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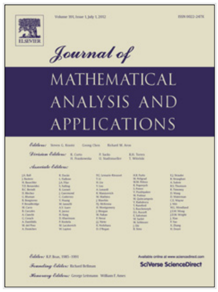

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THE WAVE EQUATION FOR THE BESSEL LAPLACIAN

ÓSCAR CIAURRI AND LUZ RONCAL

ABSTRACT. We study radial solutions of the Cauchy problem for the wave equation in the multidimensional unit ball B^d, $d \geq 1$. In this case, the operator that appears is the Bessel Laplacian and the solution $u(t,x)$ is given in terms of a Fourier-Bessel expansion. We prove that, for initial L^p data, the series converges in the L^2 norm. The analysis of a particular operator, the adjoint of the Riesz transform for Fourier-Bessel series, is needed for our purposes, and may be of independent interest. As application, certain $L^p - L^2$ estimates for the solution of the heat equation and the extension problem for the fractional Bessel Laplacian are obtained.

1. Introduction and main result

For $d \geq 1$, let $B^d = \{\mathtt{x} \in \mathbb{R}^d : |\mathtt{x}| < 1\}$ be the unit ball in $\mathbb{R}^d$. Let us consider the Cauchy problem for the wave equation in $(t, \mathtt{x}) \in \mathbb{R}^+ \times B^d$

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,\mathtt{x}) = \Delta u(t,\mathtt{x}), \\ u(0,\mathtt{x}) = F(\mathtt{x}),\ \frac{\partial}{\partial t} u(0,\mathtt{x}) = G(\mathtt{x}). \end{cases}$$

When the initial data are radial, writing $|\mathtt{x}| = x, F(\mathtt{x}) = f(x), G(\mathtt{x}) = g(x)$, and expressing the Laplacian in polar coordinates, one is led to the Cauchy problem in $(t,x) \in \mathbb{R}^+ \times (0,1)$

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) = -L_\nu u(t,x), \\ u(0,x) = f(x),\ \frac{\partial}{\partial t} u(0,x) = g(x). \end{cases} \tag{1.1}$$

Here

$$L_\nu = -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx},$$

which is symmetric and nonnegative on $C_c^2(0,1) \subset L^2((0,1), d\mu_\nu)$, where the measure is given by

$$d\mu_\nu(x) = x^{2\nu+1}\,dx$$

and the type index is $\nu = d/2 - 1$. There is no need to consider just half-integer values, so we will work with a general index $\nu > -1$. The second order partial differential operator above is called the Bessel Laplacian. The operator L_ν can be decomposed as $L_\nu = \delta_\nu^* \delta_\nu$, with

$$\delta_\nu = -\frac{d}{dx} \quad \text{and} \quad \delta_\nu^* = \frac{d}{dx} + \frac{2\nu+1}{x}. \tag{1.2}$$

Let J_ν be the Bessel function of order ν, for a fixed $\nu > -1$. It is well known that the functions

$$\phi_n^\nu(x) = d_{n,\nu}\, x^{-\nu} \lambda_{n,\nu}^{1/2} J_\nu(\lambda_{n,\nu} x), \qquad n = 1, 2, \dots,$$

form an orthonormal basis in $L^2((0,1), d\mu_\nu)$, where $\{\lambda_{n,\nu}\}_{n\geq 1}$ are the sequence of successive positive zeros of J_ν and $d_{n,\nu} = \sqrt{2}|\lambda_{n,\nu}^{1/2} J_{\nu+1}(\lambda_{n,\nu})|^{-1}$ are normalizing constants. Furthermore, the functions $\phi_n^\nu(x)$ are eigenfunctions of L_ν with the corresponding eigenvalue $\lambda_{n,\nu}^2$. The Fourier-Bessel expansion of a function f is

$$f = \sum_{n=1}^{\infty} P_n f,$$

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where P_nf is the spectral projection, $P_nf = a_n(f)\phi_n^\nu$ and $a_n(f) = \int_0^1 f(y)\phi_n^\nu(y)\,d\mu_\nu(y)$, provided the integral exists.

The solution of the Cauchy problem (1.1) can be obtained by applying a Fourier method. That solution u has the Fourier-Bessel expansion given by

$$u(t,x) = \sum_{n=1}^{\infty} \frac{\sin(t\lambda_{n,\nu})}{\lambda_{n,\nu}} P_n g(x) + \sum_{n=1}^{\infty} \cos(t\lambda_{n,\nu}) P_n f(x). \tag{1.3}$$

Formally, the solution u is given by

$$u(t,x) = (L_\nu)^{-1/2} \sin(tL_\nu^{1/2})g(x) + \cos(tL_\nu^{1/2})f(x) =: u_1(t,x) + u_2(t,x). \tag{1.4}$$

Our aim is to prove that the series given in (1.3) converges in the $L^2((0,1),d\mu_\nu)$ norm whenever $g \in L^p((0,1),d\mu_\nu)$ and $\delta_\nu f \in L^p((0,1),d\mu_\nu)$, provided p is close enough to 2.

We will prove the following theorem.

Theorem 1.1. *Assume that $g \in L^p((0,1),d\mu_\nu)$, f is such that $\delta_\nu f \in L^p((0,1),d\mu_\nu)$, and $2\frac{2\nu+3}{2\nu+5} < p \le 2$. Then,*

$$\|u_1(t,\cdot)\|_{L^2((0,1),d\mu_\nu)} \le C\|g\|_{L^p((0,1),d\mu_\nu)} \qquad \text{for } \nu \ge -1/2, \tag{1.5}$$

and

$$\|u_2(t,\cdot)\|_{L^2((0,1),d\mu_\nu)} \le C\|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)} \qquad \text{for } \nu > 0, \tag{1.6}$$

with C independent of t.

In order to prove the inequality (1.5) in Theorem 1.1, we will use E. M. Stein theory of interpolation of analytic families of operators, adapted to our measure space. We follow the ideas of S. Thangavelu in [18] and [19, Ch. 4.4] for the Hermite setting, and the original idea can be found in [17]. Nevertheless, the proof of the inequality (1.6) is new and different from the one given by Thangavelu, because his ideas cannot be applied in our setting due to a problem of convergence of some series involved. We will factorize the operator in a smart way, in which we will get some operators that can be estimated either directly, or by using the inequality (1.5), or by means of results in [1] and [2]. Precisely, the restriction $\nu > 0$ for the inequality (1.6) comes from the parameter restriction in the boundedness of the adjoint of the Riesz transform for Fourier-Bessel series.

Some applications can be deduced from our main result. We will show a relationship between the wave equation solution and the solutions of the heat equation and the extension problem for the fractional Bessel Laplacian (see section 6). As a consequence, certain $L^p - L^2$ inequalities follow for these solutions.

The study of the Cauchy problem for the wave equation in $(t,\mathtt{x})$ in $\mathbb{R}^+ \times \mathbb{R}^d$ has been thoroughly developed since 19th century. There are results on the Euclidean space, see [10] and [12], that have been extended to domains with boundary. For instance, A. Magyar [8] proved $L^q \to L^p$ bounds for the wave operator on the torus for large time, analogous to those by R. S. Strichartz in $\mathbb{R}^d$, see [16]. Magyar also established this kind of estimates for the wave equation on compact manifolds [9]. Anyway, there are still results that have not been studied in the torus. The intuition is that the analysis on the surface of the sphere is easier than the one in the torus.

Radial solutions to the wave equation were investigated, in the Euclidean space, by L. Colzani, A. Cominardi and K. Stempak [5]. They presented a method based on Fourier analysis that gave them the chance to work with special functions and study singularities of the kernels involved in the solutions. That paper could inspire further research in our context. Our result is a first step in the study of radial solutions related to the wave equation in the ball.

The paper is organized as follows. In Section 2 we collect several basic facts about Bessel functions, Hardy's inequalities and the theory of analytic families of operators. Sections 3 and 4 contain the proofs of inequalities (1.5) and (1.6) respectively. Section 5 is devoted to the treatment of the adjoint Riesz transform associated to Fourier-Bessel expansions, that is involved in the proof of (1.6). Finally, in Section 6, we study the relationship between the solution of the wave equation and solutions of other equations.

By q we will denote the conjugate value to p, $1 < p < \infty$, that is to say $\frac{1}{p} + \frac{1}{q} = 1$. Throughout this paper, the letter C will denote a positive constant which may change from one instance to another and depend on the parameters involved. We shall write $X \simeq Y$ when simultaneously $X \le CY$ and $Y \le CX$.

2. Some basic tools

The Bessel function J_ν satisfies

$$J_\nu'(z) = -\frac{\nu}{z} J_\nu(z) + J_{\nu-1}(z), \qquad J_\nu'(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z). \tag{2.1}$$

From the combination of both, we obtain

$$J_{\nu+1}(z) = \left(\frac{2\nu}{z}\right) J_\nu(z) - J_{\nu-1}(z). \tag{2.2}$$

Recall the well-known asymptotics for the Bessel functions (see [20, Chapter 7])

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + O(z^{\nu+2}), \quad |z| < 1, \tag{2.3}$$

and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \big(\cos(z + D_\nu) + O(z^{-1}) \big), \quad |z| \ge 1 \tag{2.4}$$

where $D_\nu = -(\nu\pi/2 + \pi/4)$.

We also take into account some classical estimates on the size of the Bessel function, see [14, (11.10), (11.11)],

$$|J_{a+ib}(t)| \le C_a e^{\pi|b|} t^{-1/2}, \quad t \ge 1,\ a \ge 0, \qquad |J_{a+ib}(t)| \le C_a e^{\frac{1}{2}\pi|b|} t^a, \quad t > 0,\ a \ge 0$$

and from here it is not difficult to show that

$$|t^{-(a+ib)} J_{a+ib}(t)| \le C_a e^{c|b|} (1+t)^{-a-1/2} \tag{2.5}$$

for $0 < t < \infty$. This is actually proved for $a > -1/2$ but it holds for all a by (2.2).

We will use the fact that (cf. [2, (2.6)])

$$\lambda_{n,\nu} = \mathcal{O}(n), \quad d_{n,\nu} = \mathcal{O}(1). \tag{2.6}$$

Recall the following two forms of Hardy's inequality: if $a < -1$ and $1 \le p < \infty$, then

$$\int_0^1 \left| \int_0^x f(t)\, dt \right|^p x^a\, dx \le C \int_0^1 |f(x)|^p x^{a+p}\, dx; \tag{2.7}$$

if $a > -1$ and $1 \le p < \infty$, then

$$\int_0^1 \left| \int_x^1 f(t)\, dt \right|^p x^a\, dx \le C \int_0^1 |f(x)|^p x^{a+p}\, dx. \tag{2.8}$$

We also need estimates for the L^p-norms of the functions ϕ_n^ν.

Lemma 2.1. *Let $1 \le p < \infty$ and $\nu > -1$. Then, for $\nu > -1/2$,*

$$\|\phi_n^\nu\|_{L^p((0,1),d\mu_\nu)} \simeq \begin{cases} n^{(\nu+1/2) - \frac{2(\nu+1)}{p}}, & p > \frac{2(\nu+1)}{\nu+1/2}, \\ (\log n)^{1/p}, & p = \frac{2(\nu+1)}{\nu+1/2}, \\ 1, & p < \frac{2(\nu+1)}{\nu+1/2}, \end{cases}$$

and, for $-1 < \nu \le -1/2$, $\|\phi_n^\nu\|_{L^p((0,1),d\mu_\nu)} \simeq 1$.

For $p = \infty$, $\nu > -1/2$,

$$\|\phi_n^\nu\|_{L^\infty((0,1),d\mu_\nu)} \le C n^{\nu+1/2},$$

and, for $-1 < \nu \le -1/2$, $\|\phi_n^\nu\|_{L^\infty((0,1),d\mu_\nu)} \le C$.

Proof. We have

$$\|\phi_n^\nu\|^p_{L^p((0,1),d\mu_\nu)} = \int_0^1 |\phi_n^\nu(x)|^p \, d\mu_\nu(x) = \int_0^1 (d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}|J_\nu(\lambda_{n,\nu}x)|)^p x^{2\nu+1}\, dx =: I_1 + I_2,$$

where

$$I_1 = \int_0^{1/n} (d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}|J_\nu(\lambda_{n,\nu}x)|)^p x^{2\nu+1}\, dx, \quad I_2 = \int_{1/n}^1 (d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}|J_\nu(\lambda_{n,\nu}x)|)^p x^{2\nu+1}\, dx.$$

For I_1, by (2.3) and (2.6),

$$I_1 \simeq \int_0^{1/n} n^{p/2}x^{-\nu p}(nx)^{\nu p}x^{2\nu+1}\, dx \simeq n^{p(\nu+1/2)} \int_0^{1/n} x^{2\nu+1}\, dx \simeq n^{p(\nu+1/2)-2(\nu+1)},$$

by taking into account that the last integral converges when $\nu > -1$. Concerning I_2, using (2.4) and again (2.6),

$$I_2 \simeq \int_{1/n}^1 n^{p/2}x^{-\nu p}(nx)^{-p/2}x^{2\nu+1}\, dx = \int_{1/n}^1 x^{-p(\nu+1/2)+2\nu+1}\, dx.$$

For $\nu > -1/2$, the latter integral has the size of a constant if $-p(\nu+1/2)+2\nu+1 > -1$, which is equivalent to $p < \frac{2(\nu+1)}{\nu+1/2}$, and it has the size of $n^{p(\nu+1/2)-2(\nu+1)}$ if $p > \frac{2(\nu+1)}{\nu+1/2}$. When $p = \frac{2(\nu+1)}{\nu+1/2}$, we have that $I_2 \simeq \log n$. The case $-1 < \nu < -1/2$ implies that $\nu + 1/2$ is negative, therefore we have that I_2 has the size of a constant if $-p(\nu+1/2)+2\nu+1 > -1$, which is equivalent to $p > \frac{2(\nu+1)}{\nu+1/2}$. But this last quantity is negative, and so the inequality holds for all positive p. The case $\nu = -1/2$ is obvious.

From these considerations, we obtain

$$\|\phi_n^\nu\|^p_{L^p((0,1),d\mu_\nu)} \simeq \begin{cases} n^{p(\nu+1/2)-2(\nu+1)}, & \text{if } p > \frac{2(\nu+1)}{\nu+1/2}, \\ \log n, & \text{if } p = \frac{2(\nu+1)}{\nu+1/2}, \\ 1, & \text{if } p < \frac{2(\nu+1)}{\nu+1/2}, \end{cases}$$

whenever $\nu > -1/2$, and $\|\phi_n^\nu\|^p_{L^p((0,1),d\mu_\nu)} \simeq 1$ if $-1 < \nu \le -1/2$.

On the other hand, $\|\phi_n^\nu\|_{L^\infty((0,1),d\mu_\nu)} = \sup\{|d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}J_\nu(\lambda_{n,\nu}x)| : x \in (0,1)\}$. By using the asymptotics (2.3), (2.4) and (2.6), the estimate $|\phi_n^\nu(x)| \le Cn^{\nu+1/2}$ holds for $\nu > -1/2$. Indeed, for the case $0 < x \le n^{-1}$,

$$|d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}J_\nu(\lambda_{n,\nu}x)| \le Cx^{-\nu}n^{1/2}(nx)^\nu \simeq n^{\nu+1/2},$$

and for the case $n^{-1} < x < 1$, we have that

$$|d_{n,\nu}\, x^{-\nu}\lambda_{n,\nu}^{1/2}J_\nu(\lambda_{n,\nu}x)| \le Cx^{-\nu}n^{1/2}(nx)^{-1/2} \simeq x^{-(\nu+1/2)} < n^{\nu+1/2}.$$

Note that, for $-1 < \nu \le -1/2$, $\|\phi_n^\nu\|_{L^\infty((0,1),d\mu_\nu)}$ is controlled by a constant (in the case $0 < x \le n^{-1}$, $n^{\nu+1/2}$ is obviously estimated by a constant, and in the case $n^{-1} < x < 1$ the estimate is immediate for $x^{-(\nu+1/2)}$). □

Now we are going to recall some topics about the theory of interpolation of analytic families of operators that will be adapted to our context. The definitions in this paper and interpolation theory can be seen in [13].

A family of operators $\{T(z)\}$ depending on a complex parameter z that varies in the strip $0 \le \operatorname{Re} z \le 1$ is said to be analytic if it has the following properties:

(a) For each z, $T(z)$ is a linear transformation of simple functions on $(0,1)$ to measurable functions on $(0,1)$.

(b) If f and g are simple functions on $(0,1)$, then the function

$$F(z) = \int_0^1 T(z)(g(x))f(x)\, dx$$

is analytic in $0 < \operatorname{Re} z < 1$ and continuous in $0 \le \operatorname{Re} z \le 1$.

We will say that an analytic family $\{T(z)\}$ is of admissible growth if $F(z)$ is of admissible growth; that is, if

$$\sup_{|y|\le r}\sup_{0\le x\le 1}\log|F(x+iy)|\le Ae^{ar}$$

where $a<\pi$ and A is a constant. Both A and a can depend on the functions f and g.

Stein Theorem of interpolation is the following

Theorem 2.2 (see [13]). *Let $\{T(z)\}$ be an analytic family of linear operators of admissible growth defined in the strip $0\le \mathrm{Re}(z)\le 1$. Suppose that $1\le p_0,p_1,\widetilde{p}_0,\widetilde{p}_1\le\infty$ and*

$$\frac{1}{p}=\frac{1-t}{p_0}+\frac{t}{p_1},\quad \frac{1}{\widetilde{p}}=\frac{1-t}{\widetilde{p}_0}+\frac{t}{\widetilde{p}_1},$$

where $0\le t\le 1$. Finally suppose that

$$\|T(iy)f\|_{L^{\widetilde{p}_0}((0,1),d\mu_\nu)}\le A_0(y)\|f\|_{L^{p_0}((0,1),d\mu_\nu)}$$

and

$$\|T(1+iy)f\|_{L^{\widetilde{p}_1}((0,1),d\mu_\nu)}\le A_1(y)\|f\|_{L^{p_1}((0,1),d\mu_\nu)}$$

for simple functions f verifying

$$\log|A_i(y)|\le Ae^{a|y|},\quad a<\pi,\quad i=0,1.$$

Then, we may conclude that

$$\|T(t)f\|_{L^{\widetilde{p}}((0,1),d\mu_\nu)}\le A_t\|f\|_{L^{p}((0,1),d\mu_\nu)}$$

where

$$\log A_t\le\int_{\mathbb{R}}w(1-t,y)\log A_0(y)\,dy+\int_{\mathbb{R}}w(t,y)\log A_1(y)\,dy$$

and

$$w(t,y)=\frac{\tan(\pi t/2)}{2[\tan^2(\pi t/2)+\tanh^2(\pi y/2)]\,\cos^2(\pi y/2)}.$$

3. Proof of inequality (1.5)

In order to get (1.5), we are going to prove

$$\|L_\nu^{-1/2}\sin(tL_\nu^{1/2})g\|_{L^2((0,1),d\mu_\nu)}\le C\|g\|_{L^p((0,1),d\mu_\nu)}.$$

We would like to embed $L_\nu^{-1/2}\sin(tL_\nu^{1/2})$ into an analytic family of operators. We define the operators $S_t(\alpha)$, for α in the strip $0\le\mathrm{Re}(\alpha)\le(2\nu+3)/2$, as

$$S_t(\alpha)=\left(\frac{\pi}{2}\right)^{1/2}t(tL_\nu^{1/2})^{\alpha-(\nu+1)}J_{(\nu+1)-\alpha}(tL_\nu^{1/2}). \tag{3.1}$$

These operators form an analytic family of operators and it can be checked that this family is admissible, by using (2.5). When $\alpha=\frac{2\nu+1}{2}$ we have

$$S_t\left(\frac{2\nu+1}{2}\right)=\left(\frac{\pi}{2}\right)^{1/2}t(tL_\nu^{1/2})^{-1/2}\left(\frac{2}{\pi}\right)^{1/2}\frac{\sin(tL_\nu^{1/2})}{(tL_\nu^{1/2})^{1/2}}=L_\nu^{-1/2}\sin(tL_\nu^{1/2}),$$

so that

$$u_1(t,x)=S_t\left(\frac{2\nu+1}{2}\right)g(x).$$

For $S_t(\alpha)$, we will need the following

Proposition 3.1. *Let* $\nu > -1$. *Then,*

$$\left\| S_t\left(\frac{2\nu+3}{2}+iz\right)g\right\|_{L^2((0,1),d\mu_\nu)} \le Ce^{c|z|}t\|g\|_{L^2((0,1),d\mu_\nu)}, \tag{3.2}$$

and

$$\|S_t(-s+iz)g\|_{L^2((0,1),d\mu_\nu)} \le Ce^{c|z|}t^{-s-\left(\frac{2\nu+1}{2}\right)}\|g\|_{L^1((0,1),d\mu_\nu)}, \tag{3.3}$$

for every $s>0$.

Suppose we have proved Proposition 3.1. If we define

$$T_t(\alpha) = S_t\left(-s+\left(\frac{2\nu+3}{2}+s\right)\alpha\right), \quad s>0,$$

then

$$T_t(iy) = S_t\left(-s+\left(\frac{2\nu+3}{2}+s\right)iy\right).$$

Therefore, from (3.3) in Proposition 3.1 with $z=\left(\frac{2\nu+3}{2}+s\right)y$, it follows that

$$\|T_t(iy)g\|_{L^2((0,1),d\mu_\nu)} \le C\exp\left(c\left|\left(\frac{2\nu+3}{2}+s\right)y\right|\right)t^{-s-\left(\frac{2\nu+1}{2}\right)}\|g\|_{L^1((0,1),d\mu_\nu)}.$$

On the other hand,

$$T_t(1+iy) = S_t\left(\frac{2\nu+3}{2}+\left(\frac{2\nu+3}{2}+s\right)iy\right),$$

and again, from (3.2) in Proposition 3.1 with $z=\left(\frac{2\nu+3}{2}+s\right)y$,

$$\|T_t(1+iy)g\|_{L^2((0,1),d\mu_\nu)} \le C\exp\left(c\left|\left(\frac{2\nu+3}{2}+s\right)y\right|\right)t\|g\|_{L^2((0,1),d\mu_\nu)}.$$

By Theorem 2.2 we have that, for $0<\mathrm{Re}(\alpha)<1$,

$$\|T_t(\alpha)g\|_{L^2((0,1),d\mu_\nu)} \le Ct^{\alpha}t^{\left(-s-\frac{2\nu+1}{2}\right)(1-\alpha)}\|g\|_{L^p((0,1),d\mu_\nu)},$$

with p such that $\frac{1}{p}=1-\frac{\alpha}{2}$. By choosing $\alpha=\frac{2\nu+1+2s}{2\nu+3+2s}$, $s>0$ (we still have that $0<\mathrm{Re}(\alpha)<1$) we get

$$T_t\left(\frac{2\nu+1+2s}{2\nu+3+2s}\right) = S_t\left(-s+\left(\frac{2\nu+3}{2}+s\right)\left(\frac{2\nu+1+2s}{2\nu+3+2s}\right)\right) = S_t\left(\frac{2\nu+1}{2}\right),$$

and

$$\|T_t(\alpha)g\|_{L^2((0,1),d\mu_\nu)} \le C\|g\|_{L^{p_s}((0,1),d\mu_\nu)}.$$

Hence

$$\|L_\nu^{-1/2}\sin(tL_\nu^{1/2})g\|_{L^2((0,1),d\mu_\nu)} \le C\|g\|_{L^{p_s}((0,1),d\mu_\nu)},$$

where $\frac{1}{p_s}=\frac{2\nu+5+2s}{2(2\nu+3+2s)}$. If $p>\frac{2(2\nu+3)}{2\nu+5}$ is given, we can always choose $s>0$ such that $p>p_s$ and $L_\nu^{-1/2}\sin(tL_\nu^{1/2})$ is bounded from L^2 into L^{p_s}. On the other hand, it is easy to verify that $L_\nu^{-1/2}\sin(tL_\nu^{1/2})$ is bounded from L^2 to L^2. Indeed,

$$\begin{aligned}\|L_\nu^{-1/2}\sin(tL_\nu^{1/2})g\|^2_{L^2((0,1),d\mu_\nu)} &\le \sum_{n=1}^{\infty}\left(\frac{\sin(t\lambda_{n,\nu})}{\lambda_{n,\nu}}\right)^2\|P_ng\|^2_{L^2((0,1),d\mu_\nu)} \\ &\le C\sum_{n=1}^{\infty}\frac{1}{n^2}\|P_ng\|^2_{L^2((0,1),d\mu_\nu)} \le C\|g\|^2_{L^2((0,1),d\mu_\nu)}.\end{aligned}$$

By the Riesz-Thorin interpolation Theorem, it follows that $L_\nu^{-1/2}\sin(tL_\nu^{1/2})$ is bounded from L^p into L^2 for $p_s\le p\le 2$, and therefore, for $\frac{2(2\nu+3)}{2\nu+5}<p\le 2$.

Proof of Proposition 3.1. In order to prove (3.2) we first note that

$$S_t\left(\frac{2\nu+3}{2}+iz\right)=\left(\frac{\pi}{2}\right)^{1/2}t(tL_\nu^{1/2})^{1/2+iz}\left(\frac{2}{\pi}\right)^{1/2}J_{-1/2-iz}(tL_\nu^{1/2}).$$

In the spectral way,

$$\begin{aligned}\left|S_t\left(\frac{2\nu+3}{2}+iz\right)g\right| &\le \left(\frac{\pi}{2}\right)^{1/2}t\sum_{n=1}^{\infty}|(t\lambda_{n,\nu})^{1/2+iz}J_{-1/2-iz}(t\lambda_{n,\nu})P_ng|\\ &\le Cte^{c|z|}\sum_{n=1}^{\infty}|P_ng|,\end{aligned}$$

where we have used (2.5). Hence, the operator $S_t\left(\frac{2\nu+3}{2}+iz\right)$ is bounded on $L^2((0,1),d\mu_\nu)$ with the required estimate.

In order to prove (3.3), we take $\alpha=-s+iz$ in (3.1), and then

$$S_t(-s+iz)=\left(\frac{\pi}{2}\right)^{1/2}t(tL_\nu^{1/2})^{-s-(\nu+1)+iz}J_{\nu+1+s-iz}(tL_\nu^{1/2}).$$

Therefore, by using (2.5) and taking into account that

$$\|P_ng\|_{L^2((0,1),d\mu_\nu)}\le\|\phi_n^\nu\|_{L^q((0,1),d\mu_\nu)}\|g\|_{L^p((0,1),d\mu_\nu)}$$

we have

$$\begin{aligned}\|S_t(-s+iz)g\|^2_{L^2((0,1),d\mu_\nu)} &= \left\|\left(\frac{\pi}{2}\right)^{1/2}t\sum_{n=1}^{\infty}(t\lambda_{n,\nu})^{-s-(\nu+1)+iz}J_{\nu+1+s-iz}(t\lambda_{n,\nu})P_ng\right\|^2_{L^2((0,1),d\mu_\nu)}\\ &\le \frac{\pi}{2}t^2\sum_{n=1}^{\infty}|(t\lambda_{n,\nu})^{-s-(\nu+1)+iz}J_{\nu+1+s-iz}(t\lambda_{n,\nu})|^2\|P_ng\|^2_{L^2((0,1),d\mu_\nu)}\\ &\le \frac{\pi}{2}Ce^{c|z|}t^2t^{-2s-2(\nu+1)}\sum_{n=1}^{\infty}n^{-2s-2(\nu+1)-1}\|P_ng\|^2_{L^2((0,1),d\mu_\nu)}\\ &\le Ce^{c|z|}t^{-2s-2\nu-1}\sum_{n=1}^{\infty}n^{-2s-2\nu-3}\|\phi_n^\nu\|^2_{L^\infty((0,1),d\mu_\nu)}\|g\|^2_{L^1((0,1),d\mu_\nu)}.\end{aligned}$$

Now, for $\nu>-1/2$, by using Lemma 2.1, the last sum can be estimated by

$$\sum_{n=1}^{\infty}n^{-2s-2\nu-3}n^{2\nu+1}\|g\|^2_{L^1((0,1),d\mu_\nu)}\le C\|g\|^2_{L^1((0,1),d\mu_\nu)},$$

as the series converges. Analogously, for $-1<\nu\le-1/2$, and applying Lemma 2.1 in this case, we get the estimate

$$\sum_{n=1}^{\infty}n^{-2s-2\nu-3}\|g\|^2_{L^1((0,1),d\mu_\nu)}\le C\|g\|^2_{L^1((0,1),d\mu_\nu)},$$

as the series converges again, due to the fact that $-2s-2\nu-3<-1$, for $s>0$. This completes the proof of Proposition 3.1.

□

4. Proof of inequality (1.6)

Recall that, by (1.4), $u_2(t,x)=\cos(tL_\nu^{1/2})f(x)$. We can write

$$\cos(tL_\nu^{1/2})=\cos(tL_\nu^{1/2})L_\nu^{-1}L_\nu=(L_\nu^{-1/2}\cos(tL_\nu^{1/2}))L_\nu^{-1/2}\delta_\nu^*\delta_\nu. \tag{4.1}$$

We define formally the operator $\mathcal{R}_\nu^*=L_\nu^{-1/2}\delta_\nu^*$. It will be proved in Section 5 that this operator is bounded on $L^p((0,1),d\mu_\nu)$. In this way, once we prove that

$$\|L_\nu^{-1/2}\cos(tL_\nu^{1/2})f\|_{L^2((0,1),d\mu_\nu)}\le C\|f\|_{L^p((0,1),d\mu_\nu)}, \tag{4.2}$$

it follows from (4.1) that

$$\|\cos(tL_\nu^{1/2})f\|_{L^2((0,1),d\mu_\nu)} \le C\|(\mathcal{R}_\nu^*)\delta_\nu f\|_{L^p((0,1),d\mu_\nu)} \le C\|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)}.$$

In order to establish (4.2), first we need to express the operator in a proper way. We write

$$L_\nu^{-1/2}\cos(tL_\nu^{1/2}) = -2\big(\sin\big(tL_\nu^{1/2}/2\big)\big)\big(L_\nu^{-1/2}\sin\big(tL_\nu^{1/2}/2\big)\big) + L_\nu^{-1/2}.$$

Note that $\sin\big(tL_\nu^{1/2}/2\big)$ is a bounded operator on L^2. Besides, by (1.5), $\big(L_\nu^{-1/2}\sin\big(tL_\nu^{1/2}/2\big)\big)$ verifies $L^p - L^2$ estimates for $\frac{2(2\nu+3)}{2\nu+5} < p \le 2$. So it is enough to verify that $L_\nu^{-1/2}$ sends L^p into L^2.

Lemma 4.1. *Let $\nu \ge -1/2$ and $f \in L^p$, for $\frac{2(2\nu+3)}{2\nu+5} < p \le 2$. Then,*

$$\|L_\nu^{-1/2}f\|_{L^2((0,1),d\mu_\nu)} \le C\|f\|_{L^p((0,1),d\mu_\nu)}.$$

Proof. Note that

$$\begin{aligned}\|L_\nu^{-1/2}f\|^2_{L^2((0,1),d\mu_\nu)} &= \left\|\sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}}P_n(f)\right\|^2_{L^2((0,1),d\mu_\nu)} \\ &\le \sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}^2}\|P_n f\|^2_{L^2((0,1),d\mu_\nu)} \le \sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}^2}\|\phi_n^\nu\|^2_{L^q((0,1),d\mu_\nu)}\|f\|^2_{L^p((0,1),d\mu_\nu)}.\end{aligned}$$

Since $\frac{2(2\nu+3)}{2\nu+5} < p \le 2$, we have that $2 \le q < \frac{2(2\nu+3)}{2\nu+1}$. Besides, it is easy to check that $2 < \frac{2(\nu+1)}{\nu+1/2} < \frac{2\nu+3}{\nu+1/2}$. Now, for $\nu > -1/2$, we distinguish three cases. First, when $2 < q < \frac{2(\nu+1)}{\nu+1/2}$. In this case, by Lemma 2.1, $\|\phi_n^\nu\|_{L^q((0,1),d\mu_\nu)} \simeq C$, so

$$\sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}^2}\|\phi_n^\nu\|^2_{L^q((0,1),d\mu_\nu)}\|f\|^2_{L^p((0,1),d\mu_\nu)} \simeq C\sum_{n=1}^\infty \frac{1}{n^2}\|f\|^2_{L^p((0,1),d\mu_\nu)} \le C\|f\|^2_{L^p((0,1),d\mu_\nu)}.$$

For the second case, when $\frac{2(\nu+1)}{\nu+1/2} < q < \frac{2\nu+3}{\nu+1/2}$, by Lemma 2.1 it is verified that $\|\phi_n^\nu\|_{L^q((0,1),d\mu_\nu)} \simeq n^{(\nu+1/2)-\frac{2(\nu+1)}{q}}$. Now

$$\sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}^2}\|\phi_n^\nu\|^2_{L^q((0,1),d\mu_\nu)}\|f\|^2_{L^p((0,1),d\mu_\nu)} \simeq C\sum_{n=1}^\infty n^{2\nu-1-\frac{4(\nu+1)}{q}}\|f\|^2_{L^p((0,1),d\mu_\nu)},$$

and the series $\sum_{n=1}^\infty n^{2\nu-1-\frac{4(\nu+1)}{q}}$ converges in the corresponding range of q. Finally, for the case $q = \frac{2(\nu+1)}{\nu+1/2}$, Lemma 2.1 says that $\|\phi_n^\nu\|_{L^q((0,1),d\mu_\nu)} \simeq (\log n)^{1/q}$. Therefore,

$$\begin{aligned}\sum_{n=1}^\infty \frac{1}{\lambda_{n,\nu}^2}\|\phi_n^\nu\|^2_{L^q((0,1),d\mu_\nu)}\|f\|^2_{L^p((0,1),d\mu_\nu)} &\simeq C\sum_{n=1}^\infty \frac{1}{n^2}(\log n)^{2/q}\|f\|^2_{L^p((0,1),d\mu_\nu)} \\ &= C\sum_{n=1}^\infty \frac{(\log n)^{\frac{\nu+1/2}{\nu+1}}}{n^2}\|f\|^2_{L^p((0,1),d\mu_\nu)},\end{aligned}$$

and the series converges. For $\nu = -1/2$, an analogous reasoning and the estimate for the L^q-norm of $\phi_n^{-1/2}$ yields the result.

Remark 4.2. Lemma 4.1 can be extended to the range $-1 < \nu < -1/2$, for $1 < p \le 2$, by using Lemma 2.1 for the corresponding values of ν.

$\square$

5. The operator $\mathcal{R}_\nu^*$

We have defined formally the operator $\mathcal{R}_\nu^*$ as

$$\mathcal{R}_\nu^* = L_\nu^{-1/2}\delta_\nu^*.$$

We provide the definition of $\mathcal{R}_\nu^*$ by

$$\mathcal{R}_\nu^* f(x) = \sum_{n=1}^{\infty} \frac{a_n(f)}{\lambda_{n,\nu}} \delta_\nu^* \phi_n^\nu(x), \quad f \in L^2((0,1), d\mu_\nu).$$

First, let us prove that this operator is bounded in $L^2((0,1), d\mu_\nu)$.

Lemma 5.1. *The operator $\mathcal{R}_\nu^*$ is bounded in $L^2((0,1), d\mu_\nu)$, for $\nu > 0$.*

Proof. A computation that uses (2.1) and (2.2) shows that

$$\delta^* \phi_n^\nu(x) = \lambda_{n,\nu} d_{n,\nu} \lambda_{n,\nu}^{1/2} x^{-\nu} \left(\left(1 + \frac{1}{2\nu}\right) J_{\nu-1}(\lambda_{n,\nu} x) + \frac{1}{2\nu} J_{\nu+1}(\lambda_{n,\nu} x) \right).$$

Therefore, we can write

$$\mathcal{R}_\nu^* f(x) = C_\nu \sum_{n=1}^{\infty} a_n(f) \widehat{\phi}_{n,\nu}(x) + C_\nu' \sum_{n=1}^{\infty} a_n(f) \widetilde{\phi}_{n,\nu}(x),$$

where $\widehat{\phi}_{n,\nu}(x) = d_{n,\nu}\lambda_{n,\nu}^{1/2} x^{-\nu} J_{\nu-1}(\lambda_{n,\nu} x)$ and $\widetilde{\phi}_{n,\nu}(x) = d_{n,\nu}\lambda_{n,\nu}^{1/2} x^{-\nu} J_{\nu+1}(\lambda_{n,\nu} x)$. By [2, Lemma 2.4], the functions $\{\widetilde{\phi}_{n,\nu}\}_{n\geq 1}$ form an orthonormal basis in $L^2((0,1), d\mu_\nu)$, and by [2, Lemma 6.1], the functions $\{\widehat{\phi}_{n,\nu}\}_{n\geq 1}$ form an orthonormal basis in $L^2((0,1), d\mu_\nu)$, for $\nu > 0$. Therefore, by Minkowski inequality, we have the result. □

Now we prove the L^p boundedness for $\mathcal{R}_\nu^*$.

Proposition 5.2. *Let $\nu > 0$, $0 < x, y < 1$ and $1 < p < \infty$. There exists a constant C such that for all $f \in L^p((0,1), d\mu_\nu)$, the inequality*

$$\|\mathcal{R}_\nu^* f\|_{L^p((0,1),d\mu_\nu)} \leq C \|f\|_{L^p((0,1),d\mu_\nu)}$$

holds, with C independent of f.

Proof. By (1.2), the operator $\mathcal{R}_\nu^* f$ can be written as

$$\mathcal{R}_\nu^* f(x) = -\sum_{n=1}^{\infty} \frac{a_n(f)}{\lambda_{n,\nu}} \delta_\nu \phi_n^\nu(x) + C_\nu \sum_{n=1}^{\infty} \frac{a_n(f)}{\lambda_{n,\nu} x} \phi_n^\nu(x), \quad f \in L^p((0,1), d\mu_\nu).$$

The first summand coincides with the operator $-\mathcal{R}_\nu^1 f$ defined in [1] that is shown to be bounded in $L^p((0,1), d\mu_\nu)$, see [1, Theorem 1]. Let $0 < r < 1$, for the second summand it is possible to check that

$$\sum_{n=1}^{\infty} \frac{a_n(f)}{\lambda_{n,\nu} x} \phi_n^\nu(x) = \int_0^1 S_\nu(x,y) f(y)\, d\mu_\nu(y),$$

where

$$S_\nu(x,y) = \lim_{r\to 1} S_\nu(r,x,y) \tag{5.1}$$

and

$$S_\nu(r,x,y) = \sum_{n=1}^{\infty} r^n (\lambda_{n,\nu} x)^{-1} \phi_n^\nu(x) \phi_n^\nu(y) = (xy)^{-\nu} x^{-1} \sum_{n=1}^{\infty} r^n d_{n,\nu}^2 J_\nu(\lambda_{n,\nu} x) J_\nu(\lambda_{n,\nu} y).$$

This kernel can be identified with $(xy)^{-\nu} x^{-1} P_{0,\nu,\nu}(r,x,y)$ where $P_{0,\nu,\nu}(r,x,y)$ is the kernel that appears in [1, Section 3]. Then, by [1, Proposition 6] therein, we get

$$|S_\nu(r,x,y)| \leq C(xy)^{-\nu} x^{-1} \begin{cases} x^{-\nu-1} y^\nu, & 0 < y < x/2, \\ (xy)^{-1/2} \log\left(\frac{x}{|x-y|}\right), & x/2 \leq y \leq \min\{1, 3x/2\}, \\ x^\nu y^{-\nu-1}, & \min\{1, 3x/2\} \leq y < 1. \end{cases} \tag{5.2}$$

Finally, the existence of the limit (5.1) is a known result that can be found in [1, Proposition 7] (see the proofs in [3, Proposition 4.2] or [2, Proposition 3.3]). Moreover, $S_\nu(x,y)$ satisfies the same inequality as $S_\nu(r,x,y)$ in (5.2).

Take $f \in L^p((0,1),d\mu_\nu)$. It is sufficient to verify that the quantity

$$\int_0^1 \left| \int_0^1 S_\nu(x,y) f(y) y^{2\nu+1}\, dy \right|^p x^{2\nu+1}\, dx$$

is bounded by $C\|f\|_{L^p((0,1),d\mu_\nu)}$. To check this, split the inner integration onto the intervals $(0,x/2)$, $(x/2,\min\{1,3x/2\})$ and $(\min\{1,3x/2\},1)$, and consider each of the resulting integrals separately. For the first integral, by using (5.2) and Hardy's inequality (2.7), we show that

$$\begin{aligned}\int_0^1 \left| \int_0^{x/2} S_\nu(x,y) f(y) y^{2\nu+1}\, dy \right|^p x^{2\nu+1}\, dx &\le C \int_0^1 \left| \int_0^{x/2} x^{-2\nu-2} f(y) y^{2\nu+1}\, dy \right|^p x^{2\nu+1}\, dx \\ &= \int_0^1 \left| \int_0^{x/2} f(y) y^{2\nu+1}\, dy \right|^p x^{-p(2\nu+2)+2\nu+1}\, dx \\ &\le C \int_0^1 |f(x)|^p x^{2\nu+1}\, dx,\end{aligned}$$

and analogously for the third integral, by (5.2) and (2.8) we obtain

$$\int_0^1 \left| \int_{\min\{1,3x/2\}}^1 S_\nu(x,y) f(y) y^{2\nu+1}\, dy \right|^p x^{2\nu+1}\, dx \le C \int_0^1 |f(x)|^p x^{2\nu+1}\, dx.$$

For the second integral we apply the estimate in (5.2) and the task reduces to bounding the quantity

$$\int_0^1 \left(\int_{x/2}^{\min\{1,3x/2\}} \frac{|f(y)|}{x} \log\left(\frac{x}{|x-y|}\right) dy \right)^p x^{2\nu+1}\, dx$$

by $C\|f\|_{L^p((0,1),d\mu_\nu)}$. But this integral can be treated by copying the argument of [11, p. 39]. □

6. Relationship between the wave equation for the Bessel Laplacian and other equations

Consider the wave equation given in (1.1) with initial data $f \not\equiv 0$ and $g = 0$. In this case, the solution of the wave equation is $u(t,x) = \cos(tL_\nu^{1/2})f(x)$. We are going to relate this solution with the solutions of other two equations. Then, we will deduce L^2 estimates for those solutions from the results obtained for the wave equation solution, whenever the function f considered is in the Sobolev space $W^{1,p}((0,1),d\mu_\nu)$.

The heat equation. We consider the heat equation associated to the Bessel Laplacian

$$\begin{cases} \frac{\partial}{\partial t} v(t,x) = -L_\nu v(t,x), & x \in (0,1),\ t > 0, \\ v(0,x) = f(x). \end{cases} \tag{6.1}$$

The relationship between the solutions of (1.1) with $g = 0$ and (6.1) is expressed through an abstract formula (see for instance [7, p. 120]). For completeness, we show it and give the proof in the following lemma.

Lemma 6.1. *Let $x \in (0,1)$, $t > 0$, $f \in L^2((0,1),d\mu_\nu)$ and $u(s,x)$ be the solution of (1.1) with $g = 0$. Then,*

$$v(t,x) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} u(s,x)\, ds \tag{6.2}$$

is the solution of the initial value problem (6.1). The identity above is understood in $L^2((0,1),d\mu_\nu)$.

Proof. Since $f \in L^2((0,1), d\mu_\nu)$, the solution

$$u(t,x) = \sum_{n=1}^{\infty} \cos(t\lambda_{n,\nu}) P_n f(x)$$

is well defined as a sum in $L^2((0,1), d\mu_\nu)$. It is enough to prove the result for functions of the form

$$f = \sum_{n=1}^{N} P_n f,$$

then the result for functions in $L^2((0,1), d\mu_\nu)$ is obtained by a standard density argument. Observe that the right hand side of (6.2) is well defined:

$$\frac{1}{\sqrt{\pi t}} \int_0^\infty \left| e^{-\frac{s^2}{4t}} u(s,x) \right| ds \leq \frac{1}{\sqrt{\pi t}} \sum_{n=1}^{N} |P_n f(x)| \int_0^\infty e^{-\frac{s^2}{4t}} ds = \sum_{n=1}^{N} |P_n f(x)| < \infty,$$

and

$$v(t,x) = \frac{1}{\sqrt{\pi t}} \sum_{n=1}^{N} P_n f(x) \int_0^\infty e^{-\frac{s^2}{4t}} \cos(s\lambda_{n,\nu})\, ds.$$

With this, by integrating by parts twice, we have

$$\begin{aligned} -L_\nu v(t,x) &= -\frac{1}{\sqrt{\pi t}} \sum_{n=1}^{N} \lambda_{n,\nu}^2 P_n f(x) \int_0^\infty e^{-\frac{s^2}{4t}} \cos(s\lambda_{n,\nu})\, ds \\ &= \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} \frac{\partial^2}{\partial s^2} u(s,x)\, ds = \frac{1}{\sqrt{\pi t}} \int_0^\infty \frac{\partial^2}{\partial s^2} e^{-\frac{s^2}{4t}} u(s,x)\, ds \\ &= \frac{1}{\sqrt{\pi t}} \int_0^\infty \left(\frac{s^2}{4t^2} - \frac{1}{2t} \right) e^{-\frac{s^2}{4t}} u(s,x)\, ds. \end{aligned}$$

On the other hand,

$$\frac{\partial}{\partial t} v(t,x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\partial}{\partial t} \frac{e^{-\frac{s^2}{4t}}}{t^{1/2}} u(s,x)\, ds = \frac{1}{\sqrt{\pi}} \int_0^\infty \left(\frac{s^2}{4t^2} - \frac{1}{2t} \right) \frac{e^{-\frac{s^2}{4t}}}{t^{1/2}} u(s,x)\, ds.$$

Therefore $\frac{\partial}{\partial t} v(t,x) = -L_\nu v(t,x)$; besides, with the change of variables $z = \frac{s^2}{4t}$,

$$\begin{aligned} v(t,x) &= \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-z} u(2\sqrt{tz}, x) \frac{dz}{z^{1/2}} t^{1/2} \\ &= \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-z} u(2\sqrt{tz}, x) \frac{dz}{z^{1/2}} t^{1/2}, \end{aligned}$$

thus $v(0,x) = f(x) \frac{\Gamma(1/2)}{\sqrt{\pi}} = f(x)$. □

We can state, for the solution of the heat equation, the following corollary:

Corollary 6.2. *Let $\nu > 0$ and f be such that $\delta_\nu f$ is in $L^p((0,1), d\mu_\nu)$, $2\frac{2\nu+3}{2\nu+5} < p \leq 2$. Then, the inequality*

$$\sup_{t>0} \|v(t,\cdot)\|_{L^2((0,1),d\mu_\nu)} \leq C \|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)}$$

holds, with a constant C independent of f.

Proof. The proof of this result is easily obtained by using the previous lemma, Minkowski's integral inequality and (1.6) in Theorem 1.1. So,

$$\begin{aligned} \|v(t,\cdot)\|_{L^2((0,1),d\mu_\nu)} &\leq C \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} \|u(s,\cdot)\|_{L^2((0,1),d\mu_\nu)}\, ds \\ &\leq C \|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)} \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} ds = C \|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)}, \end{aligned}$$

for all $t > 0$. □

The extension problem for the fractional powers of the Bessel Laplacian. Let $0<\sigma<1$ and L_ν be the Bessel Laplacian acting on the variable x. Let w be the solution to the extension problem

$$\begin{cases} -L_\nu w(x,y)+\frac{1-2\sigma}{y}\frac{\partial}{\partial y}w(x,y)+\frac{\partial^2}{\partial y^2}w(x,y)=0, & x\in(0,1),\ y>0,\\ w(x,0)=f(x), & \text{on } (0,1). \end{cases} \tag{6.3}$$

The solution w of the partial differential equation (6.3) characterizes the fractional Bessel Laplacian, namely

$$-y^{1-2\sigma}\frac{\partial}{\partial y}w(x,y)\big|_{y=0}=c_\sigma L_\nu^\sigma f(x),$$

where c_σ is a constant depending on σ. The fractional powers L_ν^σ can be defined in a spectral way, see [15] for this result. The extension problem for $-\Delta$ instead of L_ν was first introduced by Cafarelli and Silvestre [4] and then extended to more general second order partial differential operators, see [15]. It is known (see [15, Theorem 1.1]) that a solution of the extension problem is given by

$$w(x,y)=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{y^2}{4t}}v(t,x)\frac{dt}{t^{1+\sigma}}, \tag{6.4}$$

whenever $f\in\operatorname{Dom}(L_\nu^\sigma)$. The relationship between the solution of the extension problem (6.3) and the solution of the wave equation (1.1) with $g=0$ is shown in [6], but we state the result and give the proof, by completeness.

Lemma 6.3. *Let $0<\sigma<1$, $x\in(0,1), y>0$ and $f\in\operatorname{Dom}(L_\nu^\sigma)$. Then, a solution of the extension problem can be written as*

$$w(x,y)=\frac{2\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(\sigma)}\int_0^\infty\frac{y^{2\sigma}}{(y^2+s^2)^{\sigma+1/2}}u(s,x)\,ds,$$

where $u(s,x)$ is the solution of the equation (1.1) with $g=0$. The identity above is understood in $L^2((0,1),d\mu_\nu)$.

Proof. Applying (6.2) in (6.4) and with the change of variable $z=\frac{y^2+s^2}{4s}$, we obtain

$$\begin{aligned} w(x,y)&=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-\frac{y^2}{4t}}\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}u(s,x)\,ds\frac{dt}{t^{1+\sigma}}\\ &=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)\sqrt{\pi}}\int_0^\infty\int_0^\infty\frac{e^{-\frac{y^2+s^2}{4t}}}{t^{\sigma+1/2}}\frac{dt}{t}u(s,x)\,ds\\ &=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)\sqrt{\pi}}\int_0^\infty\int_0^\infty e^{-z}\frac{(4z)^{\sigma+1/2}}{(y^2+s^2)^{\sigma+1/2}}\frac{dz}{z}u(s,x)\,ds\\ &=\frac{\Gamma(\sigma+1/2)4^{\sigma+1/2}}{4^\sigma\Gamma(\sigma)\sqrt{\pi}}\int_0^\infty\frac{y^{2\sigma}}{(y^2+s^2)^{\sigma+1/2}}u(s,x)\,ds. \end{aligned}$$

Besides, making the change $\bar{z}=\frac{s^2}{y^2}$ one has

$$w(x,y)=\frac{\Gamma(\sigma+1/2)4^{\sigma+1/2}}{4^\sigma\Gamma(\sigma)\sqrt{\pi}}\frac{1}{2}\int_0^\infty\frac{\bar{z}^{-1/2}}{(1+\bar{z})^{\sigma+1/2}}u(y\sqrt{\bar{z}},x)\,d\bar{z}.$$

Therefore, $w(x,0)=f(x)$, since

$$\int_0^\infty\frac{\bar{z}^{-1/2}}{(1+\bar{z})^{\sigma+1/2}}\,d\bar{z}=\frac{\Gamma(1/2)\Gamma(\sigma)}{\Gamma(\sigma+1/2)}.$$

□

In the end, we have the following result concerning the solution of the extension problem:

Corollary 6.4. *Let $\nu > 0$ and f be such that $\delta_\nu f$ is in $L^p((0,1), d\mu_\nu)$, $2\frac{2\nu+3}{2\nu+5} < p \le 2$. Then,*

$$\sup_{y>0} \|w(\cdot, y)\|_{L^2((0,1),d\mu_\nu)} \le C\|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)},$$

with a constant C independent of f.

Proof. In order to obtain the proof of this corollary, we use Lemma 6.3, Minkowski's integral inequality and (1.6) in Theorem 1.1. In this way,

$$\begin{aligned}\sup_{y>0} \|w(\cdot, y)\|_{L^2((0,1),d\mu_\nu)} &\le C \frac{2\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(\sigma)} \sup_{y>0} \int_0^\infty \frac{y^{2\sigma}}{(y^2+t^2)^{\sigma+1/2}} \|u(t,\cdot)\|_{L^2((0,1),d\mu_\nu)}\, dt \\ &\le C\|\delta_\nu f\|_{L^p((0,1),d\mu_\nu)} \frac{2\Gamma(\sigma+1/2)}{\sqrt{\pi}\Gamma(\sigma)} \sup_{y>0} \int_0^\infty \frac{y^{2\sigma}}{(y^2+t^2)^{\sigma+1/2}}\, dt,\end{aligned}$$

and the last integral was computed some lines above, in the proof of Lemma 6.3. From this, it follows the desired result. □

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