



Pólya–Radoux type results for some arithmetical functions



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ABSTRACT

Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$ and $\lim_{x \rightarrow \infty} h(x) = \infty$. We prove that if $a : \mathbb{N} \rightarrow [0, \infty)$ and $k \geq 1$ then, $\sum_{n \leq x} a(n) \sim [h(x)]^k$ if and only if for every function $f : [0, 1] \rightarrow \mathbb{R}$ such that $x^{k-1}f(x)$ is Riemann integrable the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{[h(x)]^k} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) = k \int_0^1 x^{k-1} f(x) dx.$$

Applications in the case of the prime numbers are given.

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1. Introduction and notation

It is well known that if $f : [0, 1] \rightarrow \mathbb{R}$ is Riemann integrable, then $\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f\left(\frac{n}{x}\right) = \int_0^1 f(x) dx$. G. Pólya, in a classical paper from 1917, see [6] and also [4], using the prime number theorem, showed that if $f : [0, 1] \rightarrow \mathbb{R}$ is a Riemann integrable function on $[0, 1]$, then $\lim_{x \rightarrow \infty} \frac{\ln x}{x} \sum_{p \leq x, p \text{ prime}} f\left(\frac{p}{x}\right) = \int_0^1 f(x) dx$. In 1977, see [7], Ch. Radoux showed that $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n f\left(\frac{k}{n}\right) \varphi(k) = \frac{6}{\pi^2} \int_0^1 xf(x) dx$, for all functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $xf(x)$ is continuous on $[0, 1]$, φ is Euler's totient function. Very recently, using a different approach than that of Pólya's, in [2] various extensions of Pólya's and Radoux's theorems are shown. In this paper we continue this study by showing some results of the same kind, [Theorem 1](#), [Corollaries 1 and 3](#). Let us fix some notation and notions. By e we denote the Euler number and $\ln x = \log_e x$. Let $a \in \mathbb{R} \cup \{-\infty\}$, $g : (a, \infty) \rightarrow \mathbb{R}$ be such that there exists $b \geq a$ with $g(x) \neq 0$ for any $x \in (b, \infty)$. If $f : (a, \infty) \rightarrow \mathbb{R}$ is a function, the notation $f(x) \sim g(x)$ means that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$. Throughout this paper, we use the notation

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$\sum_{n \leq x}$ to mean $\sum_{n \leq x; n \in \mathbb{N}}$. All notation and notions used and not defined in this paper are standard, see e.g. [1,9].

2. The preliminary results

We begin with a result analogous to Proposition 1 from [2].

Lemma 1. *Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$ and $\lim_{x \rightarrow \infty} h(x) = \infty$. Let also $g : \mathbb{N} \rightarrow [0, \infty)$ be a function and $m \geq 1$. If $\sum_{n \leq x} g(n) \sim [h(x)]^m$, then for all $\alpha > -m$ we have*

$$\sum_{x_0 < n \leq x} g(n) [h(n)]^\alpha \sim \frac{m}{m + \alpha} [h(x)]^{m+\alpha}.$$

Proof. Let $x > x_0$. By the Abel summation formula, see [1, p. 78] we have

$$\begin{aligned} \sum_{x_0 < n \leq x} g(n) [h(n)]^\alpha &= \left(\sum_{n \leq x} g(n) \right) [h(x)]^\alpha - \left(\sum_{n \leq x_0} g(n) \right) [h(x_0)]^\alpha \\ &\quad - \alpha \int_{x_0}^x \left(\sum_{n \leq t} g(n) \right) [h(t)]^{\alpha-1} h'(t) dt \end{aligned}$$

Since $\lim_{x \rightarrow \infty} h(x) = \infty$ and $\alpha > -m$ the integral $\int_{x_0}^\infty [h(t)]^{m+\alpha-1} h'(t) dt$ is divergent. By hypothesis $\left(\sum_{n \leq t} g(n) \right) [h(t)]^{\alpha-1} h'(t) \sim [h(t)]^m [h(t)]^{\alpha-1} h'(t)$ and then, as is well known, see [3, Proposition 6, p. 230], or [5, Exercise 1.1(a), p. 53]

$$\begin{aligned} \int_{x_0}^x \left(\sum_{n \leq t} g(n) \right) [h(t)]^{\alpha-1} h'(t) dt &\sim \int_{x_0}^x [h(t)]^m [h(t)]^{\alpha-1} h'(t) dt \\ &= \frac{([h(x)]^{m+\alpha} - [h(x_0)]^{m+\alpha})}{m + \alpha} \sim \frac{[h(x)]^{m+\alpha}}{m + \alpha}. \end{aligned}$$

Then, by hypothesis and $\lim_{x \rightarrow \infty} [h(x)]^{m+\alpha} = \infty$ (since $\lim_{x \rightarrow \infty} h(x) = \infty$, $\alpha > -m$), we deduce

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sum_{x_0 < n \leq x} g(n) [h(n)]^\alpha}{[h(x)]^{m+\alpha}} &= \lim_{x \rightarrow \infty} \frac{\sum_{n \leq x} g(n)}{[h(x)]^m} \\ &\quad - \lim_{x \rightarrow \infty} \frac{\left(\sum_{n \leq x_0} g(n) \right) [h(x_0)]^\alpha}{[h(x)]^{m+\alpha}} - \alpha \lim_{x \rightarrow \infty} \frac{\int_{x_0}^x \left(\sum_{n \leq t} g(n) \right) [h(t)]^{\alpha-1} h'(t) dt}{[h(x)]^{m+\alpha}} \\ &= \frac{m}{m + \alpha}. \quad \square \end{aligned}$$

Proposition 1. *Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$ and $\lim_{x \rightarrow \infty} h(x) = \infty$. Let also $a : \mathbb{N} \rightarrow [0, \infty)$ be a function and $k \geq 1$. The following assertions are equivalent:*

- (i) $\sum_{n \leq x} a(n) \sim [h(x)]^k$.
- (ii) For all $\alpha > -k$ we have $\sum_{x_0 < n \leq x} a(n) [h(n)]^\alpha \sim \frac{k}{k+\alpha} [h(x)]^{k+\alpha}$.
- (iii) $\sum_{x_0 < n \leq x} \frac{a(n)}{[h(n)]^{k-1}} \sim kh(x)$.

Proof. (i) \Rightarrow (ii) was shown in [Lemma 1](#).

(ii) \Rightarrow (iii). Take in (ii) $\alpha = -k + 1$.

(iii) \Rightarrow (i). Let $g(n) = \begin{cases} 0 & \text{if } n \leq x_0 \\ \frac{a(n)}{[h(n)]^{k-1}} & \text{if } n > x_0 \end{cases}$. Then by (iii), $\sum_{n \leq x} g(n) \sim kh(x)$. From [Lemma 1](#) applied to $m = 1$ and $\alpha = k - 1 > -1$ we deduce $\sum_{x_0 < n \leq x} g(n) [h(n)]^{k-1} \sim [h(x)]^k$ that is $\sum_{x_0 < n \leq x} a(n) \sim [h(x)]^k$. Now since $\sum_{n \leq x} a(n) = \sum_{x_0 < n \leq x} a(n) + \sum_{n \leq x_0} a(n)$ and $\lim_{x \rightarrow \infty} [h(x)]^k = \infty$ ($\lim_{x \rightarrow \infty} h(x) = \infty$, $k \geq 1$) we get (i). \square

We write $C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ continuous}\}$, which is a real linear space with respect to usual addition and scalar multiplication for functions and a Banach space with respect to the uniform norm i.e. $\|f\|_u = \sup_{x \in [0, 1]} |f(x)|$, $f \in C[0, 1]$. Also for $a > 0$ we write $\mathcal{F}[a, \infty) := \{f : [a, \infty) \rightarrow \mathbb{R}\}$ which is a real linear space with respect to usual addition and scalar multiplication for functions. We need the following result whose proof can be found in [\[2, Theorem 2\]](#).

Lemma 2. Let $T : C[0, 1] \rightarrow \mathcal{F}[a, \infty)$ be a linear operator with the properties that there are $x_0 \geq 1$ and $L > 0$ such that

$$|(T(f)(x))| \leq L \|f\|_u \text{ for each } f \in C[0, 1] \text{ and each } x \geq x_0$$

and $V : C[0, 1] \rightarrow \mathbb{R}$ a bounded linear functional. Further, suppose that there exists $A \subset C[0, 1]$, A dense in $C[0, 1]$, such that

$$\lim_{x \rightarrow \infty} (T(f)(x)) = V(f) \text{ for each } f \in A.$$

Then

$$\lim_{x \rightarrow \infty} (T(f))(x) = V(f) \text{ for each } f \in C[0, 1].$$

3. The main result

The next theorem is the main result of this paper. As it was pointed to us by the referee, our method of the proof resembles Karamata's method in Tauberian theory, see [\[10, pp. 227–229\]](#).

Theorem 1. Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$, $\lim_{x \rightarrow \infty} h(x) = \infty$ and let $a : \mathbb{N} \rightarrow [0, \infty)$ be a function. The following assertions are equivalent:

- (i) $\sum_{n \leq x} a(n) \sim h(x)$ (equivalent $\sum_{x_0 < n \leq x} a(n) \sim h(x)$, since $\lim_{x \rightarrow \infty} h(x) = \infty$).
- (ii) For each continuous function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) = \int_0^1 f(x) dx.$$

(iii) For each Riemann integrable function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) = \int_0^1 f(x) dx.$$

Proof. (i) \Rightarrow (ii). We define $T : C[0, 1] \rightarrow \mathcal{F}[x_0, \infty)$ and $V : C[0, 1] \rightarrow \mathbb{R}$ by

$$T(f)(x) = \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \text{ and } V(f) = \int_0^1 f(x) dx.$$

We must prove that for each $f \in C[0, 1]$ we have

$$\lim_{x \rightarrow \infty} T(f)(x) = V(f). \quad (1)$$

Let $P(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k$ be a polynomial. From (i) and Lemma 1 we have

$$\begin{aligned} \lim_{x \rightarrow \infty} T(P)(x) &= \lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) P\left(\frac{h(n)}{h(x)}\right) \\ &= \sum_{i=0}^k a_i \lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} \left(\frac{h(n)}{h(x)}\right)^i = \sum_{i=0}^k \frac{a_i}{i+1} = \int_0^1 P(x) dx \end{aligned}$$

i.e. the relation (1) is true for all polynomials. We will apply Lemma 2 in which we take $A = \{P : [0, 1] \rightarrow \mathbb{R} \mid P \text{ is a polynomial}\}$. By the well known Weierstrass–Bernstein approximation theorem, the set A is dense in $C[0, 1]$. Since by (i), $\sum_{n \leq x} a(n) \sim h(x)$ and $\lim_{x \rightarrow \infty} h(x) = \infty$ we deduce $\sum_{x_0 < n \leq x} a(n) \sim h(x)$ thus there exist $x_1 > x_0$ such that

$$\frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) \leq 2 \text{ for each } x \geq x_1. \quad (2)$$

Let $f \in C[0, 1]$. For each $x \geq x_1$ by (2) and $a(n) \geq 0$ we have

$$|T(f)(x)| \leq \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) \left| f\left(\frac{h(n)}{h(x)}\right) \right| \leq \frac{\|f\|_u}{h(x)} \sum_{x_0 < n \leq x} a(n) \leq 2 \|f\|_u$$

since h is increasing and positive on $[x_0, \infty)$ and $0 < \frac{h(n)}{h(x)} \leq 1$. The hypotheses in Lemma 2 are satisfied and thus (1) holds for each $f \in C[0, 1]$.

(ii) \Rightarrow (iii). Let $f : [0, 1] \rightarrow \mathbb{R}$ be a Riemann integrable function. Let also $\varepsilon > 0$. From Lemma 4 in [2] there exists continuous functions $\varphi, \psi : [0, 1] \rightarrow \mathbb{R}$ such that

$$\varphi(x) \leq f(x) \leq \psi(x), \forall x \in [0, 1] \quad (3)$$

$$\int_0^1 (\psi(x) - \varphi(x)) dx \leq \varepsilon. \quad (4)$$

From (3) we deduce

$$\begin{aligned} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) \varphi\left(\frac{h(n)}{h(x)}\right) &\leq \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \\ &\leq \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) \psi\left(\frac{h(n)}{h(x)}\right), \quad \forall x > 1. \end{aligned}$$

Since φ, ψ are continuous from (ii) we get

$$\begin{aligned} \int_0^1 \varphi(x) dx &\leq \liminf_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \\ &\leq \limsup_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \leq \int_0^1 \psi(x) dx \end{aligned}$$

i.e. by (3) and (4)

$$\begin{aligned} \int_0^1 f(x) dx - \varepsilon &\leq \liminf_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \\ &\leq \limsup_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) \leq \int_0^1 f(x) dx + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we get (ii).

(iii) \Rightarrow (i). Take in (iii) $f(x) = 1$. \square

Remark 1. Let us note that if in [Theorem 1](#) $\lim_{x \rightarrow \infty} \frac{xh'(x)}{h(x)} = 1$, we get Theorem 2 in [\[2\]](#) for $k = 0$.

4. Applications

Corollary 1. Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$, $\lim_{x \rightarrow \infty} h(x) = \infty$, let also $a : \mathbb{N} \rightarrow [0, \infty)$ be a function and $k \geq 1$. The following assertions are equivalent:

- (i) $\sum_{n \leq x} a(n) \sim [h(x)]^k$.
- (ii) $\sum_{x_0 < n \leq x} \frac{a(n)}{[h(n)]^{k-1}} \sim kh(x)$.
- (iii) For each continuous function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} \frac{a(n)}{[h(n)]^{k-1}} f\left(\frac{h(n)}{h(x)}\right) = k \int_0^1 f(x) dx.$$

- (iv) For each Riemann integrable function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < n \leq x} \frac{a(n)}{[h(n)]^{k-1}} f\left(\frac{h(n)}{h(x)}\right) = k \int_0^1 f(x) dx.$$

(v) For each function $f : [0, 1] \rightarrow \mathbb{R}$ such that $x^{k-1}f(x)$ is Riemann integrable the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{[h(x)]^k} \sum_{x_0 < n \leq x} a(n) f\left(\frac{h(n)}{h(x)}\right) = k \int_0^1 x^{k-1} f(x) dx.$$

Proof. (i) \Leftrightarrow (ii) was shown in Proposition 1.

(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) was shown in Theorem 1.

(iv) \Rightarrow (v). It follows from (iv) applied to $x^{k-1}f(x)$ instead of $f(x)$.

(v) \Rightarrow (i). It follows from (v) applied to $g(x) = 1$. \square

We give now some concrete examples. We note that all these examples do not follow from Theorem 2 in [2], see Remark 1. By $d : \mathbb{N} \rightarrow \mathbb{N}$ we denote the divisor function, $d(n) = \text{card} \{d \in \mathbb{N} \mid d \mid n\}$.

Corollary 2.

(i) If $f : [0, 1] \rightarrow \mathbb{R}$ is such that $xf(x)$ is Riemann integrable, then

$$\lim_{x \rightarrow \infty} \frac{1}{\ln^2 x} \sum_{n \leq x} \frac{d(n)}{n} f\left(\frac{\ln n}{\ln x}\right) = \int_0^1 xf(x) dx.$$

(ii) If $f : [0, 1] \rightarrow \mathbb{R}$ is such that $x^2f(x)$ is Riemann integrable, then

$$\lim_{x \rightarrow \infty} \frac{1}{\ln^3 x} \sum_{n \leq x} \frac{d(n) \ln n}{n} f\left(\frac{\ln n}{\ln x}\right) = \int_0^1 x^2 f(x) dx.$$

Proof. Since $\sum_{n \leq x} \frac{d(n)}{n} = \frac{1}{2} \ln^2 x + 2C \ln x + O(1)$, see [1, Exercise 2, p. 70], and $\sum_{n \leq x} \frac{d(n) \ln n}{n} = \frac{1}{3} \ln^3 x + C \ln^2 x + O(\ln x)$, C is the Euler constant, from Theorem 1 we get the statement. \square

In the sequel by \mathbb{P} we denote the set of all prime numbers and the notation $\sum_{p \leq x} \cdots$ means $\sum_{p \leq x, p \text{ prime}} \cdots$.

Corollary 3. Let $h : [1, \infty) \rightarrow [0, \infty)$ be a function with the properties that there exists $x_0 \geq 1$ such that $h \in C^1([x_0, \infty))$, $h(x_0) > 0$, $h'(x) > 0$ for all $x \geq x_0$, $\lim_{x \rightarrow \infty} h(x) = \infty$, let also $v : \mathbb{P} \rightarrow [0, \infty)$ be a function and let $k \geq 1$. The following assertions are equivalent:

(i) $\sum_{p \leq x} v(p) \sim [h(x)]^k.$

(ii) $\sum_{x_0 < p \leq x} \frac{v(p)}{[h(p)]^{k-1}} \sim kh(x).$

(iii) For each continuous function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < p \leq x} \frac{v(p)}{[h(p)]^{k-1}} f\left(\frac{h(p)}{h(x)}\right) = k \int_0^1 f(x) dx.$$

(iv) For each Riemann integrable function $f : [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{h(x)} \sum_{x_0 < p \leq x} \frac{v(p)}{[h(p)]^{k-1}} f\left(\frac{h(p)}{h(x)}\right) = k \int_0^1 f(x) dx.$$

(v) For each function $f: [0, 1] \rightarrow \mathbb{R}$ such that $x^{k-1}f(x)$ is Riemann integrable the following equality holds

$$\lim_{x \rightarrow \infty} \frac{1}{[h(x)]^k} \sum_{x_0 < p \leq x} v(p) g\left(\frac{h(p)}{h(x)}\right) = k \int_0^1 x^{k-1} g(x) dx.$$

Proof. Let us define $a(n) = \begin{cases} v(n) & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$. Then $\sum_{p \leq x} v(p) = \sum_{n \leq x} a(n)$ and $\sum_{x_0 < p \leq x} v(p) w(p, x) = \sum_{x_0 < n \leq x} a(n) w(n, x)$, where $w: \mathbb{N} \times [0, 1] \rightarrow \mathbb{R}$. The statement follows from [Corollary 1](#). \square

Corollary 4. For each Riemann integrable function $f: [0, 1] \rightarrow \mathbb{R}$ the following equality holds

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{\ln x} \sum_{p \leq x} \frac{\ln p}{p} f\left(\frac{\ln p}{\ln x}\right) &= \int_0^1 f(x) dx. \\ \lim_{x \rightarrow \infty} \frac{1}{\ln(\ln x)} \sum_{3 \leq p \leq x} \frac{1}{p} f\left(\frac{\ln(\ln p)}{\ln(\ln x)}\right) &= \int_0^1 f(x) dx. \\ \lim_{x \rightarrow \infty} \frac{1}{\ln(\ln x)} \sum_{3 \leq p \leq x} \left[\ln\left(1 - \frac{1}{p}\right) \right] f\left(\frac{\ln(\ln p)}{\ln(\ln x)}\right) &= - \int_0^1 f(x) dx. \end{aligned}$$

Proof. Recall the Mertens asymptotic formula $\sum_{p \leq x} \frac{\ln p}{p} \sim \ln x$, $\sum_{p \leq x} \frac{1}{p} \sim \ln(\ln x)$ and $-\sum_{p \leq x} \ln\left(1 - \frac{1}{p}\right) \sim \ln(\ln x)$, see [\[1,9\]](#). The statement follows from these evaluations and [Corollary 3](#). \square

Let us note that in [\[8, Proposition 4\]](#) it is given a different proof of the first limit in [Corollary 4](#).

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References

- [1] T.M. Apostol, Introduction to Analytic Number Theory, Undergrad. Texts Math., Springer, 1998.
- [2] M. Bănescu, D. Popa, New extensions of some classical theorems in number theory, J. Number Theory 133 (13) (2013) 3771–3795.
- [3] N. Bourbaki, Functions of a Real Variable: Elementary Theory, Trans. from the 1976 French original by Philip Spain, Springer, Berlin, 2004.
- [4] E. Landau, Sur quelques problèmes relatifs à la distribution des nombres premiers, Bull. Soc. Math. France 28 (1900) 25–38, <http://www.numdam.org/>.
- [5] B.M. Makarov, M.G. Goluzina, A.A. Lodkin, A.N. Podkoryotov, Selected problems in real analysis, Transl. Math. Monogr., vol. 107, Amer. Math. Soc., 1992.
- [6] G. Pólya, Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen, Göttingen Nachr. (1917) 149–159.
- [7] Ch. Radoux, Note sur le comportement asymptotique de l'indicateur d'Euler, Ann. Soc. Sci. Bruxelles, Sér. I 91 (1977) 13–18.
- [8] T. Tao, Mertens's theorems, <http://terrytao.wordpress.com/2013/12/11/mertens-theorems/>.
- [9] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, Cambridge Stud. Adv. Math., vol. 46, 1995.
- [10] E.C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford University Press, London, 1975, X, 454 p.