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Strong type of Steckin inequality for the linear combination of Bernstein operators

Linsen Xie

Department of Mathematics, Lishui University, Zhejiang, Lishui 323000, PR China

HIGHLIGHTS

- We obtain a new Steckin inequality for the combinations of Bernstein operators.
- This inequality gives the optimal approximation rate.
- We give the strong converse result for the combinations of Bernstein operators.
- This result covers the direct estimate, the converse result and the saturation.

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ABSTRACT

In this paper we obtain a new strong type of Steckin inequality for the linear combinations of Bernstein operators, which gives the optimal approximation rate. Moreover, a method to prove lower estimates for linear operators is introduced. As a result the lower estimate for the linear combinations of Bernstein operators is obtained by using the Ditzian–Totik modulus of smoothness.

1. Introduction and main results

Bernstein operators on $f \in C[0,1]$ are given by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \qquad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

The relation between their rate of approximation and the smoothness of the function approximated has been investigated in many papers. In many cases, such investigations lead to the so-called direct estimate, the converse result and the saturation. Among these investigations, the Berens–Lorentz theorem [2] is an early influential converse result. M. Becker [1], Z. Ditzian [3–5], and V. Totik [12] later proved this type of converse results for various cases. In 1992 D.X. Zhou [19] proved the final case by confirming the conjecture of Z. Ditzian [5]. In 1994, Ditzian [6] obtained the upper pointwise estimate for approximation by the Bernstein operators. The saturation for these operators was found in [11,18]. In 1994, V. Totik [13], H.B. Knoop and X.L. Zhou [10] obtained the strong converse inequality for approximation by the Bernstein operators respectively. In other words, for some constant $C > 0$ independent of f and n, one has

$$C^{-1}\omega_{\varphi}^{2}\left(f, n^{-1/2}\right) \le \|B_n f - f\| \le C\omega_{\varphi}^{2}\left(f, n^{-1/2}\right), \quad \forall f \in C[0,1]. \tag{1.1}$$

Here $\omega_{\varphi}^{2}(f,t)$ denotes the second order modulus of smoothness with weight function $\varphi(x) = (x(1-x))^{1/2}$ (see [7, pp. 7–8] for details), and $\|f\| = \|f\|_{C[0,1]}$.

E-mail address: linsenxie@tom.com.

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Because of the Korovkin theorem (see e.g. [11]), the optimal rate of the convergence for a positive operator cannot be faster than that of C^2 functions. Therefore, in order to obtain more efficient approximation operators one has to consider non-positive operators. To this end, in [7, p. 116] the following combinations of Bernstein operators were introduced for $r \geq 1$ by

$$B_{n,r}(f,x) = \sum_{i=0}^{r-1} c_i(n) B_{n_i}(f,x),$$

where n_i and $c_i(n)$ satisfy

$$\text{(a)}\; n = n_0 < \cdots < n_{r-1} \leq Kn; \qquad \text{(b)}\; \sum_{i=0}^{r-1} |c_i(n)| \leq C;$$

$$\text{(c)}\; \sum_{i=0}^{r-1} c_i(n) = 1; \qquad \text{(d)}\; \sum_{i=0}^{r-1} c_i(n) n_i^{\rho} = 0, \quad \rho = 1, 2, \ldots, r-1.$$

It was shown by Z. Ditzian and V. Totik [7, p. 11] in 1987 that

$$\|B_{n,r}(f) - f\| \leq C\left(\omega_{\varphi}^{2r}\left(f, n^{-1/2}\right) + n^{-r}\|f\|\right), \tag{1.2}$$

and for $0 < \alpha < 2r$

$$\|B_{n,r}(f) - f\| = \mathcal{O}(n^{-\alpha/2}) \Leftrightarrow \omega_{\varphi}^{2r}(f,t) = \mathcal{O}(t^{\alpha}), \tag{1.3}$$

where $\omega_{\varphi}^{2r}(f,t)$ is the $2r$-th modulus of smoothness with the step-weight function $\varphi(x) = \sqrt{x(1-x)}$ (see [7, pp. 7–8]).

The results above are the so-called norm direct and converse results. In [7, p. 122] they also obtained a strong type of Steckin inequality for the combinations of Bernstein operators. In 1995, D.X. Zhou [20] extended the Berens–Lorentz theorem to higher order of smoothness, by proving pointwise direct and converse results for these combinations by the r-th classical modulus of smoothness $\omega_r(f,t)$, where r is the number of terms in the combinations. In 1999 and 2000, L.S. Xie [17] and S.S. Guo et al. [8] respectively obtained similar results as mentioned above for the combinations of Bernstein operators by the $2r$-th Ditzian–Totik modulus of smoothness $\omega_{\varphi}^{2r}(f,t)$. The similar results on pointwise simultaneous approximation by the combinations can be found in [15]. However, the saturation problem for all $r \geq 1$ was first solved in [16]. It is necessary to mention some notations here.

For $k = 1, 2, \ldots$ let

$$a_{j,k} = j a_{j,k-1} + (k-1) a_{j-1,k-2} \tag{1.4}$$

with

$$a_{0,k} = 0, \qquad a_{1,k} = 1, \qquad a_{k,2k} = (2k-1)!!,$$

where $1 < j < [k/2]$ if k is even and $1 < j \leq [k/2]$ otherwise. The differential operators needed are given by

$$P_r(D) = \sum_{j=1}^{r} \frac{a_{j,r+j}}{(r+j)!}(x(1-x))^j (1-2x)^{\delta_{r+j}} D^{r+j},$$

where $\delta_j = 0$ if j is even and $\delta_j = 1$ otherwise. We use these differential operators to define the K-functional, namely,

$$K(f,r,t) = \inf_{g}\left\{\|f-g\| + t^{2r}\|P_r(D)g\| + t^{2r+1}\|\varphi^{2r+1} g^{(2r+1)}\|\right\},$$

where as usual $\varphi(x) = \sqrt{x(1-x)}$ and $g \in C^{2r+1}[0,1]$. We proved in [16, p. 87] the following

Theorem 1.1. *For fixed $r \geq 1$ the following statement is true:*

$$\|B_{n,r}(f) - f\| = \mathcal{O}(n^{-r}) \Longleftrightarrow K(f,r,t) = \mathcal{O}(t^{2r}).$$

In [16] we also studied the problem of whether one can replace $K(f,r,t) = \mathcal{O}(t^{2r})$ by $\omega_{\varphi}^{2r}(f,t) = \mathcal{O}(t^{2r})$. Thus, let

$$\sigma(x) = \frac{a_{1,r+1}}{(r+1)!} + \sum_{j=1}^{r-1} \frac{a_{j+1,r+j+1}}{(r+j+1)!} x(x-1)\cdots(x-j+1).$$

We have (see [16, p. 88])

Theorem 1.2. *Let $r \geq 1$ be fixed. If $-1 \notin \{\operatorname{Re} x : \sigma(x) = 0\}$, then there holds for $f \in C[0,1]$*

$$K(f,r,t) = \mathcal{O}(t^{2r}) \Longleftrightarrow \omega_{\varphi}^{2r}(f,t) = \mathcal{O}(t^{2r}).$$

Let Π_n be a set of algebraic polynomials with degree n, and

$$E_n(f) = \inf_{P \in \Pi_n} \|f - P\|.$$

In this paper we will prove a new strong type of Steckin inequality for $B_{n,r}(f)$, i.e.,

Theorem 1.3. *For fixed $r \geq 1$ there is a constant $C > 0$ such that for $f \in C[0,1]$ and $n = 1, 2, \ldots$*

$$\|B_{n,r}(f) - f\| \leq C\left(K(f, r, n^{-1/2}) + n^{-r}E_r(f)\right) \tag{1.5}$$

and

$$K(f, r, n^{-1/2}) \leq C\left(n^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}\|B_{k,r}(f) - f\| + n^{-r}E_r(f)\right). \tag{1.6}$$

Remark 1. Theorem 1.3 implies the saturation for the combination of Bernstein operators. We know that the classic Steckin inequality for operators does not give an optimal approximation rate, while (1.5) and (1.6) imply the result of Theorem 1.1. In fact, if $K(f, r, t) = \mathcal{O}(t^{2r})$, by (1.5) we have

$$\|B_{n,r}(f) - f\| = \mathcal{O}(n^{-r}).$$

On the other hand, if $\|B_{k,r}(f) - f\| = \mathcal{O}(k^{-r})$, by (1.6) we get

$$\begin{aligned} K(f, r, n^{-1/2}) &\leq C\left(n^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}\cdot k^{-r} + n^{-r}E_r(f)\right) \\ &\leq Cn^{-r}, \end{aligned}$$

which implies $K(f, r, t) = \mathcal{O}(t^{2r})$.

The following theorem gives the so-called strong converse result for approximation by $B_{n,r}(f)$.

Theorem 1.4. *Let $r > 1$ be fixed. If $-1 \notin \{\mathrm{Re}\, x : \sigma(x) = 0\}$, then there holds for all $f \in C[0,1]$ and all $n = 1, 2, \ldots$*

$$\max_{k\geq n}\|B_{k,r}(f) - f\| + n^{-r}E_r(f) \asymp \omega_{\varphi}^{2r}(f, n^{-1/2}) + n^{-r}E_r(f), \tag{1.7}$$

where the symbol $X \asymp Y$ means that there exists a positive constant M independent of n and f such that $M^{-1}Y \leq X \leq MY$.

Remark 2. The equivalence (1.7) covers the direct estimate (1.2), the converse result (1.3) and the saturation (see Theorems 1.1 and 1.2) for norm approximation by these combinations. In fact, from (1.7), we have

$$\|B_{n,r}(f) - f\| \leq \max_{k\geq n}\|B_{k,r}(f) - f\| \leq C\left(\omega_{\varphi}^{2r}\left(f, n^{-1/2}\right) + n^{-r}\|f\|\right),$$

which is the direct norm estimate for $B_{n,r}(f)$. Secondly, if $\omega_{\varphi}^{2r}(f, t) = O(t^{\alpha})$ for $0 < \alpha \leq 2r$, by the above inequality we have

$$\|B_{n,r}(f) - f\| \leq Cn^{-\alpha/2}. \tag{1.8}$$

On the other hand, if $\|B_{n,r}(f) - f\| = O\left(n^{-\alpha/2}\right)$ for $0 < \alpha \leq 2r$, by (1.7) we have

$$\omega_{\varphi}^{2r}\left(f, n^{-1/2}\right) \leq C\left(\max_{k\geq n}\|B_{k,r}(f) - f\| + n^{-r}E_r(f)\right).$$

Setting $\|B_{n_0,r}(f) - f\| = \max_{k\geq n}\|B_{k,r}(f) - f\|$ with $n_0 \geq n$,

$$\begin{aligned} \omega_{\varphi}^{2r}\left(f, n^{-1/2}\right) &\leq C\left(\|B_{n_0,r}(f) - f\| + n^{-r}E_r(f)\right) \\ &\leq Cn^{-\alpha/2}, \end{aligned}$$

which, combining (1.8), implies the norm converse result (the case $0 < \alpha < 2r$) and the saturation (the case $\alpha = 2r$) for $B_{n,r}(f)$.

Remark 3. The so-called Berens–Lorentz type theorems are the pointwise converse results for the operators. The complete characterization given by D.X. Zhou [19] is a case in point. However, as the simple example $f(x) = x^2$, with $B_n(f, x) - f(x) = \frac{x(1-x)}{n}$, but $\omega_{\varphi}^{2}\left(f, n^{-1/2}\right) = \frac{1}{2n}$, shows, it is impossible to have pointwise lower estimate like (1.1) for Bernstein operators. The same is true with the combinations of these operators.

Theorems 1.3 and 1.4 will be proved in Section 2. Throughout this paper, C denotes a positive constant independent of n and x, whose value may be different in different situations.

2. Proof of Theorems 1.3 and 1.4

We begin with the following

Lemma 2.1. *For $P_n \in \Pi_n$ satisfying*

$$\|P_n - f\| \le CE_n(f),$$

we have

$$\|f - P_n\| + n^{-2r}\|P_r(D)P_n\| + n^{-2r-1}\|\varphi^{2r+1}P_n^{(2r+1)}\| \asymp K(f, r, n^{-1}). \tag{2.1}$$

Proof. Clearly, we need only to verify the following three inequalities:

$$\|f - P_n\| \le MK(f, r, n^{-1}), \qquad \|\varphi^{2r+1}P_n^{(2r+1)}\| \le Mn^{2r+1}K(f, r, n^{-1})$$

and

$$\|P_r(D)P_n\| \le Mn^{2r}K(f, r, n^{-1}). \tag{2.2}$$

The first two are evident as

$$\omega_\varphi^{2r+1}(f, t) \le CK(f, r, t), \qquad E_n(f) \le C\omega_\varphi^{2r+1}(f, n^{-1})$$

and

$$\|\varphi^{2r+1}P_n^{(2r+1)}\| \le Cn^{2r+1}\omega_\varphi^{2r+1}(f, n^{-1}),$$

which can be deduced immediately from the definition of $K(f, r, t)$, (7.2.2) and (7.3.1) in [7]. To prove (2.2) we choose $g \in C^{2r+1}[0, 1]$ such that

$$\|f - g\| + n^{-2r}\|P_r(D)g\| + n^{-2r-1}\|\varphi^{2r+1}g^{(2r+1)}\| \le 2K(f, r, n^{-1}). \tag{2.3}$$

We may assume $n = 2^m,\ P_{2^j} \in \Pi_{2^j},\ j = m + 1, \ldots,$ and

$$\|P_{2^j} - g\| = E_{2^j}(g), \quad j = m + 1, \ldots.$$

Thus we have

$$g - P_{2^m} = \sum_{j=m}^{\infty}(P_{2^{j+1}} - P_{2^j}).$$

From Theorem 7.2.1 in [7] and (2.3), we conclude

$$\begin{aligned}\|P_{2^{j+1}} - P_{2^j}\| &\le C\left(2^{-j}\right)^{2r+1}\|\varphi^{2r+1}g^{(2r+1)}\|\\ &\le C(2^{-j})^{2r+1}(2^m)^{2r+1}K(f, r, 2^{-m}), \quad j = m + 1, \ldots.\end{aligned}$$

Obviously, this estimate holds also for $j = m$. Hence, by using the Bernstein inequality, we get finally

$$\begin{aligned}\|P_r(D)(g - P_{2^m})\| &\le C\sum_{j=m}^{\infty} 2^{2rj}2^{-2rj-j}2^{2rm+m}K(f, r, 2^{-m})\\ &\le C2^{2rm}K(f, r, 2^{-m}),\end{aligned}$$

which implies (2.2). $\square$

Denote $E_n = [a/n, 1 - a/n]$ for fixed $a > 0$. For the moments of the operator $B_{n,r}$ we have shown in [16, p. 89]:

Lemma 2.2. *Let $x \in E_n$, $C(n) = \sum_{i=0}^{r-1} c_i(n)n_i^{-r}$ and $a_{j,k}$ given by (1.4). Then for some $\epsilon_k \in Lip1$ satisfying $\epsilon_k(0) = \epsilon_k(1) = 0$, $\epsilon_{2r-1}(x) \equiv 0$ and $\epsilon_{2r}(x) \equiv 0$, we have*

$$B_{n,r}((\cdot - x)^{r+1}, x) = C(n)x(1 - x)(1 - 2x)^{\delta_{r+1}}a_{1,r+1},$$

and for $r + 2 \le k \le 2r$

$$B_{n,r}((\cdot - x)^k, x) = C(n)(x(1 - x))^{k-r}(1 - 2x)^{\delta_k}(a_{k-r,k} + \epsilon_k(x)) + \mathcal{O}\left(\frac{\varphi^{2(k-r-1)}(x)}{n^{r+1}}\right).$$

Moreover, following the notations of Lemma 2.2 we have (see [16, p. 92])

Lemma 2.3. *Let $P \in \prod_m$ with $m \le \sqrt{n}$, then the following inequality is true for all $x \in [0, 1]$:*

$$\left|B_{n,r}(P,x) - P(x) - C(n)\sum_{i=1}^{r}\frac{(x(1-x))^i}{(r+i)!}(1-2x)^{\delta_{r+i}}\left(a_{i,r+i}+\epsilon_{r+i}(x)\right)P^{(r+i)}(x)\right| \le C_1\left(n^{-r-1/2}\|\varphi^{2r+1}P^{(2r+1)}\| + n^{-r-1}\|P\|\right), \tag{2.4}$$

where C_1 is a positive constant independent of P and n, and $\epsilon_{r+1}(x) \equiv 0$.

Let q be a given algebraic polynomial, and $\bar{q} = \{\mathrm{Re}\, x : q(x) = 0\}$. We need also the results concerning the following differential operator (see [14]): let

$$P(D) = \sum_{i=0}^{l}\alpha_i(x)(x(1-x))^i D^{l+i},$$

where $\alpha_i \in \mathrm{Lip}\,\delta,\ i = 0, 1, \ldots, l$, for some $0 < \delta \le 1$ and $\alpha_l(x) \ne 0$ for $x \in [0, 1]$. Let further

$$\sigma_0(x) = \alpha_0(0) + \sum_{i=1}^{l}\alpha_i(0)x(x-1)\cdots(x-i+1)$$

and

$$\sigma_1(x) = \alpha_0(1) + \sum_{i=1}^{l}(-1)^i\alpha_i(1)x(x-1)\cdots(x-i+1).$$

We have (see [9, pp. 80–81] and [14, p. 251])

Lemma 2.4. *Let $\alpha \ge 0$ be fixed. Then there is a constant $A > 0$ such that for all $P \in \Pi_n$ and all $n = 1, 2, \ldots,$ there hold*

$$\|\varphi^{2l+2\alpha+1}P^{(2l+1)}\| \le An(\|\varphi^{2\alpha}P(D)P\| + \|\varphi^{2\alpha}P\|)$$

and if $-\alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$

$$\|\varphi^{2l+2\alpha}P^{(2l)}\| \le A(\|\varphi^{2\alpha}P(D)P\| + \|\varphi^{2\alpha}P\|). \tag{2.5}$$

Moreover, if $\sigma_0(x) = x\sigma_{0,0}(x)$, $\sigma_1(x) = x\sigma_{1,1}(x)$ and $0 \notin \overline{\sigma}_{0,0} \cup \overline{\sigma}_{1,1}$, then (2.5) also holds for $\alpha = 0$.

We are now in the position to verify Theorem 1.3.

Proof of Theorem 1.3. Let $P_m \in \Pi_m$ with $m = [\sqrt{n}]$ satisfy

$$\|P_m - f\| = E_m(f).$$

By (7.2.2) and (7.3.1) in [7] we have

$$\|P_m - f\| \le C\omega_\varphi^{2r+1}(f, n^{-1/2}) \tag{2.6}$$

and

$$\|\varphi^{2r+1}P_m^{(2r+1)}\| \le Cm^{2r+1}\omega_\varphi^{2r+1}(f, n^{-1/2}). \tag{2.7}$$

As (see (2.1.4) in [7])

$$\omega_\varphi^{2r+1}(f,t) \asymp \inf_g\left(\|f-g\| + t^{2r+1}\|\varphi^{2r+1}g^{(2r+1)}\|\right),$$

we conclude by the definition of $K(f, r, t)$ that

$$\omega_\varphi^{2r+1}(f, n^{-1/2}) \le CK(f, r, n^{-1/2}). \tag{2.8}$$

By (2.6) and (2.8), we have

$$\begin{aligned}\|B_{n,r}(f) - f\| &\le 2\|f - P_m\| + \|B_{n,r}(P_m) - P_m\| \\ &\le CK(f, r, n^{-1/2}) + \|B_{n,r}(P_m) - P_m\|.\end{aligned} \tag{2.9}$$

Using (2.4), (2.7) and (2.8), we obtain for $x \in [0, 1]$

$$\begin{aligned}|B_{n,r}(P_m, x) - P_m(x)| &\le |C(n)|\left|\sum_{i=1}^{r}\frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}}\left(a_{i,r+i}+\epsilon_{r+i}(x)\right)P_m^{(r+i)}(x)\right| \\ &\quad + C\left(K(f, r, n^{-1/2}) + n^{-r-1}\|P_m\|\right).\end{aligned} \tag{2.10}$$

We know that $\epsilon_{2r-1}(x) \equiv \epsilon_{2r}(x) \equiv 0$, and $\epsilon_{r+i} \in \mathrm{Lip}\, 1$ with $\epsilon_{r+i}(0) = \epsilon_{r+i}(1) = 0$ for $i = 1, 2, \ldots, r-2$. Thus, there is $C > 0$ such that $|\epsilon_{r+i}(x)| \le C\varphi^2(x)$ for $x \in [0, 1]$ and $i = 1, 2, \ldots, r-2$. In what follows we should prove

$$\|\varphi^{2i+2}P_m^{(r+i)}\| \le C\left(\|P_r(D)P_m\| + \|P_m\|\right) \quad \text{for } i = 1, 2, \ldots, r-2. \tag{2.11}$$

Indeed, we may assume $m = 2^k$. The set $\overline{\sigma}_0 \cup \overline{\sigma}_1$ for $P_r(D)$ has only finite elements. We have $0 < \alpha < 1/2$ satisfying $-\alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$. Let $P_{2^j} \in \Pi_{2^j}, j = 0, 1, \ldots, k-1$, be the best approximation of P_m with the weight $\varphi^{2\alpha}$. Then, we have from Theorem 8.2.1 in [7]

$$\|\varphi^{2\alpha}(P_m - P_{2^j})\| \le C2^{-2jr}(\|\varphi^{2\alpha+2r}P_m^{(2r)}\| + \|P_m\|), \quad j = 0, 1, \ldots, k-1.$$

Consequently, we conclude from (8.1.4) and (8.1.3) in [7]

$$\begin{aligned}\|\varphi^{2i+2}P_m^{(r+i)}\| &\le \sum_{j=0}^{k-1}\|\varphi^{2i+2}(P_{2^{j+1}} - P_{2^j})^{(r+i)}\| \\ &\le C\sum_{j=0}^{k-1}2^{(2r-1)j}\|\varphi(P_{2^{j+1}} - P_{2^j})\| \\ &\le C\sum_{j=0}^{k}2^{-j}(\|\varphi^{2\alpha+2r}P_m^{(2r)}\| + \|P_m\|) \\ &\le C(\|\varphi^{2\alpha+2r}P_m^{(2r)}\| + \|P_m\|).\end{aligned}$$

On the other hand, as $-\alpha \notin \overline{\sigma}_0 \cup \overline{\sigma}_1$ we get by (2.5) with $l = r$

$$\|\varphi^{2\alpha+2r}P_m^{(2r)}\| + \|P_m\| \le A(\|P_r(D)P_m\| + \|P_m\|).$$

Thus, (2.11) follows from the last two displays. From (2.11), we obtain

$$\left|\sum_{i=1}^{r}\frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}}\left(a_{i,r+i} + \epsilon_{r+i}(x)\right)P_m^{(r+i)}(x)\right| \le C(\|P_r(D)P_m\| + \|P_m\|).$$

Therefore, as $C(n) \asymp n^{-r}$, we conclude from (2.10) and (2.1)

$$\begin{aligned}\|B_{n,r}(P_m) - P_m\| &\le Cn^{-r}(\|P_r(D)P_m\| + \|P_m\|) + CK(f, r, n^{-1/2}) \\ &\le C\left(K(f, r, n^{-1/2}) + n^{-r}\|f\|\right).\end{aligned} \tag{2.12}$$

Combining (2.12) and (2.9) we have finally

$$\|B_{n,r}(f) - f\| \le C\left(K(f, r, n^{-1/2}) + n^{-r}\|f\|\right).$$

This inequality implies (1.5), since for any $P \in \Pi_r$

$$\|B_{n,r}(f-P) - (f-P)\| = \|B_{n,r}(f) - f\| \quad \text{and} \quad K(f-P, r, n^{-1/2}) = K(f, r, n^{-1/2}).$$

To show (1.6) we notice that if we define

$$\tilde{P}_r(D) = \sum_{i=1}^{r}\frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}}\left(a_{i,r+i} + \epsilon_{r+i}(x)\right)D^{r+i},$$

then the set $\overline{\sigma}_0 \cup \overline{\sigma}_1$ for $\tilde{P}_r(D)$ is the same as for $P_r(D)$. Consequently, (2.11) holds also for $\tilde{P}_r(D)$ instead of $P_r(D)$. Thus,

$$\left\|\sum_{i=0}^{r}\frac{\varphi^{2i}(x)}{(r+i)!}(1-2x)^{\delta_{r+i}}\epsilon_{r+i}(x)P_m^{(r+i)}(x)\right\| \le C\left(\|\tilde{P}_r(D)P_m\| + \|P_m\|\right).$$

Therefore,

$$\|P_r(D)P_m\| \le C\left(\|\tilde{P}_r(D)P_m\| + \|P_m\|\right). \tag{2.13}$$

On the other hand, we have

$$K(f, r, n^{-1/2}) \le C\left(\|f - P_m\| + n^{-r}\|P_r(D)P_m\| + n^{-r-1/2}\|\varphi^{2r+1}P_m^{(2r+1)}\|\right).$$

We already know from (2.6) and (2.7)

$$\|f - P_m\| + n^{-r-1/2}\|\varphi^{2r+1}P_m^{(2r+1)}\| \le C\omega_\varphi^{2r+1}(f, n^{-1/2}),$$

and from (2.13), (2.4), (2.6) and (2.7)

$$n^{-r}\|P_r(D)P_m\| \le C\left(\|B_{n,r}(P_m)-P_m\| + \omega_\varphi^{2r+1}(f,n^{-1/2}) + n^{-r}\|f\|\right).$$

Hence, there holds

$$K(f,r,n^{-1/2}) \le C\left(\|B_{n,r}(P_m)-P_m\| + \omega_\varphi^{2r+1}(f,n^{-1/2}) + n^{-r}\|f\|\right).$$

Following from Theorem 9.3.6 of [7]

$$\omega_\varphi^{2r+1}(f,n^{-1/2}) \le Cn^{-r-1/2}\left(\sum_{k=1}^{n} k^{r-1/2}\|B_{k,r}(f)-f\| + \|f\|\right)$$

and from (2.6)

$$\|B_{n,r}(P_m)-P_m\| \le \left(\|B_{n,r}(f)-f\| + C\omega_\varphi^{2r+1}(f,n^{-1/2})\right).$$

Consequently, we obtain

$$K(f,r,n^{-1/2}) \le C\left(\|B_{n,r}(f)-f\| + n^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}\|B_{k,r}(f)-f\| + n^{-r}\|f\|\right).$$

Multiplying by $n^{r-1/2}$ on both sides of the above inequality and taking the sum from N to $2N$, we obtain by the monotonicity of $K(f,r,n^{-1/2})$

$$N^{r+1/2}K(f,r,(2N)^{-1/2}) \le C\left(\sum_{k=1}^{2N} k^{r-1/2}\|B_{k,r}(f)-f\| + N^{1/2}\|f\|\right),$$

which obviously implies (1.6). □

Next we should apply Theorem 1.3 and Lemma 2.4 to verifying Theorem 1.4.

Proof of Theorem 1.4. It is easy to see that for $P_r(D)$ the functions $\sigma_0(x) = x\sigma(x-1)$ and $\sigma_1(x) = (-1)^r x\sigma(x-1)$. As $-1 \notin \{\mathrm{Re}\, x : \sigma(x) = 0\}$, the operator $P_r(D)$ satisfies the condition of Lemma 2.4 for $\alpha = 0$. We obtain from (2.5) that there is a constant $A > 0$ such that for all $P \in \Pi_n$ and all $n = 1, 2, \ldots$

$$\|\varphi^{2r}P^{(2r)}\| \le A(\|P_r(D)P\| + \|P\|). \tag{2.14}$$

On the other hand, if $P_m \in \Pi_m$ with $m = [\sqrt{n}]$ is such that $\|f - P_m\| = E_m(f)$, then

$$\omega_\varphi^{2r}(f,m^{-1}) \le C\left(\|f-P_m\| + m^{-2r}\|\varphi^{2r}P_m^{(2r)}\|\right).$$

Thus, we have from (2.14) and (2.1)

$$\begin{aligned}\omega_\varphi^{2r}(f,m^{-1}) &\le C(\|f-P_m\| + m^{-2r}\|P_r(D)P_m\| + m^{-2r}\|f\|)\\ &\le C\left(K(f,r,m^{-1}) + m^{-2r}\|f\|\right),\end{aligned}$$

which obviously implies

$$\omega_\varphi^{2r}(f,m^{-1}) + m^{-2r}E_r(f) \le C(K(f,r,m^{-1}) + m^{-2r}E_r(f)).$$

Therefore, it follows from (1.6) that

$$\omega_\varphi^{2r}(f,n^{-1/2}) + n^{-r}E_r(f) \le C\left(n^{-r-1/2}\sum_{k=1}^{n} k^{r-1/2}\|B_{k,r}(f)-f\| + n^{-r}E_r(f)\right).$$

Consequently, for $\tau = 0, 1/4$ we obtain from the last display

$$\omega_\varphi^{2r}(f,n^{-1/2}) + n^{-r}E_r(f) \le Cn^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|B_{k,r}(f)-f\| + n^{-r}E_r(f)).$$

On the other hand, let $J(f,t) = \omega_\varphi^{2r}(f,t) + t^{2r}E_r(f)$, then $J(f,\lambda t) \le C\lambda^{2r}J(f,t)$ for $\lambda \ge 1$. We conclude from (1.2)

$$n^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|B_{k,r}(f)-f\| + n^{-r}E_r(f)) \le CJ(f,n^{-1/2}).$$

Combining the last two displays, we obtain finally for $\tau = 0$ and $1/4$

$$n^{-r-\tau}\max_{1\le k\le n} k^{r+\tau}(\|B_{k,r}(f)-f\| + n^{-r}E_r(f)) \asymp J(f,n^{-1/2}). \tag{2.15}$$

Hence, there holds

$$n^{-r}\max_{1\le k\le n} k^r(\|B_{k,r}(f)-f\|+n^{-r}E_r(f)) \asymp n^{-r-1/4}\max_{1\le k\le n} k^{r+1/4}(\|B_{k,r}(f)-f\|+n^{-r}E_r(f)).$$

Assuming $1\le k_0\le n$ satisfies

$$\max_{1\le k\le n} k^{r+1/4}(\|B_{k,r}(f)-f\|+n^{-r}E_r(f)) = {k_0}^{r+1/4}(\|B_{k_0,r}(f)-f\|+n^{-r}E_r(f)),$$

we have for some $C_0>0$

$$\begin{aligned} n^{-r}k_0^r(\|B_{k_0,r}(f)-f\|+n^{-r}E_r(f)) &\le n^{-r}\max_{1\le k\le n} k^r(\|B_{k,r}(f)-f\|+n^{-r}E_r(f)) \\ &\le C_0 n^{-r-1/4}k_0^{r+1/4}(\|B_{k_0,r}(f)-f\|+n^{-r}E_r(f)), \end{aligned}$$

which gives $k_0\ge nC_0^{-4}$. Therefore we have from (2.15)

$$\begin{aligned} J(f,n^{-1/2}) &\le Cn^{-r-1/4}\max_{1\le k\le n} k^{r+1/4}(\|B_{k,r}(f)-f\|+n^{-r}E_r(f)) \\ &\le Cn^{-r-1/4}{k_0}^{r+1/4}(\|B_{k_0,r}(f)-f\|+n^{-r}E_r(f)) \\ &\le C\left(\max_{k\ge nC_0^{-4}}\|B_{k,r}(f)-f\|+n^{-r}E_r(f)\right). \end{aligned}$$

The property of $J(f,n^{-1/2})$ implies

$$J(f,(nC_0^{-4})^{-1/2})\le C\left(\max_{k\ge nC_0^{-4}}\|B_{k,r}(f)-f\|+n^{-r}E_r(f)\right).$$

The desired assertion follows from this estimate and (1.2). □

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