$|I_{1,2}| \leq \frac{a}{20} \int_0^t \left\| \sqrt{\psi} U(\tau) \right\|^2 d\tau + O(1) \eta (\sup |\delta_y|^4 + 1) \int_0^t (\|\delta_y(\tau)\|^2 + \|\delta_t(\tau)\|^2) d\tau \tag{3.10}$$

and

$$I_{1,3} = -\frac{1}{2} \int_0^t \iint f''(\phi) \phi' U^2 \, d\xi dy d\tau \geq \frac{a}{2} \int_0^t \left\| \sqrt{\psi} U(\tau) \right\|^2 d\tau. \tag{3.11}$$

Here we have used the assumption [\(1.3\)](#).

Secondly, from [\(2.22\)](#) and the *a priori* assumption [\(3.4\)](#) and its consequence [\(3.5\)](#), we can estimate  $I_{1,4}$  as follows

$$\begin{aligned}
 |I_{1,4}| & \leq \sup |U| \sup |q| \int_0^t \|U_\xi(\tau)\|^2 d\tau \\
 & \quad + O(1) \eta \int_0^t \iint |U \delta_y \psi| \, d\xi dy d\tau + \frac{1}{2} \int_0^t \iint |U^2 g''(\phi) \delta_y \psi| \, d\xi dy d\tau \\
 & \leq \sup |U| \sup |q| \int_0^t \|U_\xi(\tau)\|^2 d\tau + O(1) (1 + \sup |\delta_y|) \eta \int_0^t \left( \|\delta_y(\tau)\|^2 + \left\| \sqrt{\psi} U(\tau) \right\|^2 \right) d\tau.
 \end{aligned} \tag{3.12}$$

And from [Lemma 2.6](#),  $I_{1,5}$  and  $I_{1,6}$  can be controlled as

$$\begin{aligned}
 |I_{1,5}| & \leq (\eta^{-1} \sup |U_\xi|^2 + \eta) \sup_{y \in \mathbf{R}} \left\{ \int U_\xi^2 \, d\xi \right\} \int_0^t \left( \left\| \sqrt{\psi} U(\tau) \right\|^2 + \|U_\xi(\tau)\|^2 \right) d\tau \\
 & \quad + \sup |U|^2 \int_0^t \|\delta_y(\tau)\|^2 d\tau,
 \end{aligned} \tag{3.13}$$

$$\begin{aligned}
 |I_{1,6}| & \leq \frac{b}{20} \int_0^t \|U_y(\tau)\|^2 d\tau + O(1) \eta \int_0^t \left\| \sqrt{\psi} U(\tau) \right\|^2 d\tau \\
 & \quad + O(1) (\eta^{-1} \sup |U_\xi|^2 + 1) \sup_{y \in \mathbf{R}} \left\{ \int w U_\xi^2 \, d\xi \right\} \int_0^t \|U_\xi(\tau)\|^2 d\tau.
 \end{aligned} \tag{3.14}$$

Finally, similar to the estimates above, we have

$$|I_{1,7}| \leq \frac{a}{20} \int_0^t \left\| \sqrt{\psi} U(\tau) \right\|^2 d\tau + O(1)\eta \left( 1 + \sup |\delta_y|^2 \right) \int_0^t \|\delta_y(\tau)\|^2 d\tau. \quad (3.15)$$

Inserting (3.8)–(3.15) into (3.7), (3.6) follows immediately from the *a priori* assumption (3.4), the Sobolev inequality provided that we choose  $\eta > 0$  sufficiently small and this proves Lemma 3.2.  $\square$

To control the terms appeared in the right hand side of (3.6), we have the following estimates on  $\delta(t, y)$

**Lemma 3.2.** *Under the assumptions given in Proposition 3.1, there exist a positive constant  $C_4 > 0$ , which is independent of  $T, \xi, y$  and  $\eta$ , and a sufficiently small constant  $\eta_3 > 0$  such that for any  $\eta < \eta_3$  and  $t \in [0, T]$ ,  $\delta(t, y)$  satisfies*

$$\|\delta(t)\|_1^2 + \int_0^t \|\delta_y(\tau)\|^2 d\tau \leq C_4 \left( \|\delta_0\|_1^2 + \eta^{2\bar{p}} \int_0^t \left( \left\| \sqrt{\psi} U(\tau) \right\|^2 + \|U_\xi(\tau)\|^2 \right) d\tau \right). \quad (3.16)$$

**Proof.** Multiplying (2.32) by  $2\delta(t, y)$  and integrating the resulting equation with respect to  $t, y$  over  $[0, t] \times \mathbf{R}$ , we can get that

$$\begin{aligned} \|\delta(t)\|_1^2 + \int_0^t \|\delta_y(\tau)\|^2 d\tau &\leq \|\delta_0\|_1^2 + \int_0^t \int (\alpha^2 + \beta^2) dy d\tau \\ &\leq O(1) \left( \|\delta_0\|_1^2 + \left( \eta + \eta^{-2} \sup |U_\xi|^2 \sup_{y \in \mathbf{R}} \int U_\xi^2 d\xi \right) \int_0^t \left( \left\| \sqrt{\psi} U(\tau) \right\|^2 + \|U_\xi(\tau)\|^2 \right) d\tau \right). \end{aligned} \quad (3.17)$$

Here we have used (2.36), (2.37) in Lemma 2.6.

Now the *a priori* assumption (3.4) and its consequence (3.5) tell us that

$$\eta^{-2} \sup |U_\xi|^2 \sup_{y \in \mathbf{R}} \int w U_\xi^2 d\xi \leq O(1)\eta^{-2} \|U_\xi(t)\|_2^2 \|U_\xi(t)\|_{H_w^1}^2 \leq O(1)\eta^{2\bar{p}}. \quad (3.18)$$

Inserting (3.18) into (3.17) yields (3.16) and this completes the proof of Lemma 3.2.  $\square$

Combining Lemma 3.1 with Lemma 3.2, we have

**Lemma 3.3.** *Under the assumptions listed in Proposition 3.1, there exists a sufficiently small constant  $\eta_4 > 0$  such that for any  $\eta < \eta_4$  and  $t \in [0, T]$ ,  $U(t, \xi, y)$ ,  $\delta(t, y)$  satisfy*

$$\begin{aligned} \|U(t)\|_1^2 + \eta \|\delta(t)\|_1^2 + \int_0^t \left( \left\| \sqrt{\psi} U(\tau) \right\|^2 + \|\nabla U(\tau)\|^2 + \eta \|\delta_y(\tau)\|^2 \right) d\tau \\ \leq C_5 \left( \|U_0\|_1^2 + \eta \int_0^t \|\delta_t(\tau)\|_1^2 d\tau \right), \end{aligned} \quad (3.19)$$

where the constant  $C_5 > 0$  is independent of  $T$  and  $\eta$ .

Next we turn to deduce the desired higher-order energy type estimates on  $U_\xi(t, \xi, y)$  and  $\delta(t, y)$ . To this end, we have from (2.25), (2.29) and (2.32) that

$$\begin{aligned} U_{\xi t} + [sU_{\xi\xi\xi\xi} + \rho_1 U_{\xi\xi\xi y} + \rho_2 U_{\xi\xi y y} + \rho_3 U_{\xi y y y}] - (U_{\xi\xi\xi t} + 2\rho U_{\xi\xi y t} + (1 + \rho^2) U_{\xi y y t}) \\ - \phi'(\alpha_y + \beta_y) + [\rho_1 \phi''' \delta_y + \rho_2 (\phi''' \delta_y^2 + \phi'' \delta_{yy}) + \rho_3 (\phi''' \delta_y^3 + 3\phi'' \delta_y \delta_{yy})] \\ - [\phi''' \delta_t + 2\rho (\phi''' \delta_y \delta_t + \phi'' \delta_{yt}) + (1 + \rho^2) (\phi''' \delta_y^2 \delta_t + 2\phi'' \delta_y \delta_{yt} + \phi' \delta_t \delta_{yy})] \\ + (f'(\phi) U_\xi)_\xi + (q U_\xi^2)_\xi + g'(\phi) \phi' \delta_y + (g'(\phi) U_\xi)_y + (r U_\xi^2)_y \\ = U_{\xi\xi\xi} + 2\rho U_{\xi\xi y} + (1 + \rho^2) U_{\xi y y} + 2\rho \phi'' \delta_y + (1 + \rho^2) \phi'' \delta_y^2. \end{aligned} \quad (3.20)$$

And we can obtain the following estimate on  $U_\xi(t, \xi, y)$

**Lemma 3.4.** *Under the assumptions stated in Proposition 3.1, there exist a positive constant  $C_6 > 0$  and a sufficiently small constant  $\eta_5 > 0$  such that for any  $\eta < \eta_5$  and  $t \in [0, T]$ ,  $U(t, \xi, y)$  satisfies*

$$\|U_\xi(t)\|_1^2 + \int_0^t \|\nabla U_\xi(\tau)\|^2 d\tau \leq C_6 \left( \|U_0\|_1^2 + \|U_{0\xi}\|_1^2 + \eta \|\delta_0\|_1^2 + \eta \int_0^t \|\delta_t(\tau)\|_1^2 d\tau \right). \quad (3.21)$$

Here  $C_6$  is independent of  $T$  and  $\eta$ .

**Proof.** Multiplying (3.20) by  $U_\xi(t, \xi, y)$  and integrating the resulting equation with respect to  $\xi, y, t$  over  $\mathbf{R}^2 \times [0, t]$  yield

$$\begin{aligned} & \int \int A_3(t, \xi, y) d\xi dy + \underbrace{\int_0^t \int \int A_4(\tau, \xi, y) d\xi dy d\tau + \int_0^t \int \int (\phi'' \delta_y U_{\xi y} + \phi' U_\xi (\alpha + \beta)) d\xi dy d\tau}_{I_{2,1}} \\ & + \underbrace{\int_0^t \int \int U_\xi \phi''' [\rho_1 \delta_y + \rho_2 \delta_y^2 + \rho_3 \delta_y^3 - \delta_t - 2\rho \delta_y \delta_t - (1 + \rho^2) \delta_y^2 \delta_t] d\xi dy d\tau}_{I_{2,2}} \\ & + \underbrace{\int_0^t \int \int U_\xi \phi'' [\rho_2 \delta_{yy} + 3\rho_3 \delta_y \delta_{yy} - 2\rho \delta_{yt} - (1 + \rho^2)(2\delta_y \delta_{yt} + \delta_t \delta_{yy})] d\xi dy d\tau}_{I_{2,3}} \\ & - \underbrace{\int_0^t \int \int U_{\xi\xi} (f'(\phi) U_\xi + q U_\xi^2 + g(\phi) \delta_y) d\xi dy d\tau}_{I_{2,4}} - \underbrace{\int_0^t \int \int U_{\xi y} (g'(\phi) U_\xi + r U_\xi^2) d\xi dy d\tau}_{I_{2,5}} \\ & = \underbrace{\int \int A_3(0, \xi, y) d\xi dy + \int_0^t \int \int U_\xi (2\rho \phi'' \delta_y + (1 + \rho^2) \phi'' \delta_y^2) d\xi dy d\tau}_{I_{2,6}}. \end{aligned} \quad (3.22)$$

Here

$$\begin{cases} A_3 = \frac{1}{2} (U_\xi^2 + U_{\xi\xi}^2 + 2\rho U_{\xi\xi} U_{\xi y} + (1 + \rho^2) U_{\xi y}^2), \\ A_4 = U_{\xi\xi}^2 + 2\rho U_{\xi\xi} U_{\xi y} + (1 + \rho^2) U_{\xi y}^2. \end{cases}$$

Similar to that of (3.8), we can obtain that

$$\begin{aligned} A_3 & \geq \frac{1}{2} (U_\xi^2 + b(U_{\xi\xi}^2 + U_{\xi y}^2)), \\ A_4 & \geq b (U_{\xi\xi}^2 + U_{\xi y}^2). \end{aligned} \quad (3.23)$$

Now we estimate the terms  $I_{2,1} - I_{2,6}$  term by term. Firstly, by Lemma 2.4, (2.36), (2.37) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |I_{2,1}| & \leq \frac{b}{20} \int_0^t \|U_{\xi y}(\tau)\|^2 d\tau + O(1) \int_0^t \int \int (\psi^2(\alpha + \beta)^2 + \eta(U_\xi^2 + \|\delta_y(\tau)\|^2)) d\xi dy d\tau \\ & \leq \frac{b}{20} \int_0^t \|U_{\xi y}(\tau)\|^2 d\tau + O(1) (\eta + \sup |U_\xi|^2) \int_0^t \left( \|\sqrt{\psi} U(\tau)\|^2 + \|U_\xi(\tau)\|^2 + \|\delta_y(\tau)\|^2 \right) d\tau. \end{aligned} \quad (3.24)$$

And it is easy to see that

$$|I_{2,2}| \leq O(1) \eta \int_0^t (\|U_\xi(\tau)\|^2 + \|\delta_y(\tau)\|^2 + \|\delta_t(\tau)\|^2) d\tau. \quad (3.25)$$

Here we have used the *a priori* assumption (3.4) and the Sobolev inequality  $\sup |\delta_y| \leq \|\delta\|_2$ .

As for  $I_{2,3}$ , by the Cauchy-Schwarz inequality, we can get by integrating it by parts that

$$|I_{2,3}| \leq \frac{b}{20} \int_0^t \|U_{\xi y}(\tau)\|^2 d\tau + O(1) \eta \int_0^t (\|U_\xi(\tau)\|^2 + \|\delta_y(\tau)\|^2 + \|\delta_{yt}(\tau)\|^2) d\tau. \quad (3.26)$$

On the other hand,  $I_{2,4} - I_{2,6}$  can be controlled as follows

$$|I_{2,4}| \leq \frac{b}{20} \int_0^t \|U_{\xi\xi}(\tau)\|^2 d\tau + O(1) (\eta^2 + \sup |U_\xi|^2) \int_0^t (\|\delta_y(\tau)\|^2 + \|U_\xi(\tau)\|^2) d\tau, \quad (3.27)$$

$$|I_{2,5}| \leq \frac{b}{20} \int_0^t \|U_{\xi y}(\tau)\|^2 d\tau + O(1) (\eta^2 + \sup |U_\xi|^2) \int_0^t \|U_\xi(\tau)\|^2 d\tau, \quad (3.28)$$

$$|I_{2,6}| \leq O(1)\eta \int_0^t (\|\delta_y(\tau)\|^2 + \|U_\xi(\tau)\|^2) d\tau. \quad (3.29)$$

Putting (3.22)–(3.29) together, (3.21) follows from the *a priori* assumption (3.4) and (3.19). Lemma 3.4 is proved.  $\square$

The next lemma is to bound  $\sup |\delta_y|$  and we can get the following result

**Lemma 3.5.** *Under the assumptions stated in Proposition 3.1, there exist a  $T$  and  $\eta$  independent positive constant  $C_7$  and a sufficiently small constant  $\eta_5 > 0$  such that for any  $\eta < \eta_5$  and  $t \in [0, T]$ ,  $\delta(t, y)$  satisfies*

$$\begin{aligned} \|U(t)\|_1^2 + \|U_\xi(t)\|_1^2 + \eta \|\delta(t)\|_2^2 + \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_t(\tau)\|_1^2) \right) d\tau \\ \leq C_7 (\eta \|\delta_0\|_2^2 + \|U_0\|_1^2 + \|U_{0\xi}\|_1^2). \end{aligned} \quad (3.30)$$

**Proof.** Multiplying (2.32) by  $-2\delta_{yy}(t, y)$  and integrating the resulting equation with respect to  $y, t$  over  $\mathbf{R} \times [0, t]$ , we have from the *a priori* assumption (3.4), (2.38) and (2.39) that

$$\begin{aligned} \|\delta_y(t)\|_1^2 + \int_0^t \|\delta_{yy}(\tau)\|^2 d\tau &\leq \|\delta_{0y}\|_1^2 + \int_0^t \int (\alpha_y^2 + \beta_y^2) dy d\tau \\ &\leq O(1) \left( \|\delta_{0y}\|_1^2 + \eta^{2\bar{p}} \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|^2 + \|U_{\xi y}(\tau)\|^2 \right) d\tau \right). \end{aligned} \quad (3.31)$$

Similarly, we multiply (2.32) by  $2\delta_t(t, y)$  and integrate the resultant equation with respect to  $y, t$  over  $\mathbf{R} \times [0, t]$  to get

$$\begin{aligned} \|\delta_y(t)\|^2 + \int_0^t \|\delta_t(\tau)\|_1^2 d\tau &\leq (1 + \rho^2) \|\delta_{0y}\|^2 + \rho_3^2 \int_0^t \|\delta_{yy}(\tau)\|^2 d\tau + \int_0^t \int (\alpha_y^2 + \beta_y^2) dy d\tau \\ &\leq O(1) \left( \|\delta_{0y}\|_1^2 + \eta^{2\bar{p}} \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|^2 + \|U_{\xi y}(\tau)\|^2 + \|\delta_{yy}(\tau)\|^2 \right) d\tau \right), \end{aligned} \quad (3.32)$$

which together with (3.19), (3.21) and (3.31) derives (3.30) if we choose  $\eta$  sufficiently small. This completes the proof of Lemma 3.5.  $\square$

On the other hand, we multiply (2.33) by  $-2U_{yy}(t, \xi, y)$ , integrate the resultant equation with respect to  $\xi, y, t$  over  $\mathbf{R}^2 \times [0, t]$  and obtain by exploiting similar arguments used in Lemma 3.4 that

$$\begin{aligned} \|U_y(t)\|_1^2 + \int_0^t \|\nabla U_y(\tau)\|^2 d\tau \\ \leq O(1) \left( \|U_{0y}\|_1^2 + \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|^2 + \|(V_y, W_y)(\tau)\|^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_{yt}(\tau)\|^2) \right) d\tau \right) \\ \leq C_8 (\|U_0\|_2^2 + \eta \|\delta_0\|_2^2), \end{aligned} \quad (3.33)$$

where we have used (2.46), (2.47) and (3.30).

The combination of (3.30) and (3.33) yields

$$\begin{aligned} \|U(t)\|_2^2 + \eta \|\delta(t)\|_2^2 + \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|_1^2 + \|U_{yt}(\tau)\|_1^2 \right) d\tau \\ \leq O(1) \left( \|U_{0y}\|_1^2 + \int_0^t \left( \|\sqrt{\psi}U(\tau)\|^2 + \|\nabla U(\tau)\|_1^2 + \eta \|\delta_y(\tau)\|_1^2 + \eta \|\delta_t(\tau)\|_1^2 \right) d\tau \right) \\ \leq C_9 (\|U_0\|_2^2 + \eta \|\delta_0\|_2^2). \end{aligned} \quad (3.34)$$

Having obtained the above estimates, it is easy to get by multiplying (2.33) by  $U_t(t, \xi, y)$  that

$$\begin{aligned} \int_0^t \|U_t(\tau)\|_1^2 d\tau &\leq O(1) \int_0^t \left( \|\nabla U(\tau)\|_1^2 + \|\sqrt{\psi}U(\tau)\|^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_t(\tau)\|_1^2) \right) d\tau \\ &\leq C_{10} (\|U_0\|_2^2 + \eta \|\delta_0\|_2^2). \end{aligned} \quad (3.35)$$

In order to bound  $\sup_{y \in \mathbf{R}} \int_{\mathbf{R}} w U_\xi^2 d\xi$ , we need to estimate  $\|\sqrt{w}U_\xi(t)\|_2$  and the following lemma is concerned with the estimate on  $\|U_\xi(t)\|_{H_w^1}$ .

**Lemma 3.6.** Under the assumptions given in Proposition 3.1, there exist a positive constant  $C_{11}$  and a sufficiently small constant  $\eta_6 > 0$  such that if  $\eta \leq \eta_6$ , we have for any  $t \in [0, T]$  that  $U(t, \xi, y)$  verifies

$$\|U_\xi(t)\|_{H_w^1}^2 + \int_0^t \|\nabla U_\xi(\tau)\|_{L_w^2}^2 d\tau \leq C_{11} \left( \|U_{0\xi}\|_{H_w^1}^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2 \right). \quad (3.36)$$

**Proof.** Noticing that  $|w'(\xi)| \leq 1$ ,  $|w''(\xi)| \leq 1$ , we have by multiplying (3.20) by  $wU_\xi(t, \xi, y)$  and integrating the resulting equation with respect to  $\xi, y, t$  over  $\mathbf{R}^2 \times [0, t]$  that

$$\begin{aligned} & \frac{b}{2} \|U_\xi(t)\|_{H_w^1}^2 + \underbrace{\int_0^t \iint w (\phi'' \delta_y U_\xi + \phi' U_{\xi y}) (\alpha + \beta) d\xi dy d\tau}_{I_{3,1}} \\ & + \underbrace{\int_0^t \iint [U_\xi w (s U_{\xi\xi\xi} + \rho_1 U_{\xi\xi y} + \rho_2 U_{\xi y y} + \rho_3 U_{\xi y y}) - U_{\xi t} (\rho w_\xi U_{\xi y} + (w_\xi U_\xi)_\xi)] d\xi dy d\tau}_{I_{3,2}} \\ & + \underbrace{\int_0^t \iint w U_\xi \phi''' [\rho_1 \delta_y + \rho_2 \delta_y^2 + \rho_3 \delta_y^3 - \delta_t - 2\rho \delta_y \delta_t - (1 + \rho^2) \delta_y^2 \delta_t] d\xi dy d\tau}_{I_{3,3}} \\ & + \underbrace{\int_0^t \iint w U_\xi \phi'' [\rho_2 \delta_{yy} + 3\rho_3 \delta_y \delta_{yy} - 2\rho \delta_{yt} - (1 + \rho^2)(2\delta_y \delta_{yt} + \delta_t \delta_{yy})] d\xi dy d\tau}_{I_{3,3}} \\ & + \underbrace{\int_0^t \iint w U_\xi ((g'(\phi) U_\xi)_y + (f'(\phi) U_\xi + g(\phi) \delta_y)_\xi) d\xi dy d\tau}_{I_{3,4}} \\ & + \underbrace{\int_0^t \iint w U_\xi (q U_\xi^2)_\xi d\xi dy d\tau}_{I_{3,5}} + \underbrace{\int_0^t \iint w U_\xi (r U_\xi^2)_y d\xi dy d\tau}_{I_{3,6}} \\ & \leq \frac{1 + \rho^2}{2} \|U_{0\xi}\|_{H_w^1}^2 - \int_0^t \iint w [U_{\xi\xi}^2 + 2\rho U_{\xi\xi} U_{\xi y} + (1 + \rho^2) U_{\xi y}^2] d\xi dy d\tau \\ & - \underbrace{\int_0^t \iint w_\xi U_\xi [U_{\xi\xi} + \rho U_{\xi y}] d\xi dy d\tau}_{I_{3,7}} + \underbrace{\int_0^t \iint w U_\xi (2\rho \phi'' \delta_y + (1 + \rho^2) \phi'' \delta_y^2) d\xi dy d\tau}_{I_{3,8}}. \end{aligned} \quad (3.37)$$

Now we deal with  $I_{3,1} - I_{3,8}$  term by term. From Lemma 2.3 and the *a priori* assumption (3.4), we get

$$\begin{aligned} |I_{3,1}| & \leq \frac{b}{20} \int_0^t \|\nabla U_\xi(\tau)\|_{L_w^2}^2 d\tau + \sup(w\sqrt{\psi}) \sup|\delta_y| \int_0^t \left( \iint \psi (\alpha + \beta)^2 d\xi dy + \|U_\xi(\tau)\|^2 \right) d\tau \\ & + O(1) \int_0^t \int w \psi^2 d\xi \int (\alpha^2 + \beta^2) dy d\tau \\ & \leq \frac{b}{20} \int_0^t \|\nabla U_\xi(\tau)\|_{L_w^2}^2 d\tau + O(1) \int_0^t \left( \|\sqrt{\psi} U(\tau)\|^2 + \|U_\xi(\tau)\|^2 + \|\delta_y(\tau)\|^2 \right) d\tau. \end{aligned} \quad (3.38)$$

For  $I_{3,2} - I_{3,4}$ , we get by integrations by parts and the Cauchy-Schwarz inequality that

$$|I_{3,2}| \leq O(1) \int_0^t (\|U_\xi(\tau)\|_1^2 + \|U_{\xi t}(\tau)\|^2) d\tau, \quad (3.39)$$

$$|I_{3,3}| \leq O(1) \int_0^t \left[ \|U_\xi(\tau)\|_{L_w^2}^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_t(\tau)\|_1^2) \right] d\tau, \quad (3.40)$$

$$|I_{3,4}| \leq O(1) \eta \int_0^t \|\nabla U_\xi(\tau)\|_{L_w^2}^2 d\tau + O(1) \int_0^t (\|U_\xi(\tau)\|^2 + \eta \|\delta_y(\tau)\|^2) d\tau. \quad (3.41)$$

We will use the same method as in [4] to control the term  $I_{3,5}$ . Firstly, it is easy to see that

$$I_{3,5} = - \int_0^t \iint w_\xi U_\xi^3 q \, d\xi dy d\tau - \int_0^t \iint w U_\xi^2 U_{\xi\xi} q \, d\xi dy d\tau. \quad (3.42)$$

To estimate the second term on the right hand side of (3.42), we set  $Q(\phi, U_\xi) = \int_0^{U_\xi} q(\phi, v) v^2 \, dv$ . Then  $Q_{U_\xi}(\phi, U_\xi) = q U_\xi^2$ ,  $|Q_\phi(\phi, U_\xi)| \leq O(1)|U_\xi|^3$  and consequently

$$\begin{aligned} \left| \int_0^t \iint w U_\xi^2 U_{\xi\xi} q \, d\xi dy d\tau \right| &= \left| \int_0^t \iint w Q_\xi(\phi, U_\xi) \, d\xi dy d\tau - \int_0^t \iint w Q_\phi(\phi, U_\xi) \phi' \, d\xi dy d\tau \right| \\ &= \left| \int_0^t \iint w_\xi Q(\phi, U_\xi) \, d\xi dy d\tau + \int_0^t \iint w Q_\phi(\phi, U_\xi) \phi' \, d\xi dy d\tau \right| \\ &\leq O(1) \sup |U_\xi| \int_0^t \|U_\xi(\tau)\|^2 \, d\tau, \end{aligned} \quad (3.43)$$

which together with (3.42) gives

$$|I_{3,5}| \leq O(1) \sup |U_\xi| \int_0^t \|U_\xi(\tau)\|^2 \, d\tau. \quad (3.44)$$

Similarly,  $I_{3,6}$  can be estimated as

$$|I_{3,6}| \leq O(1) \sup |U_\xi| \sup |\delta_y| \int_0^t \|U_\xi(\tau)\|^2 \, d\tau. \quad (3.45)$$

Finally, for  $I_{3,7}$  and  $I_{3,8}$ , we can deduce that

$$|I_{3,7}| \leq O(1) \int_0^t \|U_\xi(\tau)\|_1^2 \, d\tau, \quad (3.46)$$

$$|I_{3,8}| \leq O(1) \int_0^t (\|U_\xi(\tau)\|^2 + \eta \|\delta_y(\tau)\|^2) \, d\tau. \quad (3.47)$$

Substituting (3.38)–(3.41) and (3.44)–(3.47) into (3.37), (3.36) follows from the *a priori* assumption (3.4), (3.34) and Lemma 3.5. Lemma 3.6 is proved.  $\square$

Now we continue with the estimates on  $\|U_{\xi\xi}(t)\|_{H_w^1}$

**Lemma 3.7.** *Under the assumptions given in Proposition 3.1, there exist a positive constant  $C_{12}$  and a sufficiently small constant  $\eta_7 > 0$  such that for any  $\eta < \eta_7$  and  $t \in [0, T]$ ,  $U(t, \xi, y)$  verifies*

$$\begin{aligned} &\|U_{\xi\xi}(t)\|_{H_w^1}^2 + \int_0^t \|\nabla U_{\xi\xi}(\tau)\|_{L_w^2}^2 \, d\tau \\ &\leq C_{12} \left( \|U_{0\xi}\|_{H_w^2}^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2 + \int_0^t (\|\nabla U_{\xi\xi}(\tau)\|^2 + \|U_{\xi\xi t}(\tau)\|^2) \, d\tau \right). \end{aligned} \quad (3.48)$$

**Proof.** Differentiating (3.20) with respect to  $\xi$ , multiplying it by  $w U_{\xi\xi}(t, \xi, y)$  and integrating the resultant equation with respect to  $\xi, y, t$  over  $\mathbf{R}^2 \times [0, t]$ , we have

$$\begin{aligned} &\frac{b}{2} \|U_{\xi\xi}(t)\|_{H_w^1}^2 - \underbrace{\int_0^t \iint w U_{\xi\xi} \phi'' (\alpha_y + \beta_y) \, d\xi dy d\tau}_{I_{4,1}} \\ &+ \underbrace{\int_0^t \iint [U_{\xi\xi} w (s U_{\xi\xi\xi\xi\xi} + \rho_1 U_{\xi\xi\xi\xi y} + \rho_2 U_{\xi\xi\xi y y} + \rho_3 U_{\xi\xi y y y}) - U_{\xi\xi t} (\rho w_\xi U_{\xi\xi y} + (w_\xi U_{\xi\xi})_\xi)] \, d\xi dy d\tau}_{I_{4,2}} \\ &+ \underbrace{\int_0^t \iint w U_{\xi\xi} \phi^{(4)} [\rho_1 \delta_y + \rho_2 \delta_y^2 + \rho_3 \delta_y^3 - \delta_t - 2\rho \delta_y \delta_t - (1 + \rho^2) \delta_y^2 \delta_t] \, d\xi dy d\tau}_{I_{4,3}} \end{aligned}$$

$$\begin{aligned}
 & + \underbrace{\int_0^t \int \int w U_{\xi\xi} \phi''' [\rho_2 \delta_{yy} + 3\rho_3 \delta_y \delta_{yy} - 2\rho \delta_{yt} - (1 + \rho^2)(2\delta_y \delta_{yt} + \delta_t \delta_{yy})] d\xi dy d\tau}_{I_{4,3}} \\
 & + \underbrace{\int_0^t \int \int w U_{\xi\xi} (f'(\phi) U_\xi + q U_\xi^2 + g(\phi) \delta_y)_{\xi\xi} d\xi dy d\tau}_{I_{4,4}} + \underbrace{\int_0^t \int \int w U_{\xi\xi} (g'(\phi) U_\xi + r U_\xi^2)_{\xi y} d\xi dy d\tau}_{I_{4,5}} \\
 & \leq \frac{1 + \rho^2}{2} \|U_{0\xi\xi}\|_{H_w^1}^2 - \int_0^t \int \int w [U_{\xi\xi\xi}^2 + 2\rho U_{\xi\xi\xi} U_{\xi\xi y} + (1 + \rho^2) U_{\xi\xi y}^2] d\xi dy d\tau \\
 & - \underbrace{\int_0^t \int \int w_\xi U_{\xi\xi} [U_{\xi\xi\xi} + \rho U_{\xi\xi y}] d\xi dy d\tau}_{I_{4,6}} + \underbrace{\int_0^t \int \int w U_{\xi\xi} (2\rho \phi''' \delta_y + (1 + \rho^2) \phi''' \delta_y^2) d\xi dy d\tau}_{I_{4,7}}. \quad (3.49)
 \end{aligned}$$

From Lemma 2.3, Lemma 2.6 and the *a priori* assumption (3.4), we can obtain

$$\begin{aligned}
 |I_{4,1}| & \leq O(1) \int_0^t \|U_{\xi\xi}(\tau)\|^2 d\tau + O(1) \int_0^t \int w^2 \psi^2 d\xi \int (\alpha_y^2 + \beta_y^2) dy d\tau \\
 & \leq O(1) \int_0^t \left( \|\sqrt{\psi} U(\tau)\|^2 + \|\nabla U(\tau)\|_1^2 \right) d\tau. \quad (3.50)
 \end{aligned}$$

By integrations by parts, we can bound  $I_{4,2}$  by

$$|I_{4,2}| \leq O(1) \int_0^t (\|U_{\xi\xi t}(\tau)\|^2 + \|U_{\xi\xi}(\tau)\|_1^2) d\tau. \quad (3.51)$$

Similarly,  $I_{4,3} - I_{4,7}$  can be controlled as follows

$$|I_{4,3}| \leq O(1) \int_0^t \left( \|U_{\xi\xi}(\tau)\|_{L_w^2}^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_t(\tau)\|_1^2) \right) d\tau, \quad (3.52)$$

$$|I_{4,4}| \leq O(1) \int_0^t \left( \|U_\xi(\tau)\|^2 + \|U_{\xi\xi}(\tau)\|_{L_w^2}^2 + \eta \|\delta_y(\tau)\|^2 \right) d\tau, \quad (3.53)$$

$$|I_{4,5}| \leq \frac{b}{20} \int_0^t \|w U_{\xi\xi y}(\tau)\|^2 d\tau + O(1) \int_0^t \left( \|U_\xi(\tau)\|^2 + \|U_{\xi\xi}(\tau)\|_{L_w^2}^2 \right) d\tau, \quad (3.54)$$

$$|I_{4,6}| \leq O(1) \int_0^t \|U_{\xi\xi}(\tau)\|^2 d\tau, \quad (3.55)$$

and

$$|I_{4,7}| \leq O(1) \int_0^t \left( \|U_{\xi\xi}(\tau)\|_{L_w^2}^2 + \eta \|\delta_y(\tau)\|^2 \right) d\tau. \quad (3.56)$$

Then the desired estimate (3.48) can be derived from (3.30) and substituting (3.50)–(3.56) into (3.49). This completes the proof of Lemma 3.7.  $\square$

To control the terms appeared in the right hand side of (3.48), differentiating (3.20) with respect to  $\xi$ , multiplying it by  $U_{\xi\xi}$  and integrating the resultant equation with respect to  $\xi, y, t$  over  $\mathbf{R}^2 \times [0, t]$ , we have by employing similar arguments used in Lemma 3.7 that

$$\|U_{\xi\xi}(t)\|_1^2 + \int_0^t \|\nabla U_{\xi\xi}(\tau)\|^2 d\tau \leq C_{13} (\|U_{0\xi}\|_2^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2). \quad (3.57)$$

Similarly, we have the following estimates on  $U_{\xi y}$

$$\begin{aligned}
 & \|U_{\xi y}(t)\|_{H_w^1}^2 + \int_0^t \|\nabla U_{\xi y}(\tau)\|_{L_w^2}^2 d\tau \\
 & \leq C_{14} \left( \|U_{0\xi}\|_{H_w^2}^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2 + \int_0^t (\|\nabla U_{\xi y}(\tau)\|^2 + \|\nabla U_{\xi y t}(\tau)\|^2) d\tau \right), \quad (3.58)
 \end{aligned}$$

$$\|U_{\xi y}(t)\|_1^2 + \int_0^t \|\nabla U_{\xi y}(\tau)\|^2 d\tau \leq C_{15} (\|U_{0\xi}\|_2^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2). \quad (3.59)$$

Moreover,

$$\int_0^t \|U_{\xi t}(\tau)\|_1^2 d\tau \leq C_{16} \int_0^t (\|U(\tau)\|_2^2 + \eta (\|\delta_y(\tau)\|_1^2 + \|\delta_t(\tau)\|_1^2)) d\tau. \quad (3.60)$$

Putting (3.48), (3.57)–(3.60) together, we get the following lemma

**Lemma 3.8.** *Under the assumptions given in Proposition 3.1, there exist a positive constant  $C_{16} > 0$  and a sufficiently small constant  $\eta_7 > 0$  such that for any  $\eta < \eta_7$  and  $t \in [0, T]$ ,  $U(t, \xi, y)$  verifies*

$$\|\nabla U_\xi(t)\|_{H_w^1}^2 + \int_0^t (\|\nabla^2 U_\xi(\tau)\|_{L_w^2}^2 + \|U_{\xi t}(\tau)\|_1^2) d\tau \leq C_{17} (\|U_{0\xi}\|_{H_w^2}^2 + \|U_0\|_2^2 + \eta \|\delta_0\|_2^2). \quad (3.61)$$

With the *a priori* assumption (3.4) and the estimates above in hand, we can further derive the following estimates:

$$\begin{aligned} & \left\| \partial_\xi^{k_1+1} \partial_y^{k_2} U(t) \right\|_1^2 + \int_0^t \left\| \nabla \partial_\xi^{k_1+1} \partial_y^{k_2} U(\tau) \right\|^2 d\tau \\ & \leq O(1) \left\| \partial_\xi^{k_1+1} \partial_y^{k_2} U_0 \right\|_1^2 + C(N(t), \eta) \left[ \sum_{k'_1+k'_2 \leq k_1+k_2} \int_0^t \left\| \nabla \partial_\xi^{k'_1+1} \partial_y^{k'_2} U(\tau) \right\|^2 d\tau \right. \\ & \quad \left. + \sum_{k'_2 \leq k_2} \eta \int_0^t \left( \left\| \partial_y^{k'_2+1} \delta(\tau) \right\|^2 + \left\| \partial_y^{k'_2} \delta_t(\tau) \right\|_1^2 \right) d\tau \right], \quad k_2 \leq 4, \quad k_1 + k_2 \leq 5, \end{aligned} \quad (3.62)$$

$$\begin{aligned} \left\| \partial_y^k U(t) \right\|_1^2 + \int_0^t \left\| \nabla \partial_y^k U(\tau) \right\|^2 d\tau & \leq O(1) \left\| \partial_y^k U_0 \right\|_1^2 + C(N(t), \eta) \left[ \sum_{k' \leq k} \int_0^t \left\| \partial_\xi \partial_y^{k'} U(\tau) \right\|^2 d\tau \right. \\ & \quad \left. + \sum_{k' \leq k} \eta \int_0^t \left( \left\| \partial_y^{k'+1} \delta(\tau) \right\|^2 + \left\| \partial_y^{k'} \delta_t(\tau) \right\|_1^2 \right) d\tau \right], \quad k \leq 4, \end{aligned} \quad (3.63)$$

$$\begin{aligned} \int_0^t \left\| \partial_\xi^{k_1} \partial_y^{k_2} U_t(\tau) \right\|_1^2 d\tau & \leq C(N(t), \eta) \left[ \sum_{k'_1+k'_2 \leq k_1+k_2} \int_0^t \left( \left\| \nabla \partial_\xi^{k'_1} \partial_y^{k'_2} U(\tau) \right\|_1^2 + \left\| \sqrt{\psi} U(\tau) \right\|^2 \right) d\tau \right. \\ & \quad \left. + \sum_{k'_2 \leq k_2} \eta \int_0^t \left( \left\| \partial_y^{k'_2+1} \delta(\tau) \right\|^2 + \left\| \partial_y^{k'_2} \delta_t(\tau) \right\|_1^2 \right) d\tau \right], \quad k_2 \leq 3, \quad k_1 + k_2 \leq 4, \end{aligned} \quad (3.64)$$

$$\begin{aligned} & \left\| \partial_\xi^{k_1+1} \partial_y^{k_2} U(t) \right\|_{H_w^1}^2 + \int_0^t \left\| \nabla \partial_\xi^{k_1+1} \partial_y^{k_2} U(\tau) \right\|^2 d\tau \\ & \leq O(1) \left\| \partial_\xi^{k_1+1} \partial_y^{k_2} U_0 \right\|_1^2 + C(N(t), \eta) \left[ \int_0^t \left( \left\| \partial_\xi^{k_1+1} \partial_y^{k_2} U_t(\tau) \right\|^2 + \sum_{k'_1+k'_2 \leq k_1+k_2} \left\| \nabla \partial_\xi^{k'_1+1} \partial_y^{k'_2} U(\tau) \right\|^2 \right) d\tau \right. \\ & \quad \left. + \sum_{k'_2 \leq k_2} \eta \int_0^t \left( \left\| \partial_y^{k'_2+1} \delta(\tau) \right\|^2 + \left\| \partial_y^{k'_2} \delta_t(\tau) \right\|_1^2 \right) d\tau \right], \quad 2 \leq k_1 + k_2 \leq 4, \end{aligned} \quad (3.65)$$

$$\begin{aligned} & \left\| \partial_y^{k_2} \delta(t) \right\|_1^2 + \int_0^t \left\| \partial_y^{k_2+1} \delta(\tau) \right\|^2 d\tau \\ & \leq C(N(t), \eta) \sum_{1 \leq k'_2 \leq k_2} \int_0^t \left( \eta \left( \left\| \partial_y^{k'_2} U(\tau) \right\|^2 + \left\| \partial_\xi \partial_y^{k'_2} U(\tau) \right\|^2 \right) + \left\| \partial_y^{k'_2} \delta(\tau) \right\|^2 \right) d\tau, \quad 2 \leq k_2 \leq 4, \end{aligned} \quad (3.66)$$

and

$$\begin{aligned} \int_0^t \left\| \partial_y^{k_2} \delta_t(\tau) \right\|_1^2 d\tau & \leq C(N(t), \eta) \left[ \sum_{k'_2 \leq k_2} \int_0^t \left( \left\| \partial_y^{k'_2+1} \delta(\tau) \right\|_1^2 \right) d\tau \right. \\ & \quad \left. + \eta \sum_{1 \leq k'_2 \leq k_2} \int_0^t \left( \left\| \partial_\xi \partial_y^{k'_2} U(\tau) \right\|^2 + \left\| \partial_y^{k'_2} U(\tau) \right\|^2 + \left\| \sqrt{\psi} U(\tau) \right\|^2 \right) d\tau \right], \quad 1 \leq k_2 \leq 3. \end{aligned} \quad (3.67)$$

Here  $O(1)$  denotes some positive constant independent of  $T$  and  $\eta$  and  $C(N(t), \eta)$  stands for some polynomial of  $N(T)$  and  $\eta$  satisfying  $C(0, 0) = 0$ .

A suitable linear combination of (3.62)–(3.67) together with (3.34), (3.36) and (3.61) yields that there exists a sufficiently small constant  $\eta_0 > 0$  and a positive constant  $C_0 > 0$ , which are independent of  $T$  such that if  $0 < \eta < \eta_0$  and the *a priori* assumption (3.4) holds, then  $U(t, \xi, y)$ ,  $\delta(t, y)$  satisfy (3.2). This complete the proof of Proposition 3.1.

Now we turn to prove Theorem 3.1. In fact, based on Propositions 2.1 and 3.1, we can exploit the continuation argument to extend the unique local solutions  $U(t, \xi, y)$ ,  $\delta(t, y)$  of the Cauchy problems (2.33), (2.28), (2.32) and (2.26) constructed in Proposition 2.1 step by step to global solutions provided that the initial perturbation  $U_0(\xi, y)$  and  $\delta_0(y)$  are sufficiently small such that (3.1) holds.

As a by-product of the above argument, we can deduce that the global solutions satisfies (3.2) for all  $t \geq 0$ .

To complete the proof of Theorem 3.1, we only need to show that (3.3) holds. For this purpose, we have by Gagliardo–Nirenberg’s inequality that

$$\begin{aligned} \|\partial_\xi^{k_1} \partial_y^{k_2} U(t)\|_{L^\infty} &\leq O(1) \|\partial_\xi^{k_1} \partial_y^{k_2} U(t)\|^{\frac{1}{2}} \|\nabla^2 \partial_\xi^{k_1} \partial_y^{k_2} U(t)\|^{\frac{1}{2}}, \quad k_1 + k_2 \leq 4, \quad k_2 \leq 3, \\ \|\partial_y^k \delta(t)\|_{L^\infty} &\leq \|\partial_y^k \delta(t)\|^{\frac{1}{2}} \|\partial_y^{k+2} \delta(t)\|^{\frac{1}{2}}, \quad 1 \leq k \leq 4. \end{aligned} \quad (3.68)$$

On the other hand, we see from (3.2) that

$$\begin{aligned} \sum_{k=0}^4 \left( \|\nabla^k U_\xi(t)\|_1^2 + \|\nabla^{k+1} U(t)\|^2 + \|\nabla^k U_t(t)\|^2 + \|\nabla^k U_{\xi t}(t)\|^2 \right) &\in L^1(\mathbf{R}^+), \\ \sum_{k=0}^4 \left( \|\partial_y^{k+1} \delta(t)\|^2 + \|\partial_t^k \delta(t)\|^2 \right) &\in L^1(\mathbf{R}^+), \end{aligned} \quad (3.69)$$

which together with (3.68) implies that

$$\lim_{t \rightarrow \infty} \sup_{(\xi, y) \in \mathbf{R}^2} \sum_{k=0}^3 \left( |\nabla^k U(t, \xi, y)| + |\nabla^k U_\xi(t, \xi, y)| + |\partial_y^k \delta(t, y)| + |\partial_y^{k+1} \delta(t, y)| \right) = 0. \quad (3.70)$$

Furthermore, from (3.70), we can deduce that

$$\begin{aligned} \lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbf{R}^2} \sum_{k=0}^3 |\nabla^k(u(t, \xi, y) - \phi(\xi))| \\ \leq \lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbf{R}^2} \sum_{k=0}^3 \left( |\nabla^k(u(t, \xi, y) - \phi(\xi + \delta(t, y)))| + |\partial_y^k(\phi(\xi + \delta(t, y)) - \phi(\xi))| \right) \\ \leq O(1) \lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbf{R}^2} \sum_{k=0}^3 \left( |\nabla^k(u(t, \xi, y) - \phi(\xi + \delta(t, y)))| + |\partial_y^k \delta(t, y)| \right) \\ = 0 \end{aligned} \quad (3.71)$$

and (3.3) follows from (3.70) and (3.71). This completes the proof of Theorem 3.1.

Finally, we show that if (1.12) holds, then the assumptions imposed on  $U_0(\xi, y)$  and  $\delta_0(y)$  in Theorem 3.1 holds. In fact, let  $p' > 1$  be some constant, we see from (2.26) that

$$\begin{aligned} \sum_{\alpha \leq 5} \int |\partial_y^\alpha \delta_0|^2 dy &\leq \eta^{-2} \sum_{\alpha \leq 5} \int \left| \int (\partial_y^\alpha (u_0(x, y) - \phi(x))) dx \right|^2 dy \\ &\leq \eta^{-2} \sum_{\alpha \leq 5} \iint (1+x^2)^{\frac{p'}{2}} (\partial_y^\alpha (u_0(x, y) - \phi(x)))^2 dx dy \int (1+x^2)^{-\frac{p'}{2}} dx \\ &\leq C_1(p') \eta^{-2} \sum_{\alpha \leq 5} \iint (1+x^2)^{\frac{p'}{2}} (\partial_y^\alpha (u_0(x, y) - \phi(x)))^2 dx dy. \end{aligned} \quad (3.72)$$

On the other hand, for  $\|U_{0\xi}\|_{H_w^5}^2$ , we can deduce from (2.24) and (3.72) that

$$\begin{aligned} \sum_{|\alpha| \leq 5} \iint w |\partial^\alpha U_{0x}|^2 dx dy &\leq 2 \sum_{|\alpha| \leq 5} \iint [w |\partial^\alpha (u_0(x, y) - \phi(x))|^2 + w |\partial^\alpha (\phi(x) - \phi(x - \delta_0(y)))|^2] dx dy \\ &\leq 2 \sum_{|\alpha| \leq 5} \iint w |\partial^\alpha (u_0(x, y) - \phi(x))|^2 dx dy + O(1) \sum_{|\alpha| \leq 5} \int w \psi^2(x + \theta \delta_0) dx \int |\partial_y^{|\alpha|} \delta_0(y)|^2 dy \end{aligned}$$

$$\begin{aligned} &\leq O(1) \sum_{|\alpha| \leq 5} \iint w |\partial^\alpha (u_0(x, y) - \phi(x))|^2 dx dy + O(1) \eta^2 \sum_{|\alpha| \leq 5} \int |\partial_y^{|\alpha|} \delta_0(y)|^2 dy \\ &\leq C_2(p') \sum_{|\alpha| \leq 5} \iint (1+x^2)^{\frac{p'}{2}} |\partial^\alpha (u_0(x, y) - \phi(x))|^2 dx dy, \end{aligned} \quad (3.73)$$

where  $\theta \in (0, 1)$ .

Finally, for  $\|U_0\|_5^2$ , we have from Lemma 2.3 and (3.72) that

$$\begin{aligned} &\sum_{|\alpha| \leq 5} \iint |\partial^\alpha U_0|^2 dx dy \\ &\leq 2 \sum_{|\alpha| \leq 5} \left[ \iint_{-\infty}^0 \left| \int_{-\infty}^x \partial^\alpha (u_0(x, y) - \phi(x)) dz \right|^2 dx dy + \iint_0^\infty \left| \int_x^\infty \partial^\alpha (u_0(x, y) - \phi(x)) dz \right|^2 dx dy \right] \\ &\quad + 2 \sum_{|\alpha| \leq 5} \left[ \iint_{-\infty}^0 \left| \int_{-\infty}^x \partial^\alpha (\phi(x) - \phi(x + \delta_0)) dz \right|^2 dx dy \right. \\ &\quad \left. + \iint_0^\infty \left| \int_x^\infty \partial^\alpha (\phi(x) - \phi(x + \delta_0)) dz \right|^2 dx dy \right] \\ &\leq \bar{C}(p') \sum_{|\alpha| \leq 5} \iint_{-\infty}^0 \int_{-\infty}^x (1+z^2)^{\frac{p'}{2}} |\partial^\alpha U_{0z}(z, y)|^2 dz dx dy \\ &\quad + \bar{C}(p') \sum_{|\alpha| \leq 5} \iint_0^\infty \int_x^\infty (1+z^2)^{\frac{p'}{2}} |\partial^\alpha U_{0z}(z, y)|^2 dz dx dy \\ &\quad + O(1) \iint_{-\infty}^0 \left( \int_{-\infty}^x \eta^2 e^{-2C\eta|z|} dz + \int_x^\infty \eta^2 e^{-2C\eta|z|} dz \right) dx dy \sum_{\alpha \leq 5} \int |\partial_y^\alpha \delta_0|^2 dy \\ &\leq C_4(p') \eta^{-1} \sum_{|\alpha| \leq 5} \iint |x|(1+x^2)^{\frac{p'}{2}} |\partial^\alpha (u_0(x, y) - \phi(x))|^2 dx dy. \end{aligned} \quad (3.74)$$

Have obtained (3.72)–(3.74), it is easy to see that if (1.12) hold true, then  $U_0(\xi, y)$  and  $\delta_0(y)$  satisfy all the assumptions listed in Theorem 3.1.

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