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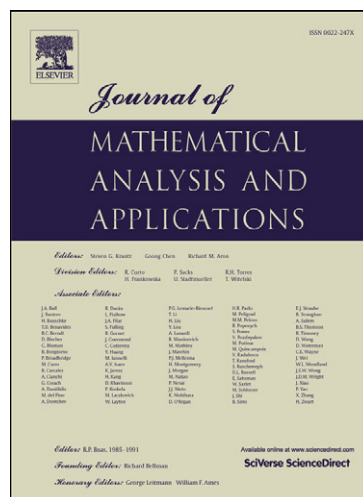
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Estimates for multiparameter maximal operators of Schrödinger type

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Abstract

Multiparameter maximal estimates are considered for operators of Schrödinger type. Sharp and almost sharp results, that extend work by Rogers and Villarroya, are obtained. We provide new estimates via the integrability of the kernel which naturally appears with a TT^* argument and discuss the behavior at the endpoints. We treat in particular the case of global integrability of the maximal operator on finite time for solutions to the linear Schrödinger equation and make some comments on an open problem.

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1. Introduction and main results

Assuming $a > 1$ and letting f belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, we set

$$S_t f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{it|\xi|^a} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Here \widehat{f} denotes the Fourier transform of the function f , defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

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We also set $u(x, t) = (2\pi)^{-n} S_t f(x)$. It then follows that $u(x, 0) = f(x)$ and in the case $a = 2$, the function u satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$. Also, more generally, if $a = 2k$ for some $k = 1, 2, 3, \dots$, then u satisfies the equation $i\partial u/\partial t = \Delta^k u$, if k is odd and $i\partial u/\partial t = -\Delta^k u$, if k is even.

We shall study the maximal function $S^* f$ defined by

$$S^* f(x) = \sup_{0 < t < 1} |S_t f(x)|, \quad x \in \mathbb{R}^n,$$

and define Sobolev spaces H_s by setting

$$H_s = \{f \in \mathcal{S}' : \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The homogeneous Sobolev spaces \dot{H}_s , for $s \in \mathbb{R}$, are defined by

$$\dot{H}_s = \{f \in \mathcal{S}' : \|f\|_{\dot{H}_s} < \infty\},$$

where

$$\|f\|_{\dot{H}_s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The inequality

$$\|S^* f\|_{L^2(B)} \leq C \|f\|_{H_s}, \quad (1)$$

for arbitrary balls B has been studied by several authors. In the case $n = 1$, it is known that (1) holds if and only if $s \geq 1/4$ (see Carleson [2], Dahlberg and Kenig [3], and Sjölin [10]). In the case $n = 2$ and $a = 2$, Lee [8], extending previous results in [18] and [17], has proved that (1) holds for $s > 3/8$. In the case $n \geq 3$, Sjölin [10] and Vega [19] proved that (1) holds for $s > 1/2$.

As is well known, the inequality (1) implies that

$$\lim_{t \rightarrow 0} \frac{1}{(2\pi)^n} S_t f(x) = f(x), \quad a.e.,$$

for every $f \in H_s$. The above estimates therefore give pointwise convergence results. In the case $a = 2$, Bourgain [1] has recently improved these results and

proved that one has convergence almost everywhere for every $f \in H_s(\mathbb{R}^n)$ if $s > 1/2 - 1/4n$. On the other hand Bourgain has also proved that one does not have convergence almost everywhere for all $f \in H_s(\mathbb{R}^n)$ if $n \geq 5$ and $s < 1/2 - 1/n$.

For $n = 1$ and $a > 1$ we set $M^*f = S^*f$ and

$$M^{**}f(x) = \sup_{t \in \mathbb{R}} |S_t f(x)|, \quad x \in \mathbb{R}.$$

In harmonic analysis considerable attention has been given to multiparameter singular integrals and related operators. Some examples of this can be seen in the work of E.M. Stein and R. Fefferman [4], [5], [6], [7]. In this paper we introduce in the same spirit multiparameter operators of Schrödinger type.

For $n \geq 2$ and a multiindex $a = (a_1, a_2, \dots, a_n)$, with $a_j > 1$, and $f \in \mathcal{S}(\mathbb{R}^n)$, we now set

$$S_t f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1 |\xi_1|^{a_1} + t_2 |\xi_2|^{a_2} + \dots + t_n |\xi_n|^{a_n})} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

where $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$. In the remaining part of this paper, S_t will be defined in this way if $n \geq 2$. Finally, we will define maximal operators for $n \geq 2$ by letting

$$M^*f(x) = \sup_{0 < t_i < 1} |S_t f(x)|, \quad x \in \mathbb{R}^n,$$

and

$$M^{**}f(x) = \sup_{t_i \in \mathbb{R}} |S_t f(x)|, \quad x \in \mathbb{R}^n.$$

In this paper we will study the inequality

$$\|M^{**}f\|_q \leq C \|f\|_{\dot{H}_s}, \quad (2)$$

as well as

$$\|M^*f\|_q \leq C \|f\|_{H_s}, \quad (3)$$

for different values of $s \in \mathbb{R}$, $1 \leq q \leq \infty$ and the multiindex a . Here we shall use the notation $\|\cdot\|_q = \|\cdot\|_{L^q(\mathbb{R}^n)}$.

We can state the following results. The first two theorems are concerned with the case $n = 1$. Some parts of them are already known, but we bring them here for the sake of completeness.

Theorem 1.1. Assume $n = 1$. Then, for every a the inequality (2) holds if and only if $4 \leq q < \infty$ and $s = 1/2 - 1/q$.

Theorem 1.1 is proved in [13], p.135.

Theorem 1.2. Assume $n = 1$. Then, for every a we have:

For $1 \leq q < 2$ (3) holds for no s .

For $q = 2$ (3) holds for $s > a/4$ and does not hold for $s < a/4$.

For $2 < q < 4$ (3) holds if and only if $s \geq 1/2 - a/4 + a/q - 1/q$.

For $4 \leq q < \infty$ (3) holds if and only if $s \geq 1/2 - 1/q$.

For $q = \infty$ (3) holds if and only if $s > 1/2$.

The case $q = 2$, $s = a/4$ in the above theorem remains open. Theorem 1.2 is well known (see [12]) except for the case $2 < q < 4$, $s = 1/2 - a/4 + a/q - 1/q$, which has been proved for $a = 2$ by Rogers and Villarroya [9] and will be proved for $a \neq 2$ in this paper.

We now consider the situation of several variables, that is, the multiparameter case.

Theorem 1.3. Assume $n \geq 2$. Then, for every a the inequality (2) holds if and only if $4 \leq q < \infty$ and $s = n(1/2 - 1/q)$.

For $n \geq 2$ and $a = (a_1, a_2, \dots, a_n)$ we set $|a| = a_1 + a_2 + \dots + a_n$.

Theorem 1.4. Assume $n \geq 2$. Then, for every a we have:

For $1 \leq q < 2$ (3) holds for no s .

For $q = 2$ (3) holds for $s > |a|/4$ and does not hold for $s < |a|/4$.

For $2 < q < 4$ (3) holds if and only if $s \geq n/2 - |a|/4 + |a|/q - n/q$.

For $4 \leq q < \infty$ (3) holds if and only if $s \geq n(1/2 - 1/q)$.

For $q = \infty$ (3) holds if and only if $s > n/2$.

In the above theorem, the case $q = 2$, $s = |a|/4$ remains open.

Now, set $a = (a_1, a_2, \dots, a_n)$ with $a_j > 1$, and set $t = (t_1, t_2, \dots, t_n)$ with $0 < t_j < 1$. Also, for $\xi_j \in \mathbb{R}$, let $e^{i|\xi_j|^{a_j}}$ have Fourier transform K^j . It is known that $K^j \in \mathcal{C}^\infty(\mathbb{R})$. The function $K_{t_j}^j$ defined as

$$K_{t_j}^j(x_j) = \frac{1}{t_j^{a_j}} K^j\left(\frac{x_j}{t_j^{a_j}}\right), \quad x_j \in \mathbb{R},$$

is then the Fourier transform of $e^{it_j|\xi_j|^{a_j}}$. Hence, $e^{it_1|\xi_1|^{a_1}} e^{it_2|\xi_2|^{a_2}} \dots e^{it_n|\xi_n|^{a_n}}$ has Fourier transform

$$K_t(x) = K_{t_1}^1(x_1) K_{t_2}^2(x_2) \dots K_{t_n}^n(x_n), \quad x \in \mathbb{R}^n,$$

with $K_t \in \mathcal{C}^\infty \cap \mathcal{S}'$. Invoking the definition of S_t , we have the identity

$$S_t f(x) = \int_{\mathbb{R}^n} K_t(y) f(x - y) dy = K_t * f(x), \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

We then set

$$T_t f(x) = (2\pi)^{-n} K_t * f(x),$$

for $f \in L^2(\mathbb{R}^n)$ with compact support. Using a standard argument relating maximal functions and pointwise convergence, one can then prove that Theorem 1.4 has the following consequence.

Corollary 1.1. *Assume that $f \in H_{n/4}(\mathbb{R}^n)$ and that f has compact support. Then*

$$\lim_{t \rightarrow 0} T_t f(x) = f(x),$$

for almost every $x \in \mathbb{R}^n$.

We remark that in the above theorems one cannot take $a = 1$. In fact, the case $a = 1$ is more related to the wave equation than to the Schrödinger equation. We also remark that maximal estimates of the above type have been used to study, among other things, nonlinear equations of Schrödinger type.

In Section 2 we shall state several lemmas. In Section 3 we will give proofs of these lemmas, whereas Section 4 will be devoted to the proof of the above theorems. In Section 5 we shall make several remarks on the inequality (3) in the open case $n = 1$, $a = 2$, $q = 2$, and $s = 1/2$.

2. Some lemmas on oscillatory integrals

In this section we will state several lemmas on oscillatory integrals that may be of interest on their own. They will be used in Section 4 to prove the above theorems.

Lemma 2.1. Assume that $a > 1$, $1/2 \leq s < 1$ and $\mu \in C_0^\infty(\mathbb{R})$. Then,

$$\left| \int_{\mathbb{R}} e^{ix \cdot \xi} e^{it|\xi|^a} |\xi|^{-s} \mu(\xi/N) d\xi \right| \leq C \frac{1}{|x|^{1-s}},$$

for $x \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$, and $N = 1, 2, 3, \dots$

Here C depends on a , s , and μ , but is independent of x , t , and N .

Lemma 2.1 is contained in [15].

Lemma 2.2. Assume that $a > 1$, $1/2 \leq \alpha \leq a/2$, $-1 < d < 1$, and $\mu \in C_0^\infty(\mathbb{R})$. Then,

$$\left| \int_{\mathbb{R}} \frac{e^{i(d|\xi|^a - x\xi)}}{(1 + \xi^2)^{\alpha/2}} \mu(\xi/N) d\xi \right| \leq C \frac{1}{|x|^\beta},$$

for $x \in \mathbb{R} \setminus \{0\}$ and $N = 1, 2, 3, \dots$, where

$$\beta = \frac{\alpha + a/2 - 1}{a - 1}.$$

Here C depends on a , α , and μ , but is independent of x , d , and N .

Observe that the definition of β implies that $1/2 \leq \beta \leq 1$. Lemma 2.2 for $a = 2$ is essentially due to Rogers and Villarroya [9]. For $a = 2$ one has $\beta = \alpha$.

Lemma 2.3. Assume that $a > 1$, $\alpha = a/2$, $-1 < d < 1$, $\mu \in C_0^\infty(\mathbb{R})$, and $\epsilon > 0$. Then, there exists a $K \in L^1(\mathbb{R})$ such that

$$\left| \int_{\mathbb{R}} \frac{e^{i(d|\xi|^a - x\xi)}}{(1 + \xi^2)^{\alpha/2} [\log(2 + \xi^2)]^{1+\epsilon}} \mu(\xi/N) d\xi \right| \leq C K(x),$$

for $x \in \mathbb{R} \setminus \{0\}$ and $N = 1, 2, 3, \dots$. Moreover, there exists a large constant C_0 such that for $|x| \geq C_0$

$$K(x) \leq C \frac{1}{|x|(\log |x|)^{1+\epsilon}},$$

whereas for $|x| < C_0$ one has

- (i) $K(x) \leq C$, if $\alpha \geq 1$, and
- (ii) $K(x) \leq C \frac{1}{|x|^{1-\alpha}}$, if $1/2 < \alpha < 1$.

Lemma 2.1 will be used in the case $4 \leq q < \infty$, while Lemma 2.2 will be used for $2 < q < 4$. Lemma 2.3, finally, will be used in the case $q = 2$.

3. Proofs of the Lemmas

Proof of Lemma 2.2. We shall use the following variants of van der Corput's Lemma (see Stein [16], p.334):

Assume $a < b$ and set $I = [a, b]$. Let $F \in C^\infty(I)$ be real valued and let $\psi \in C^\infty(I)$.

(i) Assume that $|F'(x)| \geq \gamma > 0$ for $x \in I$ and that F' is monotonic on I . Then

$$\left| \int_a^b e^{iF(x)} \psi(x) dx \right| \leq C \frac{1}{\gamma} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right),$$

where C does not depend on F , ψ and I .

(ii) Assume that $|F''(x)| \geq \gamma > 0$ for $x \in I$. Then

$$\left| \int_a^b e^{iF(x)} \psi(x) dx \right| \leq C \frac{1}{\gamma^{1/2}} \left(|\psi(b)| + \int_a^b |\psi'(x)| dx \right),$$

where C , again, does not depend on F , ψ and I .

In the proof of the lemma we may assume $d > 0$. Clearly, it suffices to estimate

$$\left| \int_0^\infty \frac{e^{i(d|\xi|^a - x\xi)}}{(1 + \xi^2)^{\alpha/2}} \mu(\xi/N) d\xi \right|.$$

Take first $|x|$ large. Set $F(\xi) = d\xi^a - x\xi$. Then $F'(\xi) = da\xi^{a-1} - x$ and $F''(\xi) = da(a-1)\xi^{a-2}$. We also set

$$\rho = (|x|/d)^{1/(a-1)}; \quad I_1 = [0, \delta\rho]; \quad I_2 = [\delta\rho, K\rho], \quad I_3 = [K\rho, \infty),$$

where δ is to be considered small and K large. On I_2 we have for a small positive constant c

$$|F''(x)| \geq cd(|x|/d)^{(a-2)/(a-1)}.$$

Setting

$$\psi(\xi) = (1 + \xi^2)^{-\alpha/2} \mu(\xi/N),$$

we have

$$\max_{I_2} |\psi| + \int_{I_2} |\psi'| d\xi \leq C (|x|/d)^{-\alpha/(a-1)},$$

since $\frac{d}{d\xi}(\mu(\xi/N)) \leq c \frac{1}{1+\xi}$, for $\xi \geq 0$. van der Corput's Lemma then gives

$$\begin{aligned} \left| \int_{I_2} e^{iF(\xi)} \psi(\xi) d\xi \right| &\leq C d^{-1/2} \left(\frac{|x|}{d} \right)^{-\frac{a-2}{2(a-1)}} \left(\frac{|x|}{d} \right)^{-\frac{\alpha}{a-1}} \\ &= C \frac{d^{\frac{\alpha-1/2}{a-1}}}{|x|^{\frac{\alpha+a/2-1}{a-1}}} \leq C \frac{1}{|x|^\beta}, \end{aligned}$$

where we have used in the last inequality that $\frac{\alpha-1/2}{a-1} \geq 0$ and that $0 < d < 1$.

On I_1 we have $\xi \leq \delta (|x|/d)^{1/(a-1)}$. Hence, $d\xi^{a-1} \leq \delta^{a-1}|x|$. It follows that $|F'(\xi)| \geq c|x|$ on I_1 . We also have

$$\max_{I_1} |\psi| + \int_{I_1} |\psi'| d\xi \leq C,$$

and van der Corput now gives

$$\left| \int_{I_1} e^{iF(\xi)} \psi(\xi) d\xi \right| \leq C \frac{1}{|x|} \leq C \frac{1}{|x|^\beta}.$$

On I_3 we have $\xi \geq K (|x|/d)^{1/(a-1)}$. Hence, $d\xi^{a-1} \geq K^{a-1}|x|$, which implies $|F'(\xi)| \geq c|x|$. Invoking van der Corput again we get

$$\left| \int_{I_3} e^{iF(\xi)} \psi(\xi) d\xi \right| \leq C \frac{1}{|x|} \leq C \frac{1}{|x|^\beta}.$$

We now consider the case of small values of x ($|x| < C_0$). We shall consider the cases $\alpha > 1$, $1/2 \leq \alpha < 1$ and $\alpha = 1$ separately. The case $\alpha > 1$ is trivial since

$$\left| \int_0^\infty e^{iF(\xi)} \psi(\xi) d\xi \right| \leq \int_0^\infty |\psi(\xi)| d\xi \leq C \leq C \frac{1}{|x|^\beta}.$$

For $1/2 \leq \alpha < 1$ we use the fact that from the Mean Value Theorem,

$$0 < (1 + \xi^2)^{\alpha/2} - \xi^\alpha \leq (\alpha/2) \xi^{2(\frac{\alpha}{2}-1)} \leq \xi^{\alpha-2}$$

and, therefore,

$$\frac{1}{\xi^\alpha} - \frac{1}{(1 + \xi^2)^{\alpha/2}} = \mathcal{O}\left(\frac{1}{\xi^{\alpha+2}}\right),$$

as $\xi \rightarrow \infty$. It follows that

$$\int_0^\infty \left| \frac{1}{\xi^\alpha} - \frac{1}{(1 + \xi^2)^{\alpha/2}} \right| d\xi < \infty.$$

and

$$\left| \int_0^\infty e^{iF(\xi)} \psi(\xi) d\xi \right| \leq C + \left| \int_0^\infty e^{iF(\xi)} \xi^{-\alpha} \mu(\xi/N) d\xi \right| \leq C \frac{1}{|x|^{1-\alpha}},$$

where we have used Lemma 2.1 (replacing the integral over \mathbb{R} with an integral only on $[0, \infty]$). Observing that $1 - \alpha \leq \beta$, this concludes the case $1/2 \leq \alpha < 1$.

For the case $\alpha = 1$ we use the argument in the proof of Lemma 2.1 in [15]. Here one obtains

$$\left| \int_0^\infty e^{iF(\xi)} \psi(\xi) d\xi \right| \leq C \log\left(\frac{1}{|x|}\right), \quad 0 < |x| \leq 1/2.$$

and

$$\left| \int_0^\infty e^{iF(\xi)} \psi(\xi) d\xi \right| \leq C, \quad 1/2 < |x| < C_0.$$

This finishes the proof of Lemma 2.2. \square

Proof of Lemma 2.3. We shall first assume that $|x|$ is large. Choose an even function $\phi_0 \in \mathcal{C}^\infty$ such that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$ and $\phi_0(\xi) = 0$ for $|\xi| \geq 1$. Set

$$\psi(\xi) = (1 + \xi^2)^{-\alpha/2} [\log(2 + \xi^2)]^{-1-\epsilon} \mu(\xi/N),$$

and $\psi_0 = \psi \phi_0$ so that $\text{supp } \psi_0 \subset [-1, 1]$. We may assume $d > 0$. Let $\rho = (|x|/da)^{1/(a-1)}$. Take C_0 large so that $|x| \geq C_0$ implies $\rho \geq 1000$. Also, take $C_0 > a^2$, K large and assume $|x| \geq C_0$. Choose $\phi_2 \in \mathcal{C}_0^\infty$ so that $\text{supp } \phi_2 \subset [\rho/4, 2K\rho]$ and $\phi_2(\xi) = 1$ for $\rho/2 \leq \xi \leq K\rho$. We may also assume that $|\phi_2'(\xi)| \leq C\xi^{-1}$ and $|\phi_2''(\xi)| \leq C\xi^{-2}$ for $\xi > 0$. Set $\phi_3 = (1 - \phi_2)\chi_{[K\rho, \infty)}$ and $\phi_1 = (1 - \phi_2 - \phi_0)\chi_{[0, \rho/2]}$.

For $j = 1, 2, 3$, define $\phi_{-j}(\xi) = \phi_j(-\xi)$ and $F(\xi) = d|\xi|^a - x\xi$. We then have

$$\int_{-\infty}^{\infty} e^{iF(\xi)} \psi(\xi) d\xi = \sum_{j=-3}^3 \int_{-\infty}^{\infty} e^{iF(\xi)} \psi(\xi) \phi_j(\xi) d\xi.$$

The estimates for $j = -1, -2, -3$ can be easily deduced from the cases $j = 1, 2, 3$, respectively. Setting $\psi_j = \psi \phi_j$, $j = 1, 2, 3$, we will only consider the integrals

$$J_j = \int e^{iF(\xi)} \psi_j(\xi) d\xi, \quad j = 0, 1, 2, 3.$$

Integrating by parts twice, we get

$$J_0 = \int e^{-ix\xi} e^{id|\xi|^a} \psi_0(\xi) d\xi = \frac{-1}{x^2} \int_{-1}^1 e^{-ix\xi} L(\xi) d\xi,$$

where

$$L(\xi) = \left(\frac{d}{d\xi} \right)^2 (e^{id|\xi|^a} \psi_0(\xi)), \quad \xi \neq 0.$$

The second integration by parts is justified since

$$\frac{d}{d\xi} (e^{id|\xi|^a} \psi_0(\xi)) = \psi_0'(\xi) e^{id|\xi|^a} + \psi_0(\xi) i d a \operatorname{sign}(\xi) |\xi|^{a-1} e^{id|\xi|^a} \equiv A(\xi) + B(\xi),$$

A and B are both continuous, A is differentiable $\forall \xi$ and B is differentiable for all $\xi \neq 0$. Moreover, $B(0) = 0$ and $B'(\xi)$ is integrable in $[-1, 0)$ and $(0, 1]$. We deduce then that

$$J_0 = \mathcal{O} \left(\frac{1}{x^2} \right),$$

since, for $-1 \leq \xi \leq 1$,

$$L(\xi) = \mathcal{O}(|\xi|^{a-2}),$$

and this says, as for B' , that L is integrable in $[-1, 1]$ when $a > 1$.

For the remaining estimates we observe that for $j = 1, 2, 3$ and $\xi \geq 1/2$

$$|\psi_j(\xi)| \leq C(1 + \xi^2)^{-\alpha/2} [\log(2 + \xi^2)]^{-1-\epsilon},$$

$$|\psi_j'(\xi)| \leq C\xi^{-1}(1 + \xi^2)^{-\alpha/2} [\log(2 + \xi^2)]^{-1-\epsilon},$$

and

$$|\psi_j''(\xi)| \leq C\xi^{-2}(1 + \xi^2)^{-\alpha/2} [\log(2 + \xi^2)]^{-1-\epsilon}.$$

On the interval $[\rho/4, 2K\rho]$ we have

$$F''(\xi) = da(a-1)\xi^{a-2} \geq cd \left(\frac{|x|}{d} \right)^{(a-2)/(a-1)},$$

for a small constant $c > 0$. Also,

$$\max |\psi_2| + \int |\psi'_2| d\xi \leq C \frac{1}{\rho^\alpha (\log \rho)^{1+\epsilon}} \leq C \left(\frac{|x|}{d} \right)^{-\alpha/(a-1)} \frac{1}{(\log |x|)^{1+\epsilon}}.$$

Using van der Corput's Lemma with the second derivative we obtain

$$\begin{aligned} |J_2| &\leq Cd^{-1/2} \left(\frac{|x|}{d} \right)^{-(a-2)/2(a-1)} \left(\frac{|x|}{d} \right)^{-\alpha/(a-1)} \frac{1}{(\log |x|)^{1+\epsilon}} \\ &= Cd^{1/2} \frac{1}{|x|(\log |x|)^{1+\epsilon}} \leq C \frac{1}{|x|(\log |x|)^{1+\epsilon}}. \end{aligned}$$

To estimate J_1 observe that $\text{supp } \psi_1 \subset [1/2, \rho/2]$. On this interval one has $da\xi^{a-1} \leq da(\rho/2)^{a-1} = 2^{1-a}|x|$ and $|F'(\xi)| = |da\xi^{a-1} - x| \geq c|x| \geq cd\xi^{a-1}$. It follows that

$$\frac{|F''(\xi)|}{|F'(\xi)|} \leq \frac{1}{\xi}, \quad \text{and} \quad \frac{|F'''(\xi)|}{|F'(\xi)|} \leq \frac{1}{\xi^2},$$

for $1/2 \leq \xi \leq \rho/2$. Integrating by parts twice we obtain

$$J_1 = \int e^{iF} \psi_1 d\xi = \int e^{iF} \frac{d}{d\xi} \left(\frac{1}{iF'} \frac{d}{d\xi} \left(\frac{\psi_1}{iF'} \right) \right) d\xi. \quad (4)$$

Now,

$$\begin{aligned} \left| \frac{d}{d\xi} \left(\frac{1}{iF'} \frac{d}{d\xi} \left(\frac{\psi_1}{iF'} \right) \right) \right| &\leq \frac{|\psi_1|}{|F'|^2} \left(\frac{|F'''|}{|F'|} + 3 \frac{|F''|^2}{|F'|^2} \right) + \frac{|\psi'_1|}{|F'|^2} \frac{|F''|}{|F'|} + \frac{|\psi''_1|}{|F'|^2} \\ &= \mathcal{O} \left(\frac{1}{|x|^2 \xi^{\alpha+2}} \right). \end{aligned}$$

Hence

$$|J_2| \leq C \int_{1/2}^{\infty} \frac{1}{|x|^2 \xi^{\alpha+2}} d\xi = \mathcal{O} \left(\frac{1}{|x|^2} \right).$$

It remains to estimate $J_3 = \int e^{iF} \psi_3 d\xi$. Here $\text{supp } \psi_3 \subset [K\rho, \infty]$, and on this interval $da\xi^{a-1} \geq K^{a-1}|x|$ and $|F'(\xi)| \geq c|x|$ and $|F'(\xi)| \geq cda\xi^{a-1}$. Using the same argument (4) as for J_1 we obtain $|J_3| \leq C/|x|^2$.

To finish with the proof of Lemma 2.3 we must consider the case $|x| < C_0$. As before, the case $\alpha \geq 1$ is trivial due to the integrability of the function ψ , and we obtain $K(x) \leq C$. When $1/2 < \alpha < 1$, the proof of Lemma 2.1 in [15] shows directly that we can take $K(x) = C/|x|^{1-\alpha}$. \square

4. Proofs of the theorems

Proof of the case $2 < q < 4$, $s = 1/2 - a/4 + a/q - 1/q$, in Theorem 1.2. Set

$$Sf(x) = \int_{\mathbb{R}} e^{it(x)|\xi|^a} e^{ix\xi} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R},$$

where $t(x)$ is measurable and $0 < t(x) < 1$. We want to prove

$$\|Sf\|_q \leq C\|f\|_{H_s} = \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.$$

We set $g(\xi) = \widehat{f}(\xi)(1 + \xi^2)^{s/2}$ and

$$Tg(x) = \int_{\mathbb{R}} e^{it(x)|\xi|^a} e^{ix\xi} (1 + \xi^2)^{-s/2} g(\xi) d\xi.$$

Then $Sf(x) = Tg(x)$ and it is sufficient to prove that

$$\|Tg\|_q \leq C\|g\|_2.$$

For $N = 1, 2, 3, \dots$ we set

$$T_N g(x) = \chi_N(x) \int_{\mathbb{R}} e^{it(x)|\xi|^a} e^{ix\xi} (1 + \xi^2)^{-s/2} \rho_N(\xi) g(\xi) d\xi.$$

Here $\chi_N(x) = \chi(x/N)$ and $\rho_N(x) = \rho(x/N)$, where χ and ρ are two cut-off functions in C_0^∞ so that $\chi(x) = \rho(x) = 1$ for $|x| \leq 1$ and $\chi(x) = \rho(x) = 0$ for $|x| \geq 2$. They are also assumed to be real-valued. It is sufficient to prove

$$\|T_N g\|_q \leq C\|g\|_2,$$

with constant C independent of N . Its adjoint has the form

$$T_N^* h(\xi) = (1 + \xi^2)^{-s/2} \rho_N(\xi) \int_{\mathbb{R}} e^{-it(x)|\xi|^a} e^{-ix\xi} \chi_N(x) h(x) dx.$$

So, it suffices to show

$$\|T_N^* h\|_2 \leq C \|h\|_{q'}, \quad N = 1, 2, 3, \dots \quad (5)$$

We observe that

$$\|T_N^* h\|_2^2 = \int \int I_N(x, y) \chi_N(x) \chi_N(y) h(x) \overline{h(y)} dx dy,$$

where

$$I_N(x, y) = \int (1 + \xi^2)^{-s} e^{i(y-x)\xi} e^{i(t(y)-t(x))|\xi|^a} \mu(\xi/N) d\xi,$$

and $\mu = \rho^2$.

The assumptions $2 < q < 4$ and $s = 1/2 - a/4 + a/q - 1/q$ imply that $1/4 < s < a/4$. Setting $\alpha = 2s$ we then have $1/2 < \alpha < a/2$. Lemma 2.2 then yields

$$|I_N(x, y)| \leq C |x - y|^{-\beta},$$

where $\beta = (\alpha + a/2 - 1)/(a - 1)$. It follows that $1/2 < \beta < 1$. Set $r = 1 - \beta$. We define a Riesz potential operator I_r by setting

$$I_r h(x) = \int_{\mathbb{R}} \frac{1}{|x - y|^{1-r}} h(y) dy, \quad x \in \mathbb{R}.$$

It is not difficult to see that $r = \frac{1}{q'} - \frac{1}{q}$, that is $\frac{1}{q} = \frac{1}{q'} - r$. It follows that,

$$\|I_r h\|_q \leq C \|h\|_{q'}.$$

Hence,

$$\begin{aligned} \|T_N^* h\|_2^2 &\leq C \int \int |x - y|^{-\beta} |h(x)| |h(y)| dx dy = C \int |h(x)| I_r(|h|)(x) dx \\ &\leq C \|h\|_{q'} \|I_r(|h|)\|_q \leq C \|h\|_{q'}^2. \end{aligned}$$

Hence (5) follows and the proof is complete. \square

Proof of Theorem 1.3. We first assume $4 \leq q < \infty$ and $s = n(1/2 - 1/q)$, that is $n/4 \leq s < n/2$ and $q = 2n/(n - 2s)$. We set

$$Sf(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1(x)|\xi_1|^{a_1} + t_2(x)|\xi_2|^{a_2} + \dots + t_n(x)|\xi_n|^{a_n})} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

where $t_i(x)$ are measurable and $t_i(x) \in \mathbb{R}$. We want to prove that

$$\|Sf\|_q \leq C \|f\|_{\dot{H}^s} = C \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \right)^{1/2}.$$

This will follow obviously from the inequality

$$\|Sf\|_q \leq C \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi_1|^{2s/n} \dots |\xi_n|^{2s/n} d\xi \right)^{1/2}.$$

Set $g(\xi) = \widehat{f}(\xi) |\xi_1|^{s/n} \dots |\xi_n|^{s/n}$ and

$$Tg(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1(x)|\xi_1|^{a_1} + t_2(x)|\xi_2|^{a_2} + \dots + t_n(x)|\xi_n|^{a_n})} g(\xi) |\xi_1|^{-s/n} \dots |\xi_n|^{-s/n} d\xi.$$

Then $Sf(x) = Tg(x)$ and the estimate we want now is

$$\|Tg\|_q \leq C \|g\|_2.$$

This can be proved by using Lemma 2.1 and applying the argument in [15]. We omit the details.

We shall now study the necessity of the conditions in Theorem 1.3. Assume that

$$\|M^{**}f\|_q \leq C \|f\|_{\dot{H}^s}.$$

Set $f_R(x) = f(Rx)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and $R > 0$. Then $\widehat{f_R}(\xi) = R^{-n} \widehat{f}(\xi/R)$ and setting $\xi = R\eta$ we obtain for $t = (t_1, t_2, \dots, t_n)$

$$\begin{aligned} S_t f_R(x) &= \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1|\xi_1|^{a_1} + t_2|\xi_2|^{a_2} + \dots + t_n|\xi_n|^{a_n})} R^{-n} \widehat{f}(\xi/R) d\xi \\ &= \int_{\mathbb{R}^n} e^{ix \cdot R\eta} e^{i(t_1 R^{a_1} |\eta_1|^{a_1} + t_2 R^{a_2} |\eta_2|^{a_2} + \dots + t_n R^{a_n} |\eta_n|^{a_n})} \widehat{f}(\eta) d\xi \\ &= S_t f(Rx), \end{aligned}$$

where $\bar{t} = (t_1 R^{a_1}, t_2 R^{a_2}, \dots, t_n R^{a_n})$. It follows that $M^{**} f_R(x) = M^{**} f(Rx)$. As in [15] one then proves that $q = 2n/(n - 2s)$ and $s \leq n/2$. A counter-example in [14], pp. 400-401, shows that the case $s = n/2$ is not possible.

It remains to prove that $s \geq n/4$. We shall use a counter-example in [10], pp. 712-713. Choose $g \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\int g(\xi) d\xi \neq 0$ and $\text{supp } g \subset [-1, 1]$. Define a function f_v for $0 < v < 1/2$ by the formula

$$\widehat{f}_v(\xi) = vg(v\xi + 1/v), \quad \xi \in \mathbb{R}.$$

In [10] it is proved that $|S_{t(x)} f_v(x)| \geq c > 0$ in a neighbourhood of $x = 0$ if $t(x)$ is suitably chosen. Here

$$S_t f_v(x) = \int_{\mathbb{R}} e^{ix\xi} e^{it|\xi|^a} \widehat{f}_v(\xi) d\xi,$$

with $a > 1$. For $n \geq 2$ we set

$$f(x) = f_v(x_1) \dots f_v(x_n).$$

Then

$$\begin{aligned} S_t f(x) &= \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1 |\xi_1|^{a_1} + t_2 |\xi_2|^{a_2} + \dots + t_n |\xi_n|^{a_n})} \widehat{f}_v(\xi_1) \dots \widehat{f}_v(\xi_n) d\xi \\ &= S_{t_1} f_v(x_1) \dots S_{t_n} f_v(x_n). \end{aligned}$$

Here

$$S_{t_j} f_v(x_j) = \int_{\mathbb{R}} e^{ix_j \xi_j} e^{it_j |\xi_j|^{a_j}} \widehat{f}_v(\xi_j) d\xi_j.$$

It follows that $M^{**} f(x) \geq c > 0$ in a neighbourhood of the origin and hence $\|M^{**} f\|_q \geq c$. On the other hand, it is easy to see that $\text{supp } \widehat{f}_v$ is included in the interval $[-1/v^2 - 1/v, -1/v^2 + 1/v]$. Also, $|\widehat{f}_v(\xi)| \leq Cv^n$ for all ξ . It follows that

$$\|f\|_{\dot{H}_s}^2 = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \leq Cv^{-n} v^{2n} v^{-4s} = Cv^{n-4s},$$

and the right hand side tends to 0 as $v \rightarrow 0$ if $n - 4s > 0$, that is $s < n/4$. Hence, the inequality $\|M^{**} f\|_q \leq C\|f\|_{\dot{H}_s}$ cannot hold for $s < n/4$. The proof of Theorem 1.3 is complete. \square

Proof of Theorem 1.4. To treat the case $1 \leq q < 2$ one can use a counter-example in [12], pp. 43 and 65. In the case $q = \infty$ we use a counter-example in [14], pp. 400-401.

The sufficiency in the case $4 \leq q < \infty$ follows from Theorem 1.3. The necessity follows from a counter-example in [12], pp. 58-59.

We then assume $2 < q < 4$. We shall prove that inequality (3) holds if $s = n/2 - |a|/4 + |a|/q - n/q$. Set

$$Sf(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{i(t_1(x)|\xi_1|^{a_1} + t_2(x)|\xi_2|^{a_2} + \dots + t_n(x)|\xi_n|^{a_n})} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n,$$

where each $t_i(x)$ is measurable and $0 < t_i(x) < 1$. We want to prove that

$$\|Sf\|_q \leq C \|f\|_{H_s} = C \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.$$

We have

$$s = n\frac{1}{2} - \frac{a_1 + \dots + a_n}{4} + \frac{a_1 + \dots + a_n}{q} - n\frac{1}{q} = s_1 \dots + s_n,$$

where $s_j = 1/2 - a_j/4 + a_j/q - 1/q$, $j = 1, 2, \dots, n$. It is sufficient to prove that

$$\|Sf\|_q \leq C \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi_1|^2)^{s_1} \dots (1 + |\xi_n|^2)^{s_n} d\xi \right)^{1/2}.$$

Define T and T_N as before. One obtains

$$\|T_N^* h\|_2^2 = \int \int I_N(x, y) \chi_N(x) \chi_N(y) h(x) \overline{h(y)} dx dy,$$

where $I_N(x, y) = I_N^1(x, y) \dots I_N^n(x, y)$, with

$$I_N^j(x, y) = \int (1 + \xi_j^2)^{-s_j} e^{i(y_j - x_j)\xi_j} e^{i(t_j(y) - t_j(x))|\xi_j|^{a_j}} \mu(\xi_j/N) d\xi_j.$$

Lemma 2.2 implies

$$|I_N^j(x, y)| \leq C |x_j - y_j|^{-\beta_j},$$

where $\beta_j = (\alpha_j + a_j/2 - 1)/(a_j - 1)$ and $\alpha_j = 2s_j$. It follows that

$$\|T_N^* h\|_2^2 \leq C \int_{\mathbb{R}^n} |h(x)| P_n P_{n-1} \dots P_1(|h|)(x) dx,$$

where

$$P_j f(x_1, \dots, x_n) = \int_{\mathbb{R}} \frac{1}{|x_j - y_j|^{\beta_j}} f(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n) dy_j.$$

We have

$$\left(\int_{\mathbb{R}} |P_j h(x)|^q dx_j \right)^{1/q} \leq C \left(\int_{\mathbb{R}} |h(x)|^{q'} dx_j \right)^{1/q'},$$

and the proof can be completed as above (see also [15], pp. 407-408).

Then assume $q = 2$. We shall prove that (3) holds for $s > |a|/4$. It is sufficient to prove that, for Sf defined as above,

$$\|Sf\|_2 \leq C_\epsilon \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + \xi^2)^{|a|/4} (\log(2 + \xi^2))^{n(1+\epsilon)} d\xi \right)^{1/2}.$$

This, in turn, will follow from the estimate

$$\|Sf\|_2 \leq C_\epsilon \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \prod_{j=1}^n [(1 + \xi_j^2)^{a_j/4} (\log(2 + \xi_j^2))^{1+\epsilon}] d\xi \right)^{1/2}. \quad (6)$$

We define T, T_N and I_N in the same way as before. Lemma 2.3 then implies

$$|I_N(x, y)| \leq K_1(x_1 - y_1) \cdots K_n(x_n - y_n),$$

with $K_j \in L^1(\mathbb{R})$ for every j . Setting $K(x) = K_1(x_1) \cdots K_n(x_n)$ we obtain

$$\begin{aligned} \|T_N^* h\|_2^2 &\leq \int |h(x)| K * |h|(x) dx \\ &\leq \|h\|_2 \|K * |h|\|_2 \leq C \|h\|_2^2. \end{aligned}$$

Hence, T is bounded on L^2 and (6) follows.

It remains to prove the necessity for $2 \leq q < 4$. We shall use a counterexample in [11], pp. 112-113. Let $\phi_j \in C_0^\infty(\mathbb{R})$ with $\text{supp } \phi_j \subset (-1, 1)$ and define

$$\widehat{f}_j(\xi_j) = \phi_j(N^{a_j/2-1} \xi_j + N^{a_j/2}), \quad \xi_j \in \mathbb{R}, \quad j = 1, 2, \dots, n.$$

Here N is large. It is easy to see that $\widehat{f_j}$ vanishes outside the interval

$$[-N - N^{1-a_j/2}, -N + N^{1-a_j/2}].$$

We also set $f(x) = f_1(x_1) \cdots f_n(x_n)$. Then

$$\|f\|_{H_s}^2 \leq C \prod_{j=1}^n (N^{1-a_j/2}) N^{2s} = C N^{n+2s-|a|/2},$$

and hence

$$\|f\|_{H_s} \leq C N^{n/2+s-|a|/4}.$$

In [11] it is proved that ϕ_j can be chosen so that

$$\sup_{0 < t_j < 1} \left| \int_{\mathbb{R}} e^{ix_j \xi_j} e^{it_j |\xi_j|^{a_j}} \widehat{f_j}(\xi_j) d\xi_j \right| \geq c N^{1-a_j/2},$$

on a set of measure $\geq c N^{a_j-1}$. It follows that

$$M^* f(x) \geq c \prod_{j=1}^n (N^{1-a_j/2}) = c N^{n-|a|/2},$$

on a set of measure $c \prod_{j=1}^n (N^{a_j-1}) = c N^{|a|-n}$. Hence,

$$\|M^* f\|_q^q \geq c N^{qn-q|a|/2} N^{|a|-n},$$

that is

$$\|M^* f\|_q \geq c N^{n-|a|/2} N^{(|a|-n)/q}.$$

Inequality (3) then implies

$$N^{n-|a|/2} N^{(|a|-n)/q} \leq C N^{n/2+s-|a|/4}.$$

Letting $N \rightarrow \infty$ we deduce that

$$n - \frac{|a|}{2} + \frac{|a|}{q} - \frac{n}{q} \leq \frac{n}{2} + s - \frac{|a|}{4},$$

that is

$$\frac{n}{2} - \frac{|a|}{4} + \frac{|a|}{q} - \frac{n}{q} \leq s.$$

The proof of Theorem 1.4 is complete. \square

5. Some remarks about the case $n = 1, a = 2, q = 2$

In this section we assume $n = 1$ and $a = 2$. It follows from the method in the proof of Theorem 1.4 that in this case one has

$$\|M^*f\|_2 \leq C_\epsilon \left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + \xi^2)^{1/2} (\log(2 + \xi^2))^{1+\epsilon} d\xi \right)^{1/2},$$

for $\epsilon > 0$

As we said above, it remains an open question whether the logarithmic factor can be removed, that is if

$$\|M^*f\|_2 \leq C\|f\|_{H_{1/2}}$$

holds. To study this problem we set

$$Sf(x) = \int_{\mathbb{R}} e^{it(x)\xi^2} e^{ix\xi} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}, \quad f \in \mathcal{S},$$

where $t(x)$ is measurable and $0 < t(x) < 1$. We are interested in the inequality

$$\int_{\mathbb{R}} |Sf(x)|^2 dx \leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^{1/2} d\xi, \quad (7)$$

According to Theorem 1.1 one has the estimate

$$\|Sf\|_4 \leq C\|f\|_{\dot{H}_{1/4}},$$

and hence

$$\int_{|x| \leq 2} |Sf(x)|^2 dx \leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{1/2} d\xi.$$

To prove (7) it is therefore sufficient to prove that

$$\int_{|x| \geq 2} |Sf(x)|^2 dx \leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^{1/2} d\xi, \quad (8)$$

Consider the phase function $\phi_x(\xi) = t(x)\xi^2 + x\xi$. Since

$$\phi'_x(\xi) = 2t(x)\xi + x,$$

the zone of “non-oscillation” for the kernel of S , that is, when $|\phi'_x(\xi)| \leq 1$, corresponds to

$$\{\xi : |2t(x)\xi + x| \leq 1\} = \left[\frac{-x-1}{2t(x)}, \frac{-x+1}{2t(x)} \right].$$

It is therefore natural to look at the operator L defined by

$$Lf(x) = \int_{\frac{x-1}{2t(x)}}^{\frac{x+1}{2t(x)}} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}, \quad f \in \mathcal{S}.$$

(For convenience, we have replaced x with $-x$.)

We will show that (8) holds with L instead of S . In fact, one has the homogeneous estimate

Theorem 5.1. *With the previous notation, we have*

$$\int_{|x| \geq 2} |Lf(x)|^2 dx \leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi| d\xi. \quad (9)$$

Proof. Setting $g(\xi) = \widehat{f}(\xi)|\xi|^{1/2}$, we see that (9) is equivalent to the inequality

$$\int_{|x| \geq 2} |Tg(x)|^2 dx \leq C \int_{\mathbb{R}} |g(\xi)|^2 d\xi,$$

where

$$Tg(x) = \int_{\frac{x-1}{2t(x)}}^{\frac{x+1}{2t(x)}} g(\xi) \frac{d\xi}{|\xi|^{1/2}}, \quad |x| \geq 2.$$

Observe that the kernel of T is

$$K_T(x, \xi) = \frac{1}{|\xi|^{1/2}} \chi_{\left[\frac{x-1}{2t(x)}, \frac{x+1}{2t(x)}\right]}(\xi) \chi_{\{|x| \geq 2\}}(x).$$

Therefore, the kernel of TT^* is

$$K(x, y) = \int K_T(x, \xi) K_T(y, \xi) d\xi = \int_{\left[\frac{x-1}{2t(x)}, \frac{x+1}{2t(x)}\right] \cap \left[\frac{y-1}{2t(y)}, \frac{y+1}{2t(y)}\right]} \frac{d\xi}{|\xi|},$$

for $|x|, |y| \geq 2$. It follows that

$$K(x, y) \leq 2 \min \left[\log \left(\frac{|x| + 1}{|x| - 1} \right), \log \left(\frac{|y| + 1}{|y| - 1} \right) \right], \quad |x|, |y| \geq 2.$$

Using that for $u \geq 2$ one has

$$\log \frac{u + 1}{u - 1} = \log \left(1 + \frac{2}{u - 1} \right) \leq \frac{2}{u - 1} \leq \frac{c}{u},$$

we get the estimate

$$|K(x, y)| \leq C \min \left(\frac{1}{|x|}, \frac{1}{|y|} \right) \sim \frac{1}{|x| + |y|}.$$

Hence,

$$|TT^*g(x)| \leq C \left(\frac{1}{|x|} \int_{|y| \leq |x|} |g(y)| dy + \int_{|y| \geq |x|} |g(y)| \frac{dy}{|y|} \right).$$

The two operators on the right hand side are easily seen to be bounded on L^p , for $1 < p < \infty$, and so we obtain the above estimates for T and L . \square

Remark. There is an important feature about the operator L that we want to point out here and that is that on scale 1 this is pointwise majorized by the above operator S . To be more precise, we claim that if f is a function so that

i) $\text{supp } \hat{f} \subset [b - 1/2, b + 1/2] \subset \left[\frac{-x-1}{2t(x)}, \frac{-x+1}{2t(x)} \right]$, for some b and some x , and

ii) \hat{f} is positive,

then

$$|Sf(x)| \geq c \int_{\frac{-x-1}{2t(x)}}^{\frac{-x+1}{2t(x)}} \hat{f}(\xi) d\xi. \quad (10)$$

The reason is simply that

$$|\phi_x(\xi) - \phi_x(b)| \leq 1/2, \quad \forall \xi \in \text{supp } \hat{f},$$

(remember that the interval $\left[\frac{-x-1}{2t(x)}, \frac{-x+1}{2t(x)} \right]$ is chosen so that $|\phi'_x(\xi)| \leq 1$ there) and so (10) follows easily.

The fact is that many counterexamples in the theory come from the behavior at the “non-oscillation” zone of the kernel defining S . Theorem 5.1 shows that if inequality (7) is not true then we should look for a more elaborate type of counterexamples.

In the same spirit of the above remark, we continue by giving a simple proof of the following result mentioned at the introduction

Theorem 5.2. *If the inequality*

$$\|Sf\|_2 \leq C\|f\|_{H_s} \quad (11)$$

holds for a constant C independent of f and $t(x)$, then we must have $s \geq 1/2$.

Proof. We take M large and set $\hat{f} = \chi_{[M, M+1]}$. We let $a = M + 1/2$ and choose x so that $-M \leq x \leq -2$. Also set $y = -x$ so that $2 \leq y \leq M$. We then take $t(x) = y/2a$. It follows that $0 < t(x) < 1$, $y/2t(x) = a$, and

$$\frac{-x-1}{2t(x)} = \frac{y}{2t(x)} - \frac{1}{2t(x)} \leq a - 1/2 = M,$$

$$\frac{-x+1}{2t(x)} = \frac{y}{2t(x)} + \frac{1}{2t(x)} \geq a + 1/2 = M + 1.$$

Hence, i) and ii) in the above remark are satisfied and (10) gives

$$|Sf(x)| \geq c \int_M^{M+1} d\xi = c.$$

It follows that

$$\int |Sf(x)|^2 dx \geq c \int_{-M}^{-2} dx \geq cM.$$

On the other hand

$$\|f\|_{H_s}^2 = \int_M^{M+1} (1 + \xi^2)^s d\xi \sim M^{2s}.$$

If (11) holds, we then get

$$cM \leq CM^{2s},$$

and it follows that $1 \leq 2s$, that is $s \geq 1/2$. □

Using the main idea in the previous proof one can also prove the following theorem.

Theorem 5.3. *Define the maximal operator U^* by setting*

$$U^*g(x) = \sup_{R>1} \int_{Rx}^{Rx+1} |g(y)| \frac{dy}{|y|^s}, \quad x \geq 2. \quad (12)$$

Then, the inequality

$$\int_2^\infty (U^*g(x))^2 dx \leq C \int_{\mathbb{R}} |g(x)|^2 dx, \quad (13)$$

holds if and only if $s \geq 1/2$.

Proof. Take M large and let $g = \chi_{[M, M+1]}$. By taking $R = M/x$ we easily see that $U^*g(x) \geq 1/(M+1)^s$ on the interval $[2, M]$. Thus, if (13) holds, then we should have

$$(M-2) \frac{1}{(M+1)^{2s}} \leq C, \quad \text{as } M \rightarrow \infty.$$

This implies $2s \geq 1$.

The fact that (13) holds if $s \geq 1/2$ follows from Theorem 5.1. □

- [1] J. Bourgain *On the Schrödinger maximal function in higher dimension*, arXiv:1201.3342 [math.AP]
- [2] L. Carleson, *Some analytical problems related to statistical mechanics*, in Euclidean Harmonic Analysis, Lecture Notes in Math. **779** (1979), 5–45.
- [3] B.E.J. Dahlberg and C.E. Kenig, *A note on almost everywhere behaviour of solutions to the Schrödinger equation*, in Harmonic Analysis, Lecture Notes in Math. **908** (1982), 205–209.

- [4] R. Fefferman, *Multiparameter Fourier analysis*. Beijing lectures in harmonic analysis (Beijing, 1984), 47–130, Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.
- [5] R. Fefferman, *Multiparameter Calderón-Zygmund theory*. Harmonic analysis and partial differential equations (Chicago, IL, 1996), 207–221, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.
- [6] R. Fefferman, *Harmonic analysis on product spaces*. Ann. of Math. (2) 126 (1987), no. 1, 109–130.
- [7] R. Fefferman and E.M. Stein, *Singular integrals on product spaces*. Adv. in Math. 45 (1982), no. 2, 117–143.
- [8] S. Lee, *On pointwise convergence of the solutions to Schrödinger equations in \mathbb{R}^2* , Int. Math. Res. Not. 2006, Art. ID 32597, 21 pp.
- [9] K.M. Rogers and P. Villarroya, *Global estimates for the Schrödinger maximal operator*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 425–435.
- [10] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), 699–715.
- [11] P. Sjölin, *Global Maximal estimates for solutions to the Schrödinger equation*, Studia Math. **110** (1994), 105–114.
- [12] P. Sjölin, *L^p maximal estimates for solutions to the Schrödinger equation*, Math. Scand. **81** (1997), 35–68.
- [13] P. Sjölin, *Homogeneous maximal estimates for solutions to the Schrödinger equation*, Bull. Inst. Math. Acad. Sinica **30** (2002), 133–140.
- [14] P. Sjölin, *Spherical Harmonics and maximal estimates for the Schrödinger equation*, Annales Acad. Scient. Fennicae, Mathematica **30** (2005), 393–406.
- [15] P. Sjölin, *Maximal estimates for solutions to the nonelliptic Schrödinger equation*, Bull. London Math. Soc. **39** (2007), 404–412.

- [16] E.M. Stein, Harmonic Analysis, Real-Variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, 1993.
- [17] T. Tao, *A sharp bilinear restriction estimate for paraboloids*, Geom. Funct. Anal. **13** (2003), 1359–1384.
- [18] T. Tao and A. Vargas, *A bilinear approach to cone multipliers. II. Applications*, Geom. Funct. Anal. **10** (2000), no. 1, 216–258.
- [19] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874–878.