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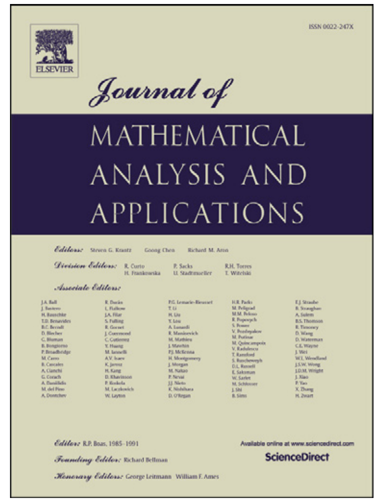

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NONLOCAL DIFFUSION EQUATIONS

VERENA BÖGELEIN, FRANK DUZAAR, PAOLO MARCELLINI, AND STEFANO SIGNORIELLO

Abstract. In this paper we present a variational approach to establish the existence and uniqueness of variational solutions to nonlocal evolutionary problems. The model we have in mind is the following nonlocal version of the parabolic p-Laplacian

$$u_t - \operatorname{div}\Big[a\big(\|Du(t)\|^p_{L^p(\Omega)}\big)\,|Du|^{p-2}\,Du\Big] = h,$$

which has recently attracted many authors. We emphasize that our approach also applies in situations where the considered functionals do not allow the derivation of the associated Euler-Lagrange equation.

1. Introduction

In this paper we are concerned with the existence of solutions to nonlocal parabolic partial differential equations, in the sense of aiming to construct solutions that inherit a certain minimizing property. Such *parabolic minimizers* or *variational solutions* are advantageous since they are likely to possess better regularity properties due to their minimizing property. In this framework we are able to give elementary proofs for existence and uniqueness for gradient flows associated to functionals depending on nonlocal quantities. The study of nonlocal problems is justified for instance by the fact that in reality measurements are made through local averages rather than pointwise. But also real-life phenomena may depend on nonlocal quantities as for example the evolution of a population whose diffusion depends on the whole population. Such problems have already been investigated for example in [8] for the case $p = 2$ and in [7] for $p > 1$: More precisely, the problem of finding a weak solution u to the Cauchy-Dirichlet problem

$$(1.1)\qquad \begin{cases} u_t - \operatorname{div}\Big[a\big(\|Du(t)\|^p_{L^p(\Omega)}\big)\,|Du|^{p-2}\,Du\Big] = h & \text{in } \Omega_T,\\ u = 0 & \text{on } \partial\Omega\times(0,T),\\ u\,(0) = u_o & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and open, $n \ge 1$, $\Omega_T := \Omega \times (0,T)$ for $T \in (0,\infty]$, $1 < p < \infty$, a is continuous and positive, the right-hand side h is an element of $W^{-1,p'}(\Omega)$, the number $p' = \frac{p}{p-1}$ being the Hölder conjugate exponent of p, and the initial value u_o is assumed to be in $W^{1,p}_0(\Omega) \cap L^2(\Omega)$. In the case $p = 2$, (1.1) is a parabolic equation with a Kirchhoff type diffusion term. Here and throughout the rest of the paper we write $u(t)$ for the restriction of u to the time slices $\Omega \times \{t\}$, i.e., for the map $u(\cdot,t)$. The authors study the problem of existence and uniqueness of solutions and their asymptotic behaviour as time t tends to ∞. The existence of weak solutions

$$u \in L^p\big(0,T;W^{1,p}_0(\Omega)\big) \cap C^0\left([0,T]\,;L^2\,(\Omega)\right)$$

is achieved in [7] by means of Galerkin's method. It is worth to keep in mind that (1.1) can be interpreted as the L^2-gradient flow to the variational functional

$$F(v) := A\bigg(\int_\Omega |Dv|^p \mathrm{d}x\bigg) - \int_\Omega hv\mathrm{d}x, \text{ where } \quad A(s) := \int_0^s a(\sigma)\mathrm{d}\sigma.$$

In this paper we do not want to argue on the level of the gradient flow associated to the above variational functional, i.e., on the level of the nonlocal diffusion equation (1.1). Instead we are going to work on the level of minimizers resp. variational solutions. This approach allows for fairly general situations, i.e., assumptions on the variational integrand, whereas additional assumptions are needed to ensure that the constructed variational solutions (parabolic minimizers) solve the corresponding evolutionary system after all. We emphasize that all our proofs are self-contained and do not use the classical existence theory for parabolic equations and systems.

More precisely, here we are concerned with an energy functional

$$F\colon W^{1,p}(\Omega, \mathbb{R}^N) \to (-\infty, \infty]$$

of the form

$$F(v) := A\bigg(\int_\Omega |Dv|^p \mathrm{d}x\bigg) + \int_\Omega S(x, K[v])\,\mathrm{d}x + \int_\Omega f(x, v, Dv)\mathrm{d}x, \tag{1.2}$$

where A, S, K and f are given functions whose properties will be explained in the following.

1.1. **Assumptions on the energy functional and definition of variational solutions.** Let $A\colon [0,\infty) \to \mathbb{R}$ be convex, increasing and such that it satisfies for some $\mu_1 \in \mathbb{R}$ and $\kappa > 0$ the coercivity condition

$$A(r) \ge \mu_1 r - \kappa \qquad \forall\, r \ge 0\,. \tag{1.3}$$

Moreover, assume that $S\colon \Omega \times \mathbb{R}^N \to [0,\infty)$ with $N \ge 1$ (i.e., the problem might be vector-valued) is a Carathéodory function such that the partial map $u \mapsto S(x,u)$ is convex for almost every $x \in \Omega$ and suppose that $K\colon L^1(\Omega, \mathbb{R}^N) \to L^1(\Omega, \mathbb{R}^N)$ is linear and bounded. For instance, the operator K could be of the form

$$K[u](x) = (k * u)(x) = \int_\Omega k(x-y)u(y)\mathrm{d}y,$$

where $k \in C^\infty(\mathbb{R}^n)$ is an integral kernel, e.g., a Gaussian filter $k \sim \exp(-c|x|^2)$ for some $c > 0$. Consider further a Carathéodory integrand $f\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to (-\infty, \infty]$ such that the partial map

$$(u, \xi) \mapsto f(x, u, \xi) \quad \text{ is convex for almost every } x \in \Omega \tag{1.4}$$

and such that for some $\mu_2 \in \mathbb{R}$, nonnegative functions $g_1 \in L^{p'}(\Omega)$ and $g_2 \in L^1(\Omega)$ the coercivity condition

$$f(x, u, \xi) \ge \mu_2 |\xi|^p - g_1(x)|u| - g_2(x) \tag{1.5}$$

is satisfied for all $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$. Throughout the paper we assume that there holds

$$\mu_1 + \mu_2 > 0. \tag{1.6}$$

Note that $[f(\cdot, v, Dv)]_- \le |\mu_2||Dv|^p + g_1|v| + g_2 \in L^1(\Omega)$ implies that the part of F containing the integrand f is well defined on $W^{1,p}(\Omega, \mathbb{R}^N)$. For the evolutionary problem considered here, we assume that the initial condition u_o satisfies

$$u_o \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^2(\Omega, \mathbb{R}^N) \tag{1.7}$$

and

$$F(u_o) = A\bigg(\int_\Omega |Du_o|^p \mathrm{d}x\bigg) + \int_\Omega S(x, K[u_o])\,\mathrm{d}x + \int_\Omega f(x, u_o, Du_o)\mathrm{d}x < \infty. \tag{1.8}$$

Corresponding to the F-energy and the initial datum u_o defined above, we are thus concerned on a purely formal level with the following Cauchy-Dirichlet problem:

$$\begin{aligned} u_t - \operatorname{div}\Big[A'\big(\|Du(t)\|^p_{L^p(\Omega)}\big)\,|Du|^{p-2}\,Du + D_\xi f(x,u,Du)\Big] \\ = -K^*\big[D_u S(x,K[u])\big] - D_u f(x,u,Du) \qquad \text{in } \Omega_T, \end{aligned} \tag{1.9}$$

and $u = u_o$ on $\partial_{\mathcal{P}}\Omega_T$. Here K^* denotes the formal adjoint operator of K and

$$\partial_{\mathcal{P}}\Omega_T = \big(\overline{\Omega}\times\{0\}\big) \cup (\partial\Omega \times (0,T))$$

stands for the parabolic boundary of Ω_T. Note that the assumptions on the integrand f in (1.4) and (1.5) allow for a large variety of interesting functionals as in [4]. For example, in the case $\mu_1 > 0$ and $S \equiv 0$, the integrand f could be of the form $-hu$ with $h \in L^{\frac{p}{p-1}}(\Omega_T)$, formally yielding $(1.1)_1$. Similarly, if f is of the form $-hu - |F|^{p-2}F\cdot\xi$ with h as before and $F \in L^p(\Omega,\mathbb{R}^{Nn})$, we get $(1.1)_1$ with an additional additive term $\operatorname{div}(|F|^{p-2}F)$ on the right-hand side. Note here, that the coercivity condition (1.5) is satisfied by means of Young's inequality for example with $\mu_2 := -\frac{1}{2}\mu_1$, $g_1 = |h|$ and $g_2 := c|F|^p$ for a constant $c = c(p,\mu_1)$, so that (1.6) holds true.

We are now in position to define the concept of variational solutions to the Cauchy-Dirichlet problem (1.9), following an idea by Lichnewsky & Temam [16] that was first used in the context of the evolutionary parametric minimal surface equation. In what follows we use the short hand notation

$$W^{1,p}_{u_o}\left(\Omega,\mathbb{R}^N\right) := u_o + W^{1,p}_0\left(\Omega,\mathbb{R}^N\right),$$

and as already mentioned the abbreviation $v(t) := v(\cdot,t)$.

Definition 1.1 (Variational Solutions)**.** Assume that the Cauchy-Dirichlet datum u_o fulfills (1.7) and (1.8). A map $u\colon \Omega_T \to \mathbb{R}^N$, $T \in (0,\infty)$, from the class

$$u \in L^p\left(0,T; W^{1,p}_{u_o}\left(\Omega,\mathbb{R}^N\right)\right) \cap C^0\left([0,T]; L^2\left(\Omega,\mathbb{R}^N\right)\right) \tag{1.10}$$

is called a *variational solution on* Ω_T to the Cauchy-Dirichlet problem (1.9) if and only if the variational inequality

$$\begin{aligned} \int_0^T F\left(u(t)\right)\mathrm{d}t \le \int_0^T \left[\int_\Omega \partial_t v\cdot(v-u)\mathrm{d}x + F(v(t))\right]\mathrm{d}t \\ + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega,\mathbb{R}^N)} \end{aligned} \tag{1.11}$$

holds true for any $v \in L^p\left(0,T;W^{1,p}_{u_o}\left(\Omega,\mathbb{R}^N\right)\right)$ with $\partial_t v \in L^2\left(\Omega_T,\mathbb{R}^N\right)$. Finally, a map $u\colon \Omega_\infty \to \mathbb{R}^N$ is termed a *global variational solution* (or variational solution on Ω_∞) if

$$u \in L^p\left(0,T;W^{1,p}_{u_o}\left(\Omega,\mathbb{R}^N\right)\right) \cap C^0\left([0,T]; L^2\left(\Omega,\mathbb{R}^N\right)\right) \qquad \forall\, T>0$$

and u is a variational solution on Ω_T for all $T > 0$. □

Note that due to (1.7) and (1.8) the time-independent extension of u_o to Ω_∞, i.e., the map $v(x,t) = u_o(x)$ for $(x,t) \in \Omega_\infty$, is an admissible comparison map in the variational inequality (1.11) on any finite cylinder Ω_T. Therefore, we have the finite energy assertion $\int_0^T F(u(t))\mathrm{d}t < \infty$ for any variational solution. Throughout the paper, we abbreviate

$$\mathcal{E}_o := \kappa + \int_\Omega \left[g_1^{p'} + g_2 + |u_o|^p + |Du_o|^p\right]\mathrm{d}x < \infty. \tag{1.12}$$

With c_o we denote the constant from §2.2, (2.3).

1.2. **The main results.** Our main results concerning the existence and regularity of variational solutions are the following:

Theorem 1.1 (Existence of variational solutions)**.** *Suppose that the energy functional F and the initial condition u_o satisfy the assumptions from Section 1.1. Then, for any $T \in (0,\infty)$ there exists a variational solution u on Ω_T in the sense of Definition 1.1.*

In the special case $\mu_2, g_1, S \equiv 0$, our existence result is similar to the one in [7]. However, in [7] the authors consider directly the differential equation (1.1) with a continuous and bounded from below and above by positive constants. In our case, the analog of the function a would be the derivative A' of A, which does not necessarily exist. However, we need to assume that A is convex, increasing and coercive. Therefore, the assumptions here and in [7] are not completely comparable. Finally, we note that the special case $p = 2$ has been considered before in [8]. However, due to its variational character our proof would not simplify much in the case $p = 2$. The reason for that stems from the use of lower semicontinuity arguments which allow to pass to the limit in the approximation scheme.

Theorem 1.2 (Uniqueness of variational solutions)**.** *Suppose that the energy functional F and the initial condition u_o satisfy the conditions from Section 1.1 and in addition that one of the following assumptions holds true:*

i) *$N \geq 1$ and either A is strictly increasing, or S is strictly convex in the second variable for almost every $x \in \Omega$ and K is injective, or $(u,\xi) \mapsto f(x,u,\xi)$ is strictly convex for almost every $x \in \Omega$.*

ii) *$N = 1$ and $A \equiv 0$ and $S \equiv 0$.*

Then, for any $T \in (0,\infty)$ the variational solution u on Ω_T from Theorem 1.1 is unique.

From the last two theorems we conclude with the following corollary on the existence of a unique global variational solution.

Corollary 1.3 (Existence and uniqueness of global variational solutions)**.** *Let the energy functional F and the initial condition u_o satisfy the assumptions from Section 1.1. Moreover, assume that one of the alternatives i) or ii) from Theorem 1.2 is satisfied. Then there exists a unique global variational solution u in the sense of Definition 1.1.*

Furthermore, we prove the following regularity properties of variational solutions:

Theorem 1.4. *Let the energy functional F and the initial condition u_o satisfy the assumptions from Section 1.1. Then, any variational solution u on Ω_T in the sense of Definition 1.1 with $T \in (0,\infty]$ satisfies*

$$\partial_t u \in L^2\left(\Omega_T, \mathbb{R}^N\right) \quad \text{and} \quad u \in C^{0,\frac{1}{2}}\left([0,\tau]\,; L^2\left(\Omega,\mathbb{R}^N\right)\right) \quad \forall\, \tau \in \mathbb{R}\cap(0,T].$$

Moreover, the a priori bound for the time derivative of u

$$\int_0^T \int_\Omega |\partial_t u|^2 \,\mathrm{d}x\mathrm{d}t \leq F(u_o) + c_o\mathcal{E}_o \tag{1.13}$$

and the energy estimate

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} F(u(t))\,\mathrm{d}t \leq F(u_o) \tag{1.14}$$

for any $0 \leq t_1 < t_2 \leq T$ hold true. In particular, for any $\tau \in \mathbb{R}\cap(0,T]$ we have the a priori bound for the L^p–$W^{1,p}$-norm of u

$$\int_0^\tau \int_\Omega \big(|u|^p + |Du|^p\big)\mathrm{d}x\mathrm{d}t \leq c\tau\big[F(u_o) + c_o\mathcal{E}_o\big], \tag{1.15}$$

where the constant $c \geq 1$ depends only on n, p, $\mu_1 + \mu_2$ and $\operatorname{diam}\Omega$.

Remark 1. The regularity requirement (1.10) from Definition 1.1 could be replaced by the stronger requirement that $u \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t u \in L^2(\Omega_T)$ and the inequality (1.11) could be replaced by

$$\int_0^T F(u(t))\,\mathrm{d}t \leq \int_0^T \left[\int_\Omega \partial_t u \cdot (v-u)\mathrm{d}x + F(v(t))\right]\mathrm{d}t. \tag{1.16}$$

In this case $u \in C^0([0,T];L^2(\Omega^*))$ would automatically be satisfied and solutions of (1.16) satisfying this stronger regularity condition could be termed *strong variational solutions*. This terminology should not be mistaken in the sense that the notion of strong solutions are often connected to classical solutions satisfying the differential equation pointwise. Since the initial condition is not anymore implicitly contained in the variational inequality (1.16), one has additionally to require $u(0) = u_o$. Theorems 1.1 and 1.4 could be joined to the statement that a unique strong variational solution exists. However, we prefer the notion of *variational solutions*, since the variational inequality (1.11) is well defined under the weaker regularity assumption (1.10). Then, Theorem 1.4 allows the interpretation of a first regularity result for variational solutions. This viewpoint is in accordance with the classical theory for non-linear parabolic equations, in which one always seeks for a natural notion for solutions under minimal regularity requirements. □

Finally, for some remarks on higher regularity properties of variational solutions we refer to §6.

1.3. **The method of proof and some remarks on previous results.** As was mentioned before, the idea of regarding weak solutions as variational solutions goes back to the work of Lichnewsky & Temam [16]. In the proof of existence, i.e., the proof of Theorem 1.1, we present a purely variational approach that has its roots in a conjecture of De Giorgi [9] on the existence of global weak solutions to the Cauchy problem for nonlinear hyperbolic wave equations. More precisely, De Giorgi suggested to infer the existence of these solutions by means of limits of minimizers of convex variational integrals on $\mathbb{R}^n \times (0,\infty)$. That this conjecture holds true for wave equations with super-critical nonlinearity of the type $u_{tt} - \Delta u = |u|^{q-2}u$ at least up to subsequences was solved by Serra and Tilli in [17]. It should be mentioned that Ilmanen gave a different proof of Brakke's existence theorem for motion by mean curvature via elliptic regularization; see [14].

The development of a related theory for evolutionary problems is therefore not far to seek. For such problems related to variational integrands $f\colon \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to (-\infty,\infty]$ (with $N \in \mathbb{N}$ such that the problem might be a vectorial one), i.e., the case $A \equiv 0$ and $S \equiv 0$, and the corresponding variational functionals

$$F(u) := \int_\Omega f(x,u,Du)\,\mathrm{d}x$$

this has already been achieved in [4]. However, compared to [4] the proofs in the present paper are different and simplified in several respects; see for instance the proof of the uniqueness via the comparison principle and the existence of the time derivative $\partial_t u$ in L^2. Here f is only assumed to be convex with respect to (u, Du) for almost every fixed x and coercive in the sense that a growth condition from below

$$f(x,u,\xi) \geq \nu\,|\xi|^p - g(x)\,(1+|u|) \qquad \forall\,(x,u,\xi) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}$$

holds true, where $\nu > 0$, $p > 1$ and $g \in L^{p'}(\Omega,[0,\infty])$ for p' the Hölder conjugate exponent of p. The associated formal parabolic system looks like

$$\partial_t u - \operatorname{div} D_\xi f(x,u,Du) = -D_u f(x,u,Du),$$

which, on account of the weak assumptions on f, generally does not have a meaning at all.

Moreover, in [5] this approach was successfully used to treat evolutionary problems for linear growth functionals in the context of image restoration problems. More precisely, the authors investigate functionals whose leading term is given by the total variation and that contain a lower order perturbation, like the general *Rudin-Osher-Fatemi models*

$$F(u) := \|Du\|(\Omega) + \int_\Omega S(x, u(x))\,\mathrm{d}x,$$

where $S\colon \Omega\times\mathbb{R} \to [0,\infty)$ is a Carathéodory integrand as above and $\|Du\|(\Omega)$ stands for the total variation of Du, or like *Tikhonov-functionals*

$$F(u) := \|Du\|(\Omega) + \tfrac{\kappa}{2}\int_\Omega \big|K[u] - u_o\big|^2\mathrm{d}x,$$

where $K : L^1(\Omega) \to L^2(\Omega)$ is linear, bounded, injective and κ is a large penalization factor.

The ideas from [4, 5] (see also [2]) can be applied in the context of this paper, i.e., to functionals with some sort of nonlocal quantities. To be more precise, we follow De Giorgi's scheme to prove the existence of solutions to evolutionary Cauchy-Dirichlet problems as in (1.9), as limit of energy functionals which in a certain sense can be interpreted as the elliptic regularization of the evolutionary problem in the spirit of Ilmanen's approach to Brakke's mean curvature flow. We consider mappings $u\colon \Omega_T \to \mathbb{R}^N$, $T > 0$, satisfying the initial-boundary condition $u = u_o$ on the parabolic boundary $\partial_{\mathcal{P}}\Omega_T$ of Ω_T, where $u_o\colon \Omega \to \mathbb{R}^N$ is a given time independent datum. For $\varepsilon \in (0,1]$ and F defined in (1.2), we investigate the strictly convex variational integrals

$$\mathcal{F}_\varepsilon(v) := \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}}\bigg[\tfrac12\int_\Omega |\partial_t v|^2\,\mathrm{d}x + \tfrac{1}{\varepsilon}F(v(t))\bigg]\,\mathrm{d}t,$$

to which the existence of unique minimizers u_ε from a suitable class can be inferred by means of classical methods of the Calculus of Variations. Formally, these minimizers satisfy the elliptic system

$$\begin{aligned}-\varepsilon\partial_{tt}u_\varepsilon + \partial_t u_\varepsilon - \operatorname{div}\Big[A'\big(\|Du_\varepsilon(t)\|^p_{L^p(\Omega)}\big)|Du_\varepsilon|^{p-2}Du_\varepsilon + D_\xi f(x,u_\varepsilon,Du_\varepsilon)\Big]&\\ = -K^*\big[D_u S(\cdot, K[u_\varepsilon])\big] - D_u f(x,u_\varepsilon,Du_\varepsilon),&\end{aligned}$$

such that the convergence of the $\mathcal{F}_\varepsilon$-minimizers u_ε to a solution u to the associated evolutionary problem (1.9) as $\varepsilon \downarrow 0$ seems natural to expect. This argument is of course purely heuristic since the derivation of the Euler-Lagrange equation is in general not possible and since it is not clear at all how to pass to the limit $\varepsilon \downarrow 0$. Therefore, remaining on the level of minimizers, one might expect that the minimization property of the approximating maps u_ε could assign to the limit u, and therefore could lead to a solution to the Cauchy-Dirichlet problem (1.9) if the data (A, S, f, K) are regular enough. In this direction, the notion of variational solutions is precisely the link between the strictly convex variational functionals $\mathcal{F}_\varepsilon$ and the evolutionary problem associated to the functionals F.

Clearly, to show subconvergence $u_\varepsilon \to u$ in an appropriate weak sense, we are in need of uniform a priori energy bounds for the sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ (cf. (5.7), (5.9) and (5.11)). These estimates will be derived in §5.3 by substantially new arguments, i.e., by means of direct comparison arguments using a time mollification procedure (cf. §2.3) to obtain comparison maps sufficiently regular with respect to time.

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2. Notations and Preliminaries

2.1. **Notations.** For $1 \le p \le \infty$, $n, N \in \mathbb{N}$ and an open set $\Omega \subset \mathbb{R}^n$, the spaces $L^p\left(\Omega, \mathbb{R}^N\right)$, $W^{1,p}\left(\Omega, \mathbb{R}^N\right)$ and $W_0^{1,p}\left(\Omega, \mathbb{R}^N\right)$ denote the usual Lebesgue and Sobolev spaces, respectively. Moreover, for $T \in (0, \infty]$, by Ω_T we denote the space-time cylinder $\Omega \times (0, T)$. Further, for a set A the characteristic function of A shall be denoted by χ_A.

2.2. **A lower bound for the functional.** In concern of the coercivity conditions in (1.3) and (1.5), both A, f and thus F could be negative. Therefore, we are going to derive a lower bound for the energy F that will be needed several times throughout the paper. We start with an application of Poincaré's inequality which leads to

$$\int_\Omega |v|^p \mathrm{d}x \le c_p \left[\int_\Omega |Dv|^p \mathrm{d}x + \|u_o\|^p_{W^{1,p}}\right] \tag{2.1}$$

for any $v \in W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)$. The constant $c_p \ge 1$ in the preceding inequality depends only on n, p and $\operatorname{diam}\Omega$. From $S \ge 0$ and the coercivity assumptions (1.3), (1.5) and (1.6) it follows for any $v \in W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)$ that

$$F(v) \ge (\mu_1 + \mu_2) \int_\Omega |Dv|^p \mathrm{d}x - \int_\Omega g_1 |v| \mathrm{d}x - \int_\Omega g_2 \mathrm{d}x - \kappa.$$

The second term can further be estimated by means of Young's and the Poincaré type inequality (2.1). For any $\lambda > 0$, this yields

$$\begin{aligned} \int_\Omega g_1 |v| \mathrm{d}x &\le \lambda \int_\Omega |v|^p \mathrm{d}x + \lambda^{-\frac{1}{p-1}} \int_\Omega g_1^{p'} \mathrm{d}x \\ &\le c_p \lambda \int_\Omega |Dv|^p \mathrm{d}x + c_p \lambda \|u_o\|^p_{W^{1,p}} + \lambda^{-\frac{1}{p-1}} \int_\Omega g_1^{p'} \mathrm{d}x. \end{aligned}$$

Setting $\lambda = \frac{\mu_1+\mu_2}{2c_p}$, the first term on the right-hand side of the previous inequality can be absorbed in the estimate for F above, i.e.,

$$\begin{aligned} F(v) &\ge \tfrac{1}{2}(\mu_1 + \mu_2) \int_\Omega |Dv|^p \mathrm{d}x \\ &\quad - \tfrac{1}{2}(\mu_1 + \mu_2)\|u_o\|^p_{W^{1,p}} - \Big(\frac{2c_p}{\mu_1 + \mu_2}\Big)^{\frac{1}{p-1}} \int_\Omega g_1^{p'} \mathrm{d}x - \int_\Omega g_2 \mathrm{d}x - \kappa \\ &\ge \tfrac{1}{2}(\mu_1 + \mu_2) \int_\Omega |Dv|^p \mathrm{d}x - c_o \mathcal{E}_o, \end{aligned} \tag{2.2}$$

for the quantity $\mathcal{E}_o$ defined in (1.12) and

$$c_o := \max\Big\{1, \tfrac{1}{2}(\mu_1 + \mu_2), \Big(\frac{2c_p}{\mu_1 + \mu_2}\Big)^{\frac{1}{p-1}}\Big\}. \tag{2.3}$$

Note that $c_o \ge 1$ depends only on n, p, $\mu_1 + \mu_2$ and $\operatorname{diam}\Omega$. From (2.2) we obtain the bound

$$0 \le \tfrac{1}{2}(\mu_1 + \mu_2) \int_\Omega |Dv|^p \mathrm{d}x \le F(v) + c_o \mathcal{E}_o$$

for any $v \in W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)$.

2.3. **Mollification in time.** In the definition of variational solutions we are not going to assume any condition on their derivative with respect to time. Therefore, we are generally not allowed to use them as comparison maps in the variational inequality (1.11) and a suitable mollification procedure in time is thus needed. To this end, for X a separable Banach space, an initial datum $v_o \in X$ and $1 \le r \le \infty$, let $v \in L^r(0,T;X)$ and define the mollification in time of v for $h \in (0,T]$ and $t \in [0,T]$ by means of

$$[v]_h(t) := \mathrm{e}^{-\frac{t}{h}} v_o + \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} v(s)\,\mathrm{d}s. \tag{2.4}$$

In the application we are going to use for instance $X = L^r(\Omega,\mathbb{R}^N)$ and the related parabolic space $L^r(0,T;L^r(\Omega,\mathbb{R}^N))$. One of the features of the mollification in time is that $[v]_h$ (formally) solves the ordinary differential equation

$$\partial_t [v]_h = \tfrac{1}{h}\big(v - [v]_h\big) \tag{2.5}$$

with initial condition $[v]_h(0) = v_o$. Note that, if $[v]_h$ does solve the ordinary differential equation (2.5) above, then clearly any common membership of both v and its regularization $[v]_h$ to a Banach space is passed also to the time derivative of $[v]_h$. This fact will be exploited in §5.3 to derive the uniform a priori bounds for the sequence of $\mathcal{F}_\varepsilon$-minimizers u_ε which later on imply in particular that the variational solution u possesses a time derivative in $L^2(\Omega_T,\mathbb{R}^N)$.

The basic properties of the mollification in time are summarized in the following lemma (cf. [15, Lemma 2.2] and [3, Appendix B] for the proofs).

Lemma 2.1. *Let X be a separable Banach space and $v_o \in X$. If $v \in L^r(0,T;X)$ for some $r \ge 1$, then also $[v]_h \in L^r(0,T;X)$, and $[v]_h \to v$ in $L^r(0,T;X)$ as $h \downarrow 0$. Further, for any $t_o \in (0,T]$ there holds*

$$\|[v]_h\|_{L^r(0,t_o;X)} \le \|v\|_{L^r(0,t_o;X)} + \Big[\tfrac{h}{r}\Big(1-\mathrm{e}^{-\frac{t_o r}{h}}\Big)\Big]^{\frac{1}{r}} \|v_o\|_X .$$

In the case $r=\infty$, the bracket $[\dots]^{1/r}$ in the preceding inequality has to be interpreted as 1. Moreover, $\partial_t[v]_h \in L^r(0,T;X)$ with

$$\partial_t[v]_h = \tfrac{1}{h}\big(v - [v]_h\big).$$

If additionally also $\partial_t v \in L^r(0,T;X)$, then

$$\partial_t[v]_h = \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}}\partial_s v(s)\mathrm{d}s$$

and

$$\|\partial_t[v]_h\|_{L^p(0,T;X)} \le \|\partial_t v\|_{L^p(0,T;X)} .$$

Finally, if $v \in C^0([0,T];X)$, then also $[v]_h \in C^0([0,T];X)$, $[v]_h(0) = v_o$ and $[v]_h \to v$ in $L^\infty([0,T];X)$ as $h \downarrow 0$. □

In the following we want to show that the time mollification of the F-energy satisfies $F([v]_h) \le [F(v)]_h$ on $[0,T]$ and $[F(v)]_h \to F(v)$ in $L^1(0,T)$ for a not relabeled subsequence as $h \downarrow 0$ if v and v_o are chosen properly. The first estimate will be used frequently throughout the paper, while the convergence is needed to show that the variational solution u is also a parabolic minimizer in the sense of Wieser [19]. Concerning the part of F involving the integrand f, we conclude similarly as in [4, Lemma 2.3] that there holds:

Lemma 2.2. *Let $T>0$ and f an integrand as in (1.4) and (1.5). Suppose further that $v \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ with $f(\cdot,v,Dv) \in L^1(\Omega_T)$ and $v_o \in W^{1,p}(\Omega,\mathbb{R}^N)$*

with $f(\cdot,v_o,Dv_o)\in L^1(\Omega)$. *Then,* $[v]_h\in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$, $f(\cdot,[v]_h,D[v]_h)\in L^1(\Omega_T)$ *and*

$$f(\cdot,[v]_h,D[v]_h)\le [f(\cdot,v,Dv)]_h \quad \textit{on } \Omega_T.$$

Moreover, we have

$$f(\cdot,[v]_h,D[v]_h)\to f(\cdot,v,Dv) \quad \textit{in } L^1(\Omega_T)$$

in the limit $h\downarrow 0$.

Proof. In the following, in order to use a variant of Lebesgue's dominated convergence theorem [12, Chap. 1.3 Thm. 4], we want to show that there exists a majorizing L^1-convergent sequence G_h with $\left|f(\cdot,[v]_h,D[v]_h)\right|\le G_h$. We are going to show that

$$G_h:=\left|[f(\cdot,v,Dv)]_h\right|+|\mu_2||D[v]_h|^p+g_1|[v]_h|+g_2$$

is an admissible choice, where, as mentioned already above, $[f(\cdot,v,Dv)]_h$ is defined as in (2.4) with v_o replaced by $f(\cdot,v_o,Dv_o)$ and v replaced by $f(\cdot,v,Dv)$. Observe that Lemma 2.1 implies the convergence

$$L^1(\Omega_T)\ni [f(\cdot,v,Dv)]_h\to f(\cdot,v,Dv) \quad \text{in } L^1(\Omega_T) \text{ as } h\downarrow 0,$$

as well as the bound

$$\left\|[f(\cdot,v,Dv)]_h\right\|_{L^1(\Omega_T)}\le \left\|f(\cdot,v,Dv)\right\|_{L^1(\Omega_T)}+h\left\|f(\cdot,v_o,Dv_o)\right\|_{L^1(\Omega)}<\infty.$$

Note further that

$$\frac{1}{h\left(1-\mathrm{e}^{-\frac{t}{h}}\right)}\int_0^t \mathrm{e}^{\frac{s-t}{h}}\,\mathrm{d}s\equiv 1,$$

i.e., the mollification $[v]_h$ may be regarded as mean with respect to the measure $\mathrm{e}^{\frac{s-t}{h}}\,\mathrm{d}s$ modulo a multiplicative factor and therefore allows for an application of Jensen's inequality. Exploiting the convexity of $(u,\xi)\mapsto f(\cdot,u,\xi)$, Jensen's inequality and (1.5), we conclude with the following pointwise estimate for almost every $x\in\Omega$

$$\begin{aligned}
-|\mu_2|&|D[v]_h|^p-g_1(x)|[v]_h|-g_2(x)\\
&\le f(x,[v]_h,D[v]_h)\\
&\le \mathrm{e}^{-\frac{t}{h}}f(x,v_o,Dv_o)+\left(1+\mathrm{e}^{-\frac{t}{h}}\right)f\left(x,\frac{1}{h(1+\mathrm{e}^{-\frac{t}{h}})}\int_0^t \mathrm{e}^{\frac{s-t}{h}}(v,Dv)(x,s)\,\mathrm{d}s\right)\\
&\le \mathrm{e}^{-\frac{t}{h}}f(x,v_o,Dv_o)+\tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}}f(x,(v,Dv)(x,s))\,\mathrm{d}s\\
&=\left[f(\cdot,v,Dv)\right]_h.
\end{aligned}$$

Since $([v]_h,D[v]_h)$ is strongly convergent in $L^p\left(\Omega_T,\mathbb{R}^N\times\mathbb{R}^{Nn}\right)$ to (v,Dv) as $h\downarrow 0$, we can extract a not relabelled subsequence such that $([v]_h,D[v]_h)\to(v,Dv)$ pointwise almost everywhere on Ω_T. Finally, the continuity of $(u,\xi)\mapsto f(\cdot,u,\xi)$ implies the pointwise almost everywhere convergence $f(\cdot,[v]_h,D[v]_h)\to f(\cdot,v,Dv)$ and the variant of the dominated convergence theorem from [12, Chap. 1.3 Thm. 4] is thus applicable. Since the same argument can be applied to any subsequence, the convergence holds for the whole sequence. □

Concerning the part of F involving the integrand S, we conclude (cf. [5, Lemma 2.5]):

Lemma 2.3. *Let* $T>0$ *and suppose that* $S\colon\Omega\times\mathbb{R}^N\to[0,\infty)$ *is a Carathéodory integrand such that the partial map* $u\mapsto S(x,u)$ *is convex for almost every* $x\in\Omega$. *Moreover, let* $K\colon L^1\left(\Omega,\mathbb{R}^N\right)\to L^1\left(\Omega,\mathbb{R}^N\right)$ *be linear and bounded and assume that*

$v \in L^1\left(\Omega_T, \mathbb{R}^N\right)$ with $S\left(\cdot, K[v]\right) \in L^1\left(\Omega_T\right)$ and $v_o \in L^1\left(\Omega, \mathbb{R}^N\right)$ with $S\left(\cdot, K[v_o]\right) \in L^1(\Omega)$. Then, $S\left(\cdot, K\left[[v]_h\right]\right) \in L^1\left(\Omega_T\right)$ and

$$S\left(\cdot, K\left[[v]_h\right]\right) \le \left[S\left(\cdot, K[v]\right)\right]_h \qquad \text{on } \Omega_T,$$

where $\left[S\left(\cdot, K[v]\right)\right]_h(t)$ is defined as in (2.4) with v_o replaced by $S\left(\cdot, K[v_0]\right)$. Moreover, we have

$$S\left(\cdot, K\left[[v]_h\right]\right) \to S\left(\cdot, K[v]\right) \quad \text{in } L^1\left(\Omega_T\right)$$

for a not relabeled subsequence of $[v]_h$ in the limit $h \downarrow 0$.

Proof. We argue similarly to the proof of Lemma 2.2 by establishing the existence of a majorizing L^1-convergent sequence G_h with $|S(\cdot, K[[v]_h])| \le G_h$. We shall show that $G_h := \big|[S(\cdot, K[v])]_h\big|$ is an admissible choice, where, $[S(\cdot, K[v])]_h$ is defined as in (2.4) with v_o replaced by $S(\cdot, K[v_o])$ and v replaced by $S(\cdot, K[v])$. By means of Lemma 2.1 there hold

$$L^1(\Omega_T) \ni [S(\cdot, \mathrm{K}[v])]_h \to S(\cdot, \mathrm{K}[v]) \quad \text{in } L^1(\Omega_T) \text{ as } h \downarrow 0,$$

and

$$\big\|[S(\cdot, K[v])]_h\big\|_{L^1(\Omega_T)} \le \big\|S(\cdot, K[v])\big\|_{L^1(\Omega_T)} + h\big\|S(\cdot, K[v_o])\big\|_{L^1(\Omega)} < \infty.$$

Using the convexity of $u \mapsto S(\cdot, u)$, the linearity of K and Jensen's inequality, we obtain the following pointwise estimate for almost every $x \in \Omega$

$$\begin{aligned} 0 &\le S(x, K[[v]_h(t)]) \\ &\le \mathrm{e}^{-\frac{t}{h}} S\big(x, K[v_o]\big) + \Big(1 + \mathrm{e}^{-\frac{t}{h}}\Big) S\bigg(x, \tfrac{1}{h(1+\mathrm{e}^{-\frac{t}{h}})} \int_0^t \mathrm{e}^{\frac{s-t}{h}} K[v(s)](x)\,\mathrm{d}s\bigg) \\ &\le \mathrm{e}^{-\frac{t}{h}} S(x, K[v_o]) + \tfrac{1}{h} \int_0^t \mathrm{e}^{\frac{s-t}{h}} S(x, K[v(s)])\,\mathrm{d}s \\ &= \big[S(x, K[v])\big]_h(t). \end{aligned}$$

Since $[v]_h$ is strongly convergent in $L^1(\Omega_T, \mathbb{R}^N)$ to v as $h \downarrow 0$, the continuity of K implies that also $K[[v]_h] \to K[v]$ in $L^1(\Omega_T, \mathbb{R}^N)$. Hence, we can extract a not relabelled subsequence such that $K[[v]_h] \to K[v]$ pointwise almost everywhere on Ω_T, and as in the proof of the preceding Lemma the continuity of $u \mapsto S(\cdot, u)$ implies the pointwise almost everywhere convergence $S(x, K[[v]_h]) \to S(x, K[v])$. The variant of the dominated convergence theorem from [12, Chap. 1.3 Thm. 4] is therefeore applicable and yields the final claim. $\square$

Concerning the part of F involving the map A we conclude (cf. [5, Lemma 2.6]):

Lemma 2.4. *Let $T > 0$ and suppose that $A\colon [0, \infty) \to \mathbb{R}$ is convex and increasing, satisfying (1.3), $v \in L^p(0, T, W^{1,p}(\Omega, \mathbb{R}^N))$ such that $A\big(\|Dv(t)\|^p_{L^p(\Omega)}\big) \in L^1(0, T)$ and $v_o \in W^{1,p}(\Omega, \mathbb{R}^N)$. Then, $A\big(\|D[v]_h(t)\|^p_{L^p(\Omega)}\big) \in L^1(0, T)$, and*

$$A\big(\|D[v]_h(t)\|^p_{L^p(\Omega)}\big) \le \big[A\big(\|Dv\|^p_{L^p(\Omega)}\big)\big]_h(t) \quad \forall\, t \in (0, T).$$

Moreover, we have

$$A\big(\|D[v]_h\|^p_{L^p(\Omega)}\big) \to A\big(\|Dv\|^p_{L^p(\Omega)}\big) \quad \text{in } L^1(0, T)$$

for a not relabeled subsequence of $[v]_h$ in the limit $h \downarrow 0$.

Proof. For the better readability we waive the reference to the domain Ω and $\mathbb{R}^{Nn}$ in the occurring L^p-norms. As in the proof of Lemma 2.4 we infer the existence of a majorizing L^1-convergent sequence G_h with $\big|A(\|D[v]_h(t)\|^p_{L^p})\big| \le G_h(t)$, to be able to apply a variant of Lebesgue's dominated convergence theorem [12, Chap. 1.3, Thm. 4]. To this end, note that for any $\varphi \in L^{p'}(\Omega, \mathbb{R}^{Nn})$, where p' is the Hölder conjugate exponent of p, there holds

$$\langle D[v]_h(t), \varphi\rangle = \mathrm{e}^{-\frac{t}{h}} \langle Dv_o, \varphi\rangle + \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} \langle Dv(s), \varphi\rangle \,\mathrm{d}s$$

such that by means of the dual characterization of $\|\cdot\|_{L^p}$, i.e., by taking the supremum over all $\varphi \in L^{p'}(\Omega, \mathbb{R}^{Nn})$ with $\|\varphi\|_{L^{p'}} \le 1$, we conclude with the following bound

$$\|D[v]_h(t)\|_{L^p} \le \mathrm{e}^{-\frac{t}{h}} \|Dv_o\|_{L^p} + \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} \|Dv(s)\|_{L^p} \mathrm{d}s.$$

Now, the convexity and the monotonicity of $\tilde{A}(s) := A(s^p)$ on $[0,\infty)$, implied by the convexity and monotonicity of both A and $s \mapsto s^p$ allow for an application of Jensen's inequality (cf. the proof of Lemma 2.4), yielding

$$\begin{aligned}
&\mu_1 \|D[v]_h(t)\|^p_{L^p} - \kappa \\
&\quad\le A\big(\|D[v]_h(t)\|^p_{L^p}\big) = \tilde{A}\big(\|D[v]_h(t)\|_{L^p}\big) \\
&\quad\le \mathrm{e}^{-\frac{t}{h}} \tilde{A}\big(\|Dv_o\|_{L^p}\big) + \Big(1-\mathrm{e}^{-\frac{t}{h}}\Big)\tilde{A}\bigg(\tfrac{1}{h(1-\mathrm{e}^{-\frac{t}{h}})}\int_0^t \mathrm{e}^{\frac{s-t}{h}} \|Dv(s)\|_{L^p}\mathrm{d}s\bigg) \\
&\quad\le \mathrm{e}^{-\frac{t}{h}} \tilde{A}\big(\|Dv_o\|_{L^p}\big) + \tfrac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}} \tilde{A}\big(\|Dv(s)\|_{L^p}\big)\mathrm{d}s \\
&\quad= \big[A\big(\|Dv\|^p_{L^p}\big)\big]_h(t),
\end{aligned}$$

where $\big[A\big(\|Dv\|^p_{L^p}\big)\big]_h(t)$ is defined according to (2.4) with v_o and v replaced by $A(\|Dv_o\|^p_{L^p})$ and $A(\|Dv(\cdot,t)\|^p_{L^p})$, respectively. The claim now follows along the lines of the proof of Lemma 2.3. □

The last three lemmata imply the following corollary:

Corollary 2.5. *Let $T > 0$, $v \in L^p(0,T,W^{1,p}(\Omega,\mathbb{R}^N))$ with $F(u(t)) \in L^1(0,T)$ and $v_o \in W^{1,p}\left(\Omega,\mathbb{R}^N\right)$ with $S(\cdot,K[v_o])$, $f(\cdot,v_o,Dv_o) \in L^1(\Omega)$. Then, $F\left([v]_h(t)\right) \in L^1(0,T)$, and*

$$F\big([v]_h(t)\big) \le [F(v)]_h(t) \qquad \forall\, t \in (0,T)\,.$$

Moreover, we have

$$F\big([v]_h(t)\big) \to F(v(t)) \qquad \text{in } L^1(0,T)$$

for a not relabeled subsequence of $[v]_h$ as $h \downarrow 0$. □

2.4. **Localization in time.** The goal here is to show that a variational solution u on some cylinder Ω_T with $T \in (0,\infty)$ is also a variational solution on any smaller cylinder Ω_τ with $\tau \in (0,T)$. To this end, for $\theta \in (0,\tau)$ consider the cut-off function

$$\xi_\theta(t) := \chi_{[0,\tau-\theta]}(t) + \tfrac{\tau-t}{\theta}\chi_{(\tau-\theta,\tau]}(t).$$

For $v \in L^p(0,\tau;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ satisfying $\partial_t v \in L^2(\Omega_\tau,\mathbb{R}^N)$ and $F\left(v(t)\right) \in L^1\left(0,\tau\right)$ (if $F(v(t)) \notin L^1(0,\tau)$, the variational inequality follows trivially) we choose

$$v_\theta := \xi_\theta v + (1-\xi_\theta)\,[u]_h$$

as comparison map in (1.11), where $[u]_h$ is defined according to (2.4) with u_o and u instead of v_o and v, respectively. Note that by extending $\xi_\theta v$ from Ω_τ to Ω_T by 0, the admissibility of v_θ is due to Lemma 2.1, i.e., $[u]_h \in L^p(0,T,W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ and $\partial_t[u]_h \in L^2(\Omega_T,\mathbb{R}^N)$. Furthermore, the fact that $u \in C^0([0,T];L^2(\Omega,\mathbb{R}^N))$ by definition implies via Lemma 2.1 that also $[u]_h \in C^0([0,T]; L^2(\Omega,\mathbb{R}^N))$. From the minimality condition (1.11) we thus infer

$$\int_0^T F(u(t))\mathrm{d}t \le \int_0^T \Big[\int_\Omega \partial_t v_\theta \cdot \big(v_\theta - u\big)\mathrm{d}x + F(v_\theta(t))\Big]\mathrm{d}t \\ + \tfrac12\|v(0)-u_o\|^2_{L^2} - \tfrac12\|([u]_h - u)(T)\|^2_{L^2}. \tag{2.6}$$

In the following we want to pass to the limit $\theta \downarrow 0$. Note in this direction that

$$\partial_t v_\theta \cdot (v_\theta - u) = \xi'_\theta \xi_\theta |v-[u]_h|^2 + \xi'_\theta (v-[u]_h)([u]_h - u) \\ + [\xi_\theta \partial_t v + (1-\xi_\theta)\partial_t [u]_h][\xi_\theta (v-[u]_h)([u]_h-u)]$$

and obviously being equal to $\partial_t v(v-u)$ if $0\le t\le \tau-\theta$ and equal to $\partial_t[u]_h([u]_h-u)$ if $\tau<t\le T$. Therefore, we have to take a look at $\int_{\tau-\theta}^{\tau}\int_\Omega \partial_t v_\theta\cdot(v_\theta-u)\mathrm{d}x\mathrm{d}t$ as $\theta\downarrow 0$. By means of an integration by parts for the first term on the right-hand side of the previous equation, using Lebesgue's differentiation theorem for the first two terms and noting that the last term is in L^1, we deduce as in [6, Lemma 3.2] that

$$\limsup_{\theta\downarrow 0}\int_0^T\int_\Omega \partial_t v_\theta (v_\theta - u)\mathrm{d}x\mathrm{d}t \\ \le \int_0^\tau\int_\Omega \partial_t v(v-u)\mathrm{d}x\mathrm{d}t + \int_\tau^T\int_\Omega \partial_t [u]_h\big([u]_h - u\big)\mathrm{d}x\mathrm{d}t \\ - \tfrac12\|(v-[u]_h)\,(\tau)\,\|^2_{L^2} + \big\|\big([u]_h-u\big)\big(v-[u]_h\big)(\tau)\big\|_{L^1}.$$

Further, by means of the convexity of F we have

$$F(v_\theta) \le F(v) + F([u]_h) \in L^1(\tau-\theta,\tau),$$

such that $\int_{\tau-\theta}^{\tau} F(v_\theta)\mathrm{d}t \to 0$ as $\theta\downarrow 0$, i.e.,

$$\lim_{\theta\downarrow 0}\int_0^T F(v_\theta(t))\mathrm{d}t = \int_0^\tau F(v(t))\mathrm{d}t + \int_\tau^T F\big([u]_h(t)\big)\mathrm{d}t.$$

In the limit $\theta\downarrow 0$ the variational inequality (2.6) thus implies

$$\int_0^\tau F(u(t))\mathrm{d}t \le \int_0^\tau \Big[\int_\Omega \partial_t v\cdot(v-u)\mathrm{d}x + F(v(t))\Big]\mathrm{d}t \\ + \tfrac12\|v(0)-u_o\|^2_{L^2} - \tfrac12\big\|(v-[u]_h)(\tau)\big\|^2_{L^2} \\ + \int_\tau^T \Big[\int_\Omega \partial_t[u]_h\cdot\big([u]_h-u\big)\mathrm{d}x + F\big([u]_h(t)\big) - F(u(t))\Big]\mathrm{d}t \\ - \tfrac12\big\|\big([u]_h-u\big)(T)\big\|^2_{L^2} + \big\|\big([u]_h-u\big)\cdot\big(v-[u]_h\big)(\tau)\big\|_{L^1}.$$

The variational inequality (1.11) on the subcylinder Ω_τ now follows in the limit $h\downarrow 0$ by taking into account $\partial_t[u]_h\cdot([u]_h-u)\le 0$ and the convergences $[u]_h\to u$ in $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ as well as $F([u]_h(t))\to F(u(t))$ in $L^1(0,T)$ as $h\downarrow 0$ by means of Lemma 2.1 and Corollary 2.5. Note that

$$\big\|\big(v-[u]_h\big)(\tau)\big\|^2_{L^2} \to \|(v-u)(\tau)\|^2_{L^2}$$

as $h\downarrow 0$, while all other terms that depend on h are either nonpositive or vanish in the limit. Since v was arbitrary, u is therefore a variational solution on the subcylinder Ω_τ.

2.5. **The initial condition.** Here we are going to show that variational solutions in the sense of Definition 1.1 fulfill the initial condition $u(0) = u_o$ in the usual L^2-sense. This follows from the fact that the difference $\|u(t) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)}$ grows at most linearly with respect to $t > 0$ (cf. the estimate (2.7) below).

Lemma 2.6. *Any variational solution u on Ω_T for some $T \in (0,\infty]$ in the sense of Definition 1.1 satisfies*

$$\lim_{t\downarrow 0} \|u(t) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} = 0.$$

Proof. From §2.4 we know that u fulfills the variational inequality (1.11) on any subcylinder Ω_τ for $\tau \in (0,T)$. Testing (1.11) with the time-independent extension of u_o, i.e., with $v(t) = u_o$ for $t \in (0,\tau]$, which is admissible due to (1.7), implies

$$\int_0^\tau F(u(t))\mathrm{d}t + \tfrac{1}{2}\|u(\tau) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} \le \int_0^\tau F(u_o)\mathrm{d}t = \tau F(u_o) < \infty$$

by means of hypothesis (1.8). The lower bound for F in (2.2) thus implies

$$\|u(\tau) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 2\tau\big[F(u_o) + c_o\mathcal{E}_o\big] \qquad \forall\, \tau \in (0,T). \tag{2.7}$$

This proves the claim of the lemma. □

3. The time derivative (Proof of Theorem 1.4)

In the definition of variational solutions we do not make any assumptions on the existence of the time derivative of u. But since the lateral boundary data are assumed to be constant with respect to time, we are able to show that the time derivative $\partial_t u$ of any such variational solution u exists in a weak sense and that it belongs to $L^2(\Omega_T,\mathbb{R}^N)$.

Recall from §2.4 that a variational solution u on Ω_T for $T \in (0,\infty]$ is also a variational solution on any subcylinder Ω_τ for $\tau \in \mathbb{R}\cap(0,T]$. Therefore, testing the variational inequality (1.11) on Ω_τ with the admissible map $v = [u]_h$, where $[u]_h$ is defined according to (2.4) with (v_o, v) replaced by (u_o, u), implies the estimate

$$\begin{aligned}
-\int_0^\tau\int_\Omega \partial_t[u]_h\cdot\big([u]_h - u\big)\mathrm{d}x\mathrm{d}t &\le \int_0^\tau \big[F\big([u]_h(t)\big) - F(u(t))\big]\mathrm{d}t\\
&\le \int_0^\tau \big[[F(u)]_h(t) - F(u(t))\big]\mathrm{d}t\\
&= -h\int_0^\tau \partial_t[F(u)]_h(t)\mathrm{d}t\\
&= h\big[F(u_o) - [F(u)]_h(\tau)\big],
\end{aligned}$$

where $[F(u)]_h$ is defined according to (2.4) with v_o and $v(t)$ replaced by $F(u_o)$ and $F(u(t))$, respectively. Note that we discarded the nonpositive term

$$-\tfrac{1}{2}\big\|\big([u]_h - u\big)(\tau)\big\|^2_{L^2(\Omega,\mathbb{R}^N)} \le 0$$

on the right-hand side of the first inequality, that we used Corollary 2.5 in the second inequality and the fact that $F(u(t)) \in L^1(0,\tau)$ in the third estimate. Now, using again Lemma 2.1 we are allowed to rewrite the difference $[u]_h - u$ appearing in the integral on the left-hand side of the previous chain of inequalities. Exploiting once again the lower bound for F from (2.2), this yields the uniform bound

$$\int_0^\tau\int_\Omega \big|\partial_t[u]_h\big|^2\mathrm{d}x\mathrm{d}t \le F(u_o) + c_o\mathcal{E}_o \qquad \forall\, h \in (0,\tau],$$

such that the time derivative $\partial_t u$ exists with $\partial_t u \in L^2(\Omega_\tau, \mathbb{R}^N)$ for all $\tau \in (0, T]$ together with the quantitative estimate

$$\int_0^\tau \int_\Omega |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \le F(u_o) + c_o \mathcal{E}_o \qquad \forall\, \tau \in (0, T].$$

Therefore, if $T < \infty$, setting $\tau = T$, or otherwise, if $T = \infty$, letting $\tau \to \infty$, shows the claim that $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ together with estimate (1.13) for any variational solution u on Ω_T and any $T \in (0, \infty]$.

Furthermore, for t_1, t_2 with $0 \le t_1 < t_2 \le T$ we have using Hölder's inequality, Fubini's theorem and the bound from above

$$\begin{aligned} \|u(t_2) - u(t_1)\|^2_{L^2(\Omega,\mathbb{R}^N)} &\le |t_2 - t_1| \int_{t_1}^{t_2} \int_\Omega |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \\ &\le |t_2 - t_1| \big[F(u_o) + c_o \mathcal{E}_o\big]. \end{aligned}$$

Now, taking $t_1 = 0$, it follows for every $t \in \mathbb{R} \cap (0, T]$ that

$$\begin{aligned} \|u(t)\|^2_{L^2(\Omega,\mathbb{R}^N)} &\le 2\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2\|u(t) - u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} \\ &\le 2\|u_o\|^2_{L^2(\Omega,\mathbb{R}^N)} + 2t\big[F(u_o) + c_o \mathcal{E}_o\big], \end{aligned}$$

thus yielding

$$u \in C^{0,\frac12}\left([0,\tau]; L^2\left(\Omega, \mathbb{R}^N\right)\right) \qquad \forall \tau \in \mathbb{R} \cap (0, T].$$

The remaining energy estimate (1.14) can be inferred by exploiting the fact $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ in the variational inequality (1.11) as follows. For any $\tau \in (0, T]$ and any

$$v \in L^p(0, \tau; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)) \quad \text{with} \quad \partial_t v \in L^2(\Omega_\tau, \mathbb{R}^N)$$

we have

$$\int_0^\tau \int_\Omega F(u(t))\mathrm{d}t \le \int_0^\tau \left[\int_\Omega \partial_t u \cdot (v - u)\mathrm{d}x + F(v(t))\right] \mathrm{d}t.$$

Now, for t_1, t_2 with $0 \le t_1 < t_2 \le \tau$ define the cut-off function

$$\xi_{t_1,t_2}(t) := \chi_{[0,t_1]}(t) + \frac{t_2 - t}{t_2 - t_1} \chi_{(t_1,t_2)}(t)$$

and choose $v = u + \xi_{t_1,t_2}([u]_h - u)$ as comparison map in the preceding minimality condition on Ω_τ. To check that v is admissible is straightforward. Now, the convexity of F and Corollary 2.5 imply

$$\begin{aligned} \int_0^{t_2} F(u(t))\mathrm{d}t \le &\int_0^{t_2} \int_\Omega \xi_{t_1,t_2} \partial_t u \cdot \big([u]_h - u\big)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^{t_2} \big[(1 - \xi_{t_1,t_2}) F(u(t)) + \xi_{t_1,t_2} [F(u)]_h(t)\big]\mathrm{d}t, \end{aligned}$$

which after rearranging terms, using Lemma 2.1 and an integration by parts gives

$$\begin{aligned} 0 &\le \int_0^{t_2} \int_\Omega \xi_{t_1,t_2} \partial_t u \cdot \big([u]_h - u\big)\mathrm{d}x\mathrm{d}t + \int_0^{t_2} \xi_{t_1,t_2} \big[[F(u(t))]_h - F(u(t))\big]\mathrm{d}t \\ &= -h \int_0^{t_2} \int_\Omega \xi_{t_1,t_2} \partial_t u \cdot \partial_t [u]_h \mathrm{d}x\mathrm{d}t - h \int_0^{t_2} \xi_{t_1,t_2} \partial_t [F(u)]_h(t)\mathrm{d}t \\ &\le -h \int_0^{t_2} \int_\Omega \xi_{t_1,t_2} \partial_t u \cdot \partial_t [u]_h \mathrm{d}x\mathrm{d}t + h \int_0^{t_2} \xi'_{t_1,t_2} [F(u)]_h(t)\mathrm{d}t + hF(u_o). \end{aligned}$$

Rearranging again terms, dividing by $h > 0$ and passing to the limit $h \downarrow 0$ then shows

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} F(u(t))\mathrm{d}t \le F(u_o) - \int_0^{t_2} \int_\Omega \xi_{t_1,t_2} |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \le F(u_o).$$

This proves the claim (1.14). Finally, (1.15) follows from the last inequality, estimate (2.2) and Poincaré's inequality. This completes the proof of Theorem 1.4.

4. Uniqueness (Proof of Theorem 1.2)

In this section we are going to show that variational solutions as in Definition 1.1 are unique provided that one of the alternatives i) and ii) in Theorem 1.2 hold true. If i) is in force, then the functional F is strictly convex. This fact will be exploited in the following lemma.

Lemma 4.1. *Suppose that the assumptions of Theorem 1.2 i) are in force. Then, for any $T \in (0,\infty]$ and any initial datum u_o as in (1.7) and (1.8) there exists at most one variational solution in the sense of Definition 1.1.*

Proof. We first show that under the assumption i), the functional F is strictly convex. Indeed, this is obvious if A is convex and strictly increasing or $(u,\xi) \mapsto f(x,u,\xi)$ is strictly convex. Therefore, it remains to consider the case when K is injective and S is strictly convex in the second variable. If $u_1 \neq u_2$, then K being injective implies $K[u_1] \neq K[u_2]$, whence yielding the strict convexity of F due to the strict convexity of the lower order term $\int_\Omega S(x, K[u](x))\mathrm{d}x$. Now, let $\tau = T$ if $T < \infty$ or $\tau \in (0,\infty)$ if $T = \infty$ and assume that

$$u_1, u_2 \in L^p\left(0,\tau; W^{1,p}_{u_o}\left(\Omega,\mathbb{R}^N\right)\right) \cap C^0\left([0,\tau]; L^2\left(\Omega,\mathbb{R}^N\right)\right)$$

are two different variational solutions as in Definition 1.1 with initial datum u_o. We add both the variational inequalities (1.11) for u_1 and u_2 and note that $\|(v - u_i)(\tau)\|^2_{L^2} \geq 0$ for $i = 1,2$. Therefore, for any comparison function

$$v \in L^p(0,T; W^{1,p}_{u_o}(\Omega,\mathbb{R}^N)) \quad \text{with} \quad \partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$$

we have

$$\begin{aligned}\int_0^\tau &\big[F(u_1(t)) + F(u_2(t))\big]\mathrm{d}t \\ &\leq 2\int_0^\tau \int_\Omega \big[\partial_t v \cdot \big(v - \tfrac12(u_1+u_2)\big)\,\mathrm{d}x + F(v(t))\big]\mathrm{d}t + \|v(0) - u_o\|^2_{L^2}.\end{aligned}$$

Since Theorem 1.4 assures that $\partial_t u_i \in L^2(\Omega_T,\mathbb{R}^N)$, $i = 1,2$, we are allowed to take the comparison map $v = \frac{u_1+u_2}{2}$. Now, as $v(0) = u_o$, this implies

$$\begin{aligned}\tfrac12\int_0^\tau \big[F(u_1(t)) + F(u_2(t))\big]\mathrm{d}t &\leq \int_0^\tau F\big(\tfrac12(u_1+u_2)(t)\big)\mathrm{d}t \\ &< \tfrac12\int_0^\tau \big[F(u_1(t)) + F(u_2(t))\big]\mathrm{d}t.\end{aligned}$$

The last inequality is due to the strict convexity of F. As the last inequality obviously yields a contradiction, we must have $u_1 \equiv u_2$, i.e., the uniqueness of variational solutions. □

If ii) is in force, then the functional F is not anymore strictly convex. Nevertheless, we are able to prove a comparison principle which in turn implies the uniqueness of variational solutions.

Lemma 4.2 (Comparison principle)**.** *Suppose that the assumptions of Theorem 1.2 ii) are in force and let $u_o, \tilde u_o$ fulfill the requirements of (1.7) and (1.8) with $u_o \leq \tilde u_o$ a.e. in Ω and $u, \tilde u$ be variational solutions in the sense of Definition 1.1 on Ω_T for some $T \in (0,\infty]$ with initial and lateral boundary values u_o and $\tilde u_o$, respectively. Then, we have*

$$u \leq \tilde u \quad \text{a.e. in } \Omega_T.$$

Proof. Let $\tau \in \mathbb{R} \cap (0,T]$. Due to §2.4 we know that u and $\tilde{u}$ are also variational solutions on the smaller cylinder Ω_τ. We define $v := \min\{u,\tilde{u}\}$ and $w := \max\{u,\tilde{u}\}$ and note that $\partial_t v, \partial_t w \in L^2(\Omega_T)$, since $\partial_t u, \partial_t \tilde{u} \in L^2(\Omega_T)$ by Theorem 1.4. Therefore, v is admissible as comparison function in the variational inequality (1.11) on Ω_τ for u and w is admissible in the variational inequality for $\tilde{u}$. Adding the two resulting inequalities and using that $v(0) = u_o$ and $w(0) = \tilde{u}_o$, we obtain

$$
\begin{aligned}
\int_0^\tau \big[F(u(t)) + F(\tilde{u}(t))\big]\mathrm{d}t \le \int_0^\tau \big[F(v(t)) + F(w(t))\big]\mathrm{d}t& \\
+ \int_0^\tau \int_\Omega \big[\partial_t v(v-u) + \partial_t w(w-\tilde{u})\big]\mathrm{d}x\mathrm{d}t& \\
- \tfrac12 \|(v-u)(\tau)\|^2_{L^2(\Omega)} - \tfrac12 \|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega)}&.
\end{aligned} \tag{4.1}
$$

In the following we estimate the terms on the right-hand side of (4.1). We start with the integral involving the functional F. Since $A \equiv 0$ and $S \equiv 0$, we have that $F(u(t)) = \int_{\Omega\times\{t\}} f(x,u,Du)\mathrm{d}x$ and therefore

$$
\int_0^\tau \big[F(v(t)) + F(w(t))\big]\mathrm{d}t \le \int_0^\tau \big[F(u(t)) + F(\tilde{u}(t))\big]\mathrm{d}t.
$$

Next, we treat the integral involving the time derivatives. Here, we first observe that $v - u = -(u-\tilde{u})_+$ and $w - \tilde{u} = (u-\tilde{u})_+$, so that

$$
\begin{aligned}
\partial_t v(v-u) + \partial_t w(w-\tilde{u}) &= \partial_t(w-v)(u-\tilde{u})_+ \\
&= \partial_t(u-\tilde{u})_+(u-\tilde{u})_+ = \tfrac12 \partial_t (u-\tilde{u})_+^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
\int_0^\tau \int_\Omega \big[\partial_t v(v-u) + \partial_t w(w-\tilde{u})\big]\mathrm{d}x\mathrm{d}t &= \tfrac12 \int_0^\tau \int_\Omega \partial_t (u-\tilde{u})_+^2 \mathrm{d}x\mathrm{d}t \\
&\le \tfrac12 \int_{\Omega\times\{\tau\}} (u-\tilde{u})_+^2 \mathrm{d}x.
\end{aligned}
$$

Next, we consider the $L^2(\Omega)$-terms, i.e., the two terms of the third line of (4.1). As before, we obtain

$$
-\tfrac12 \|(v-u)(\tau)\|^2_{L^2(\Omega)} = -\tfrac12 \int_{\Omega\times\{\tau\}} (u-\tilde{u})_+^2 \mathrm{d}x = -\tfrac12 \|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega)}.
$$

Joining the preceding estimates with (4.1), we find that

$$
\int_{\Omega\times\{\tau\}} (u-\tilde{u})_+^2 \mathrm{d}x \le 0.
$$

Since $\tau \in \mathbb{R} \cap (0,T]$ was arbitrary, this proves the claim that $u \le \tilde{u}$ a.e. in Ω_T. □

5. Proof of Theorem 1.1 and Corollary 1.3

5.1. **Convex minimization problems on Ω_T.** Here, we let $T \in (0,\infty)$ and consider for $\varepsilon \in (0,1]$ convex variational integrals of the form

$$
\mathcal{F}_\varepsilon(v) := \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \left[\tfrac12 \int_\Omega |\partial_t v|^2 \mathrm{d}x + \tfrac{1}{\varepsilon} F(v(t)) \right] \mathrm{d}t.
$$

In the following, for fixed ε we are going to look for a minimizer of this functional within a suitable function space. To this end, we define

$$
\mathcal{K}_\varepsilon := \big\{ v \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N)) \colon \partial_t v \in L^2(\Omega,\mathbb{R}^N) \big\}
$$

and let

$$
\|v\|_{\mathcal{K}_\varepsilon} := \|v\|_{L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))} + \|\partial_t v\|_{L^2(\Omega_T,\mathbb{R}^N)}.
$$

As already done several times before, we waive the reference to Ω_T and $\mathbb{R}^N$ in our notation in cases, where it is clear from the context which spaces are meant. Note that

$$e^{-\frac{T}{\varepsilon}}\|v\|_{\mathcal{K}_\varepsilon} \le \left[\int_0^T e^{-\frac{t}{\varepsilon}}\|v(t)\|^p_{W^{1,p}}\mathrm{d}t\right]^{\frac{1}{p}} + \left[\int_0^T e^{-\frac{t}{\varepsilon}}\|\partial_t v\|^2_{L^2}\mathrm{d}t\right]^{\frac{1}{2}} \le \|v\|_{\mathcal{K}_\varepsilon}.$$

The subclass of those functions $v \in \mathcal{K}_\varepsilon$ satisfying $v = u_o$ on the parabolic boundary $\partial_{\mathcal{P}}\Omega_T$ of Ω_T will be denoted by $\mathcal{K}_{\varepsilon,u_o}$. The boundary conditions must be understood in the sense of traces on the lateral boundary $\partial\Omega \times (0,T)$ and in the usual L^2-sense at the initial time. Note that for such functions $v \in \mathcal{K}_{\varepsilon,u_0}$ the L^2-bound $\|\partial_t v\|_{L^2} < \infty$ already implies that $v \in C^{0,\frac{1}{2}}([0,T]; L^2(\Omega,\mathbb{R}^N))$, showing that the initial condition $v(0) = u_o$ can be defined in the usual L^2-sense. Further, the time-independent extension of u_o to Ω_T, i.e., $v(t) = u_o$ for all $t \in (0,T]$, belongs to $\mathcal{K}_{\varepsilon,u_o}$ with $\|v\|_{\mathcal{K}_\varepsilon} = T^{1/p}\|u_o\|_{W^{1,p}}$. Finally, the subclass of mappings in $\mathcal{K}_{\varepsilon,u_o}$ with finite $\mathcal{F}_\varepsilon$-energy shall be denoted by

$$\mathcal{K}^*_{\varepsilon,u_o} := \big\{v \in \mathcal{K}_{\varepsilon,u_o} : \mathcal{F}_\varepsilon(v) < \infty\big\}.$$

Note here, that the time-independent extension v of the initial value u_o has finite $\mathcal{F}_\varepsilon$-energy

$$\mathcal{F}_\varepsilon(v) = F(u_o)\int_0^T \frac{\mathrm{e}^{-\frac{t}{\varepsilon}}}{\varepsilon}\mathrm{d}t = F(u_o)\big(1 - \mathrm{e}^{-\frac{T}{\varepsilon}}\big),$$

such that $\mathcal{K}^*_{\varepsilon,u_o} \ne \emptyset$. The following lemma ensures that minimizers of $\mathcal{F}_\varepsilon$ exist in the class $\mathcal{K}^*_{\varepsilon,u_o}$.

Lemma 5.1. *For any $\varepsilon \in (0,1]$, the functional $\mathcal{F}_\varepsilon$ admits a unique minimizer $u_\varepsilon \in \mathcal{K}^*_{\varepsilon,u_o}$.*

Proof. Applying the Poincaré type inequality (2.1) on timeslices, we deduce for any $v \in \mathcal{K}^*_{\varepsilon,u_0}$ that

$$\int_0^T\int_\Omega |v|^p\mathrm{d}x\mathrm{d}t \le c_p\bigg[\int_0^T\int_\Omega |Dv|^p\mathrm{d}x\mathrm{d}t + T\|u_o\|^p_{W^{1,p}}\bigg], \tag{5.1}$$

with $c_p \ge 1$ as in (2.1). Next, by means of the lower bound (2.2) we get

$$\begin{aligned}\int_0^T\int_\Omega |Dv|^p\mathrm{d}x\mathrm{d}t &\le \frac{2\mathrm{e}^{\frac{T}{\varepsilon}}}{\mu_1+\mu_2}\int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}}\big[F(v(t)) + c_o\mathcal{E}_0\big]\mathrm{d}t\\ &\le \frac{2\mathrm{e}^{\frac{T}{\varepsilon}}}{\mu_1+\mu_2}\big[\mathcal{F}_\varepsilon(v) + c_o\mathcal{E}_0\big].\end{aligned}$$

Inserting this into (5.1) yields

$$\begin{aligned}\int_0^T\int_\Omega |v|^p\mathrm{d}x\mathrm{d}t &\le c_p\bigg[\frac{2\mathrm{e}^{\frac{T}{\varepsilon}}}{\mu_1+\mu_2}\big[\mathcal{F}_\varepsilon(v) + c_o\mathcal{E}_0\big] + T\|u_o\|^p_{W^{1,p}}\bigg]\\ &\le c_p\mathrm{e}^{\frac{T}{\varepsilon}}\bigg[\frac{2}{\mu_1+\mu_2}\big[\mathcal{F}_\varepsilon(v) + c_o\mathcal{E}_0\big] + \|u_o\|^p_{W^{1,p}}\bigg]\\ &\le \frac{2c_p\mathrm{e}^{\frac{T}{\varepsilon}}}{\mu_1+\mu_2}\big[\mathcal{F}_\varepsilon(v) + 2c_o\mathcal{E}_0\big]\end{aligned}$$

Combining the last two inequalities leads to the following bound for the L^p–$W^{1,p}$-norm of maps in $\mathcal{K}^*_{\varepsilon,u_0}$:

$$\int_0^T\int_\Omega |v|^p + |Dv|^p\mathrm{d}x\mathrm{d}t \le \frac{4c_p\mathrm{e}^{\frac{T}{\varepsilon}}}{\mu_1+\mu_2}\big[\mathcal{F}_\varepsilon(v) + 2c_o\mathcal{E}_o\big],$$

which combined with the L^2-bound for the time derivative

$$\int_0^T \int_\Omega |\partial_t v|^2 \mathrm{d}x\mathrm{d}t \le 2\mathrm{e}^{\frac{T}{\varepsilon}} \big[\mathcal{F}_\varepsilon(v) + c_o \mathcal{E}_o\big],$$

yields the following bound for the $\|\cdot\|_{\mathcal{K}_\varepsilon}$-norm on the class $\mathcal{K}^*_{\varepsilon,u_o}$:

$$\|v\|_{\mathcal{K}_\varepsilon} \le 6c_p c_1 \mathrm{e}^{\frac{T}{\varepsilon}} \Big[\mathcal{F}_\varepsilon(v) + 2c_o\mathcal{E}_o + 1\Big].$$

Here the constant $c_1 \ge 1$ is given by

$$c_1 := \max\Big\{1, \frac{1}{\mu_1 + \mu_2}\Big\}.$$

Consider now a minimizing sequence $u_j \in \mathcal{K}^*_{\varepsilon,u_o}$, $j \in \mathbb{N}$, i.e.,

$$\lim_{j\to\infty} \mathcal{F}_\varepsilon(u_j) = \inf_{v \in \mathcal{K}^*_{\varepsilon,u_o}} \mathcal{F}_\varepsilon(v) \le \mathcal{F}_\varepsilon(u_o) = F(u_o)\big(1 - \mathrm{e}^{-\frac{T}{\varepsilon}}\big).$$

Without loss of generality, we can assume that $\mathcal{F}_\varepsilon(u_j) \le F(u_o)$. The preceding estimate then shows

$$\|u_j\|_{\mathcal{K}_\varepsilon} \le 6c_p c_1 \mathrm{e}^{\frac{T}{\varepsilon}} \Big[F(u_o) + 2c_o\mathcal{E}_o + 1\Big] \qquad \forall\, j \in \mathbb{N},$$

i.e., the minimizing sequence is uniformly bounded with respect to $\|\cdot\|_{\mathcal{K}_\varepsilon}$. Thus, there exist a map $u \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t u \in L^2(\Omega_T,\mathbb{R}^N)$ and a subsequence of $(u_j)_{j\in\mathbb{N}}$ (still denoted this way) such that

$$\begin{cases} u_j \rightharpoonup u & \text{weakly in } L^p(\Omega_T,\mathbb{R}^N) \text{ and in } L^2(\Omega_T,\mathbb{R}^N), \\ Du_j \rightharpoonup Du & \text{weakly in } L^p(\Omega_T,\mathbb{R}^{Nn}), \\ \partial_t u_j \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T,\mathbb{R}^N). \end{cases}$$

The limit map u fulfills on the lateral boundary $\partial_{\mathcal{P}}\Omega_T$ the condition that $u(t) = u_o$ for almost every $t \in (0,T)$ in the sense of traces in $W^{1,p}(\Omega,\mathbb{R}^N)$. Therefore, it remains to show that the initial condition holds true in the usual L^2 sense. To this end, we observe that

$$\|u_j(t_2) - u_j(t_1)\|^2_{L^2(\Omega)} \le (t_2 - t_1)\|\partial_t u_j\|^2_{L^2(\Omega_T)} \le 2\mathrm{e}^{\frac{T}{\varepsilon}}\big[F(u_o) + c_o\mathcal{E}_o\big](t_2 - t_1)$$

holds true for any $0 \le t_1 < t_2 \le T$. Recalling that $u_j(0) = u_o$, the last estimate with $t_1 = 0$ and the weak convergence $u_j \rightharpoonup u$ in $L^2(\Omega_T,\mathbb{R}^N)$ imply

$$\begin{aligned} \tfrac{1}{h}\int_0^h \|u(t) - u_o\|^2_{L^2(\Omega)}\,\mathrm{d}t &\le \liminf_{j\to\infty} \tfrac{1}{h}\int_0^h \|u_j(t) - u_o\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\le \mathrm{e}^{\frac{T}{\varepsilon}}\big[F(u_o) + c_o\mathcal{E}_o\big]h. \end{aligned}$$

This, however, yields $\lim_{h\downarrow 0} \frac{1}{h}\int_0^h \|u(t) - u_o\|^2_{L^2(\Omega)}\,\mathrm{d}t = 0$, so that $u(0) = u_o$ in the usual L^2-sense. In particular this implies that $u \in \mathcal{K}_{\varepsilon,u_o}$, and it remains to prove that u is indeed the unique minimizer of $\mathcal{F}_\varepsilon$ in the class $\in \mathcal{K}^*_{\varepsilon,u_o}$. This, however, follows by means of lower semicontinuity arguments for the functional $\mathcal{F}_\varepsilon$ with respect to the convergences above, i.e., by establishing

$$\mathcal{F}_\varepsilon(u) \le \liminf_{j\to\infty} \mathcal{F}_\varepsilon(u_j) = \lim_{j\to\infty} \mathcal{F}_\varepsilon(u_j) = \inf_{v\in\mathcal{K}^*_{\varepsilon,u_o}} \mathcal{F}_\varepsilon(v).$$

To this end, note that the functional $\mathcal{F}_\varepsilon$ is lower semicontinuous with respect to strong convergence in the space

$$X := \big\{v \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N)) \colon \partial_t v \in L^2(\Omega_T,\mathbb{R}^N)\big\}$$

by means of the following arguments, similar to those of [13, Thm. 4.2] for the integrand f. From a strongly converging sequence $(v_j)_{j\in\mathbb{N}}$ in the Banach space X

extract a subsequence $\left(v_j^*\right)_{j\in\mathbb{N}}$ such that on almost every time slice $t \in (0,T)$ there holds

$$v_j^*(t) \to v(t) \text{ in } W^{1,p}(\Omega,\mathbb{R}^N) \quad \text{and} \quad \partial_t v_j^*(t) \to \partial_t v(t) \text{ in } L^2(\Omega,\mathbb{R}^N).$$

Moreover, we can choose the subsequence such that also $Dv_j^*(x,t) \to Dv(x,t)$ and $\partial_t v_j^*(x,t) \to \partial_t v(x,t)$ for almost every $(x,t) \in \Omega_T$ and such that

$$\lim_{j\to\infty} \mathcal{F}_\varepsilon\left(v_j^*\right) = \liminf_{j\to\infty} \mathcal{F}_\varepsilon\left(v_j\right).$$

Using Fatou's lemma, the monotonicity and continuity of A it first follows for the part of $\mathcal{F}_\varepsilon$ involving A that

$$\begin{aligned}\int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} A\big(\|Dv(t)\|^p_{L^p(\Omega)}\big)\mathrm{d}t &= \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} A\big(\lim_{j\to\infty} \|Dv_j^*(t)\|^p_{L^p(\Omega)}\big)\mathrm{d}t \\ &= \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \lim_{j\to\infty} A\big(\|Dv_j^*(t)\|^p_{L^p(\Omega)}\big)\mathrm{d}t \\ &\le \liminf_{j\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} A\big(\|Dv_j^*(t)\|^p_{L^p(\Omega)}\big)\mathrm{d}t.\end{aligned}$$

A similar argument shows the lower semicontinuity of the part involving the time derivative, i.e.,

$$\int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\Omega |\partial_t v|^2 \mathrm{d}x\mathrm{d}t \le \liminf_{j\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\Omega |\partial_t v_j^*|^2 \mathrm{d}x\mathrm{d}t.$$

By the help of the continuity of $K\colon L^1(\Omega,\mathbb{R}^N) \to L^1(\Omega,\mathbb{R}^N)$, the continuity of $u \mapsto S(x,u)$ for almost every $x \in \Omega$ and Fatou's Lemma the same applies to the part involving S, i.e.,

$$\int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\Omega S\big(x, K[v](x,t)\big)\mathrm{d}x\mathrm{d}t \le \liminf_{j\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\Omega S\big(x, K[v_j^*](x,t)\big)\mathrm{d}x\mathrm{d}t.$$

As in [13, Thm. 4.3], the lower semicontinuity of $\mathcal{F}_\varepsilon$ with respect to the weak topology of X now follows from the lower semicontinuity of $\mathcal{F}_\varepsilon$ with respect to the strong topology of X and the convexity of the functional $\mathcal{F}_\varepsilon$ by means of [11, Thm. V.13]. This proves that u is a minimizer of $\mathcal{F}_\varepsilon$ in the class $\mathcal{K}^*_{\varepsilon,u_o}$.

The uniqueness of the minimizer u follows, because the term involving the time derivative ensures the strict convexity of the functional $\mathcal{F}_\varepsilon$. □

At this stage we want to remark that the minimizer u_ε is not only an element of L^p–$W^{1,p}$ but also a $\frac{1}{2}$-Hölder-continuous map from $[0,T]$ to L^2 due to the fact that $\partial_t u_\varepsilon \in L^2(\Omega_T,\mathbb{R}^N)$ and $u_o \in L^2(\Omega,\mathbb{R}^N)$. This follows as in the proof of Lemma 5.1 by means of Fubini's theorem and Cauchy-Schwartz inequality, i.e., for $0 \le t_1 < t_2 \le T$ we have

$$\|u_\varepsilon(t_2) - u_\varepsilon(t_1)\|^2_{L^2(\Omega)} \le (t_2 - t_1)\|\partial_t u_\varepsilon\|^2_{L^2(\Omega_T)}, \tag{5.2}$$

as well as for any $t \in [0,T]$ by setting $t_1 = 0$ and $t_2 = t$ above

$$\|u_\varepsilon(t)\|^2_{L^2(\Omega)} \le 2\|u_o\|^2_{L^2(\Omega)} + 2t\|\partial_t u_\varepsilon\|^2_{L^2(\Omega_T)} < \infty, \tag{5.3}$$

thus yielding $u_\varepsilon \in C^{0,\frac{1}{2}}([0,T];L^2(\Omega,\mathbb{R}^N))$.

5.2. **The minimality condition revisited.** For fixed $\varepsilon \in (0,1]$ consider testing functions $\varphi \in L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N))$ with $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$ and

$$\int_0^T F\left(\left(u_\varepsilon + \varphi\right)(t)\right)\mathrm{d}t < \infty. \tag{5.4}$$

Moreover, let $\xi \in W^{1,\infty}((0,T))$ with $0 \le \xi \le 1$, and $\delta \in (0, \mathrm{e}^{-\frac{T}{\varepsilon}}]$. Further, define $\sigma(t) := \delta \mathrm{e}^{\frac{t}{\varepsilon}} \xi(t)$ as well as

$$\tilde{\varphi}_{\varepsilon,\delta}(x,t) := \sigma(t)\varphi(x,t) \equiv \delta \mathrm{e}^{\frac{t}{\varepsilon}} \xi(t)\varphi(x,t), \quad (x,t) \in \Omega_T,$$

while assuming either $\xi(0) = 0$ or $\varphi(0) = 0$. Then, set

$$v_{\varepsilon,\delta}(x,t) := u_\varepsilon(x,t) + \tilde{\varphi}_{\varepsilon,\delta}(x,t) \equiv u_\varepsilon(x,t) + \delta \mathrm{e}^{\frac{t}{\varepsilon}} \xi(t)\varphi(x,t)$$

and observe that $v_{\varepsilon,\delta} \in L^p(0,T; W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ with $\partial_t v_{\varepsilon,\delta} \in L^2(\Omega_T,\mathbb{R}^N)$. To conclude that $v_{\varepsilon,\delta} \in \mathcal{K}^*_{\varepsilon,u_o}$ is an admissible comparison map in the minimality condition for u_ε, it remains to show that $\mathcal{F}_\varepsilon(v_{\varepsilon,\delta}) < \infty$. This follows from the convexity of the functional F, the fact that $v_{\varepsilon,\delta}$ is a convex combination of u_ε and $u_\varepsilon + \varphi$ on every fixed time slice $t \in [0,T]$ and the assumption (5.4). Note in this direction, that $0 \le \sigma(t) \le 1$ by the choice of δ above, such that the convexity of F implies the following estimate for the contribution of the F-part to the $\mathcal{F}_\varepsilon$-energy:

$$\begin{aligned}\int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} F(v_{\varepsilon,\delta})\mathrm{d}t &\le \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \big[(1-\sigma(t))F\big(u_\varepsilon(t)\big) + \sigma(t)F\big((u_\varepsilon+\varphi)(t)\big)\big]\mathrm{d}t \\ &\le \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} F\big(u_\varepsilon(t)\big)\mathrm{d}t + \int_0^T F\big((u_\varepsilon+\varphi)(t)\big)\mathrm{d}t < \infty.\end{aligned}$$

The minimality of u_ε thus shows that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \mathcal{F}_\varepsilon(v_{\varepsilon,\delta}) < \infty,$$

which by the convexity of F as before can be rewritten to

$$\begin{aligned}\tfrac{\delta}{\varepsilon}\int_0^T \xi(t)F\big(u_\varepsilon(t)\big)\mathrm{d}t &\le \int_0^T \mathrm{e}^{-\frac{t}{\varepsilon}} \int_\Omega \Big[\tfrac12 \delta^2 \big|\partial_t\big(\mathrm{e}^{\frac{t}{\varepsilon}}\xi\varphi\big)\big|^2 + \delta\partial_t u_\varepsilon \cdot \partial_t\big(\mathrm{e}^{\frac{t}{\varepsilon}}\xi\varphi\big)\Big]\mathrm{d}x\mathrm{d}t \\ &\quad + \tfrac{\delta}{\varepsilon}\int_0^T \xi(t)F\big((u_\varepsilon+\varphi)(t)\big)\mathrm{d}t.\end{aligned}$$

Multiplying both sides of the previous inequality by ε/δ and letting $\delta \downarrow 0$, yields

$$\begin{aligned}\int_0^T \xi(t)F\big(u_\varepsilon(t)\big)\mathrm{d}t &\le \int_0^T \xi(t)F\big((u_\varepsilon+\varphi)(t)\big)\mathrm{d}t + \int_0^T\int_\Omega \xi\partial_t u_\varepsilon\cdot\varphi\,\mathrm{d}x\mathrm{d}t \\ &\quad + \varepsilon\int_0^T\int_\Omega \big[\xi'\partial_t u_\varepsilon\cdot\varphi + \xi\partial_t u_\varepsilon\cdot\partial_t\varphi\big]\mathrm{d}x\mathrm{d}t\end{aligned} \tag{5.5}$$

for any $\xi \in W^{1,\infty}((0,T))$ with $0 \le \xi \le 1$, $\varphi \in L^p(0,T; W^{1,p}_0(\Omega,\mathbb{R}^N))$ with $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$ such that (5.4) holds true and such that either $\xi(0) = 0$ or $\varphi(0) = 0$.

5.3. **Energy estimates.** Here we are going to establish certain energy estimates for $\mathcal{F}_\varepsilon$-minimizers $u_\varepsilon \in \mathcal{K}^*_{\varepsilon,u_o}$ which will allow for an extraction of a converging subsequence in the limit $\varepsilon \downarrow 0$. To this end, define $[u_\varepsilon]_h$ according to (2.4) with u_o and u_ε instead of v_o and v, respectively. Note, that $[u_\varepsilon]_h \in L^p(0,T; W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ with $\partial_t [u_\varepsilon]_h \in L^2(\Omega_T,\mathbb{R}^N)$ and $\partial_t [u_\varepsilon]_h = \frac{1}{h}(u_\varepsilon - [u_\varepsilon]_h)$ by means of Lemma 2.1. The last identity implies even more, as the right-hand side is an element of the space $L^p(0,T; W^{1,p}_0(\Omega,\mathbb{R}^N))$ whose time derivative is an element of $L^2\left(\Omega_T,\mathbb{R}^N\right)$. Moreover, since $[u_\varepsilon](0) = u_o$ it follows that $\partial_t [u_\varepsilon](0) = \frac{1}{h}(u_o - [u_\varepsilon]_h(0)) = 0$ and from Corollary 2.5 we conclude that

$$\int_0^T F\big((u_\varepsilon - h\partial_t[u_\varepsilon]_h)(t)\big)\mathrm{d}t = \int_0^T F\big([u_\varepsilon]_h(t)\big)\mathrm{d}t < \infty.$$

In other words, we are allowed to take $\varphi = -h\partial_t\left[u_\varepsilon\right]_h$ as testing function in (5.5), which for any $\xi \in W^{1,\infty}((0,T))$ with $0 \le \xi \le 1$ implies

$$\begin{aligned}
h\int_0^T\int_\Omega &\big[(\xi+\varepsilon\xi')\partial_t u_\varepsilon \cdot \partial_t[u_\varepsilon]_h + \varepsilon\xi\partial_t u_\varepsilon \cdot \partial_{tt}[u_\varepsilon]_h\big]\mathrm{d}x\mathrm{d}t \\
&\le \int_0^T \xi(t)\big[F([u_\varepsilon]_h(t)) - F(u_\varepsilon(t))\big]\mathrm{d}t \\
&\le \int_0^T \xi(t)\big[[F(u_\varepsilon)]_h(t) - F(u_\varepsilon(t))\big]\mathrm{d}t \\
&= -h\int_0^T \xi(t)\partial_t\big[F(u_\varepsilon)\big]_h(t)\mathrm{d}t,
\end{aligned}$$

by means of Corollary 2.5 where $\left[F\left(u_\varepsilon\right)\right]_h$ is defined according to (1.11) with v_o and v replaced by $F(u_o)$ and $F(u_\varepsilon)$, respectively. The second term on the left-hand side of the previous inequality can be estimated further as follows

$$\begin{aligned}
\partial_t u_\varepsilon \cdot \partial_{tt}\left[u_\varepsilon\right]_h &= \partial_t[u_\varepsilon]_h \cdot \partial_{tt}\left[u_\varepsilon\right]_h + \big(\partial_t u_\varepsilon - \partial_t[u_\varepsilon]_h\big)\cdot\partial_{tt}[u_\varepsilon]_h \\
&= \tfrac12\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2 + \tfrac1h\big|\partial_t[u_\varepsilon]_h - \partial_t u_\varepsilon\big|^2 \\
&\ge \tfrac12\partial_t\big|\partial_t\left[u_\varepsilon\right]_h\big|^2.
\end{aligned}$$

Inserting this estimate in the inequality above and dividing by $h > 0$, we get

$$\begin{aligned}
&\int_0^T\int_\Omega\Big[(\xi+\varepsilon\xi')\partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h + \tfrac{\varepsilon}{2}\xi\partial_t\big|\partial_t[u_\varepsilon]_h\big|^2\Big]\mathrm{d}x\mathrm{d}t \\
&\qquad\le -\int_0^T \xi(t)\partial_t\big[F(u_\varepsilon)\big]_h(t)\mathrm{d}t.
\end{aligned}\tag{5.6}$$

We are now first going to choose $\xi \equiv 1$ in (5.6) to obtain by means of Fubini's theorem

$$\begin{aligned}
\int_0^T&\int_\Omega \partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h\mathrm{d}x\mathrm{d}t \\
&\le -\int_0^T \partial_t\big[F(u_\varepsilon(t))\big]_h\mathrm{d}t - \tfrac{\varepsilon}{2}\int_0^T\int_\Omega \partial_t\big|\partial_t[u_\varepsilon]_h\big|^2\mathrm{d}x\mathrm{d}t \\
&= \big[F(u_\varepsilon)\big]_h(0) - \big[F(u_\varepsilon)\big]_h(T) + \tfrac{\varepsilon}{2}\int_\Omega\Big(\big|\partial_t[u_\varepsilon]_h\big|^2(0) - \big|\partial_t[u_\varepsilon]_h\big|^2(T)\Big)\mathrm{d}x \\
&\le F(u_o) + c_o\mathcal{E}_o.
\end{aligned}$$

In the last inequality we also used the facts $[F(u_\varepsilon)]_h(0) = F\left(u_o\right)$,$\partial_t[u_\varepsilon]_h(0) = 0$, $|\partial_t[u_\varepsilon]_h|^2(T) \ge 0$ and the lower bound on F from (2.2). Taking into account that $\partial_t[u_\varepsilon]_h \to \partial_t u_\varepsilon$ in $L^2(\Omega_T,\mathbb{R}^N)$ due to Lemma 2.1, the last inequality therefore implies the following uniform bound on the time derivative

$$\int_0^T\int_\Omega |\partial_t u_\varepsilon|^2\mathrm{d}x\mathrm{d}t \le F(u_o) + c_o\mathcal{E}_o. \tag{5.7}$$

Moreover, due to (5.3) and (5.2) this implies the following uniform L^2-bound

$$\sup_{t\in[0,T]}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} \le 2\|u_o\|^2_{L^2(\Omega)} + 2T\big[F(u_o) + c_o\mathcal{E}_o\big] < \infty, \tag{5.8}$$

as well as

$$\|u_\varepsilon(t_2) - u_\varepsilon(t_1)\|_{L^2(\Omega)} \le \sqrt{F(u_o) + c_o\mathcal{E}_o}\sqrt{|t_2 - t_1|} \tag{5.9}$$

for any $t_1, t_2 \in [0,T]$. In other words, the estimates (5.8) and (5.9) imply that the family $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of minimizers is uniformly bounded in the spaces $L^2(\Omega_T,\mathbb{R}^N)$ and $C^{0,\frac12}([0,T];L^2(\Omega,\mathbb{R}^N))$.

In the following, by using again (5.6) we are going to show that the family of minimizers $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is also is uniformly bounded in $L^p{-}W^{1,p}$ with respect to $\varepsilon\in(0,1]$. To this end, for $0\le t_1<t_2\le T$ we define

$$\xi_{t_1,t_2}(t) := \chi_{[0,t_1]}(t) + \frac{t_2-t}{t_2-t_1}\chi_{(t_1,t_2)}(t)$$

and take $\xi=\xi_{t_1,t_2}$ in (5.6). Integrating by parts in the second term of the left-hand side and in the right-hand side of (5.6) then shows

$$\begin{aligned}\int_0^T\int_\Omega \xi_{t_1,t_2}(t)\partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h \mathrm{d}x\mathrm{d}t &\le F(u_o)\\ &\quad+\int_0^T \xi'_{t_1,t_2}(t)\Big[[F(u_\varepsilon)]_h(t)+\int_\Omega\Big[\tfrac{\varepsilon}{2}\big|\partial_t[u_\varepsilon]_h\big|^2-\varepsilon\partial_t u_\varepsilon\cdot\partial_t\,[u_\varepsilon]_h\Big]\mathrm{d}x\Big]\mathrm{d}t,\end{aligned}$$

such that in the limit $h\downarrow 0$ we obtain

$$\int_0^T\int_\Omega \xi_{t_1,t_2}(t)\big|\partial_t u_\varepsilon\big|^2\mathrm{d}x\mathrm{d}t \le F(u_o)+\int_0^T \xi'_{t_1,t_2}(t)\Big[F(u_\varepsilon(t))-\tfrac{\varepsilon}{2}\int_\Omega\big|\partial_t u_\varepsilon\big|^2\mathrm{d}x\Big]\mathrm{d}t.$$

Therefore, since $|\partial_t u_\varepsilon|^2\ge 0$, it follows that

$$\begin{aligned}\int_{t_1}^{t_2} F\big(u_\varepsilon(t)\big)\mathrm{d}t &\le (t_2-t_1)F(u_o)+\tfrac{\varepsilon}{2}\int_{t_1}^{t_2}\int_\Omega\big|\partial_t u_\varepsilon\big|^2\mathrm{d}x\mathrm{d}t\\ &\le \big(t_2-t_1+\tfrac{\varepsilon}{2}\big)\big[F(u_o)+c_o\mathcal{E}_o\big]\end{aligned}\tag{5.10}$$

holds true for any $0\le t_1<t_2\le T$, where in the last inequality we used the uniform bound (5.7). The lower bound on F from (2.2) then implies

$$\int_0^T\int_\Omega |Du_\varepsilon|^p\mathrm{d}x\mathrm{d}t \le \frac{2(T+1+\frac{\varepsilon}{2})}{\mu_1+\mu_2}\big[F(u_o)+c_o\mathcal{E}_o\big].$$

In combination, the Poincaré type inequality (5.1) and the preceding inequality lead to an uniform bound, i.e., independent of $\varepsilon\in(0,1]$ and the $L^p{-}W^{1,p}$-norm of u_ε, that is

$$\int_0^T\int_\Omega |u_\varepsilon|^p+|Du_\varepsilon|^p\mathrm{d}x\mathrm{d}t < \frac{(c_p+1)(2T+2+\varepsilon)}{\mu_1+\mu_2}\big[F(u_o)+2c_o\mathcal{E}_o\big].\tag{5.11}$$

5.4. **The limit procedure.** We are now going to pass to the limit $\varepsilon\downarrow 0$ in the sequence of $\mathcal{F}_\varepsilon$-minimizers $(u_\varepsilon)_{\varepsilon>0}$ on Ω_T. By means of the estimates (5.7), (5.8), (5.9) and (5.11) the sequence $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in the spaces $L^2(\Omega_T,\mathbb{R}^N)$, $C^{0,\frac12}([0,T];L^2(\Omega,\mathbb{R}^N))$ and $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$, respectively, and the sequence of the corresponding time derivatives $(\partial_t u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in the space $L^2(\Omega_T,\mathbb{R}^N)$. Therefore, there exists a map

$$u\in L^2(\Omega_T,\mathbb{R}^N)\cap C^{0,\frac12}\big([0,T];L^2(\Omega,\mathbb{R}^N)\big)\cap L^p\big(0,T;W^{1,p}(\Omega,\mathbb{R}^N)\big)$$

with $\partial_t u\in L^2(\Omega_T,\mathbb{R}^N)$ and a subsequence of u_ε (still denoted this way) such that

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } L^2(\Omega_T,\mathbb{R}^N) \text{ and } L^p(\Omega_T,\mathbb{R}^N),\\ Du_\varepsilon \rightharpoonup Du & \text{weakly in } L^p(\Omega_T,\mathbb{R}^{Nn}),\\ \partial_t u_\varepsilon \rightharpoonup \partial_t u & \text{weakly in } L^2(\Omega_T,\mathbb{R}^N).\end{cases}\tag{5.12}$$

By lower semicontinuity with respect to weak L^2-convergence and the uniform bound (5.7) it holds that

$$\int_0^T \int_\Omega |\partial_t u|^2 \mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\downarrow 0} \int_0^T \int_\Omega |\partial_t u_\varepsilon|^2 \mathrm{d}x\mathrm{d}t \le F(u_o) + c_o\mathcal{E}_o. \tag{5.13}$$

Moreover, the lower semicontinuity of the functional F (see the argument in the proof of Lemma 5.1) and (5.10) as well as the lower semicontinuity of the L^p-norm with respect to weak convergence in L^p and the uniform bound (5.11) imply

$$\int_{t_1}^{t_2} F(u(t))\mathrm{d}t \le \liminf_{\varepsilon\downarrow 0} \int_{t_1}^{t_2} F(u_\varepsilon(t))\mathrm{d}t \le (t_2 - t_1)\big[F(u_o) + c_o\mathcal{E}_o\big] \tag{5.14}$$

and

$$\|u\|^p_{L^p(0,T;W^{1,p}(\Omega))} \le \liminf_{\varepsilon\downarrow 0} \|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}(\Omega))} \le c(T+1)\big[F(u_o) + c_o\mathcal{E}_o\big] \tag{5.15}$$

for a constant c depending only on n, p, $\mu_1+\mu_2$ and $\operatorname{diam}\Omega$. Further, from the uniform Hölder estimate (5.9) and $u_\varepsilon(0) = u_o$ we conclude as in Lemma 5.1 that also $u(0) = u_o$ holds in the usual L^2-sense. Since also $u(t) = u_o$ on $\partial\Omega$ in the sense of traces for almost every $t \in (0,T)$, for u to be a variational solution as in Definition 1.1 it just remains to show, that u satisfies the minimality condition (1.11). To this end, consider without loss of generality $v \in L^p(0,T;W^{1,p}_{u_o}(\Omega,\mathbb{R}^N))$ with $\partial_t v \in L^2(\Omega_T,\mathbb{R}^N)$ and

$$\int_0^T F(v(t))\,\mathrm{d}t < \infty,$$

since otherwise (1.11) trivially holds due to (5.14) for $t_1 = 0$ and $t_2 = T$. For fixed $\theta \in (0, T/2)$ let

$$\xi_\theta(t) := \frac{t}{\theta}\chi_{[0,\theta]}(t) + \chi_{(\theta,T-\theta)}(t) + \frac{T-t}{\theta}\chi_{[T-\theta,T]}(t)$$

denote a cut-off function with respect to time. Now, fix $\varepsilon \in (0,1]$ and consider $\varphi = v - u_\varepsilon$. The properties of v and u_ε imply that $\varphi \in L^p(0,T;W^{1,p}_0(\Omega,\mathbb{R}^N))$ with $\partial_t\varphi \in L^2(\Omega_T,\mathbb{R}^N)$ and since also $\xi_\theta(0) = 0$ we are allowed to take both φ and $\xi = \xi_\theta$ in the inequality (5.5). Adding $\int_0^T F(u_\varepsilon(t))\mathrm{d}t < \infty$ on both sides shows

$$\begin{aligned}
&\int_0^T F(u_\varepsilon(t))\mathrm{d}t \\
&\quad\le \int_0^T (1-\xi_\theta(t))F(u_\varepsilon(t))\mathrm{d}t + \int_0^T\int_\Omega \xi_\theta\partial_t u_\varepsilon\cdot(v-u_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \int_0^T \xi_\theta(t)F(v(t))\mathrm{d}t + \varepsilon\int_0^T\int_\Omega \big[\xi_\theta'\partial_t u_\varepsilon\cdot(v-u_\varepsilon) + \xi_\theta\partial_t u_\varepsilon\cdot\partial_t(v-u_\varepsilon)\big]\mathrm{d}x\mathrm{d}t \\
&\quad =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{IV}_\varepsilon,
\end{aligned}$$

where the meaning of the terms $\mathrm{I}_\varepsilon - \mathrm{IV}_\varepsilon$ is obvious in this context. Note that III_ε is actually independent of ε. In the following we are first going to pass to the limit $\varepsilon\downarrow 0$. Therefore, if $\theta \ge \varepsilon$, using the estimate (5.10) it follows for the term I_ε that

$$\mathrm{I}_\varepsilon \le \int_{[0,\theta)\cup(T-\theta,T]} F(u_\varepsilon(t))\mathrm{d}t \le (2\theta+\varepsilon)\big[F(u_o)+c_o\mathcal{E}_o\big] \le 3\theta\big[F(u_o)+c_o\mathcal{E}_o\big].$$

The term II_ε can be rewritten as

$$\begin{aligned}
\mathrm{II}_\varepsilon &= \int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u_\varepsilon)\mathrm{d}x\mathrm{d}t - \tfrac12\int_0^T\int_\Omega \xi_\theta\partial_t\,|v-u_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u_\varepsilon)\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$+\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_\varepsilon|^2\mathrm{d}x\mathrm{d}t-\tfrac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u_\varepsilon|^2\mathrm{d}x\mathrm{d}t,$$

where in the last equality we performed an integration by parts. For the second term on the right-hand side Minkowski's inequality and (5.9) allow us to estimate

$$\begin{aligned}
&\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_\varepsilon|^2\mathrm{d}x\mathrm{d}t\\
&\quad\le\bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_o|^2\mathrm{d}x\mathrm{d}t\bigg)^{\frac12}+\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |u_\varepsilon-u_o|^2\mathrm{d}x\mathrm{d}t\bigg)^{\frac12}\bigg]^2\\
&\quad\le\bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_o|^2\mathrm{d}x\mathrm{d}t\bigg)^{\frac12}+\big(\tfrac{\theta}{4}\big[F(u_o)+c_o\mathcal{E}_o\big]\big)^{\frac12}\bigg]^2.
\end{aligned}$$

Therefore, by means of the weak convergence in L^2 we can pass to limit $\varepsilon\downarrow 0$ in II_ε, yielding

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0}\mathrm{II}_\varepsilon\le&\int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u)\mathrm{d}x\mathrm{d}t-\tfrac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u|^2\,\mathrm{d}x\mathrm{d}t\\
&+\bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_o|^2\mathrm{d}x\mathrm{d}t\bigg)^{\frac12}+\big(\tfrac{\theta}{4}\big[F(u_o)+c_o\mathcal{E}_o\big]\big)^{\frac12}\bigg]^2.
\end{aligned}$$

Finally, note that the integral making up the term IV_ε is uniformly bounded with respect to ε due to the uniform L^2-bounds on $\partial_t u_\varepsilon$ and u_ε in (5.7) and (5.8), respectively. Therefore, $\mathrm{IV}_\varepsilon\to 0$ as $\varepsilon\downarrow 0$. The previous estimates and the lower semicontinuity of the functional $\int_0^T F(v(t))\mathrm{d}t$ with respect to the weak convergences in (5.12) thus show

$$\begin{aligned}
\int_0^T F(u(t))\mathrm{d}t\le&\liminf_{\varepsilon\downarrow 0}\int_0^T F(u_\varepsilon(t))\mathrm{d}t\\
\le&\int_0^T\xi_\theta(t)\bigg[\int_\Omega\partial_t v\cdot(v-u)\mathrm{d}x+F(v(t))\bigg]\mathrm{d}t+3\theta\big[F(u_o)+c_o\mathcal{E}_o\big]\\
&+\bigg[\bigg(\tfrac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_o|^2\mathrm{d}x\mathrm{d}t\bigg)^{\frac12}+\big(\tfrac{\theta}{4}\big[F(u_o)+c_o\mathcal{E}_o\big]\big)^{\frac12}\bigg]^2\\
&-\tfrac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $\theta\in(0,T/2)$ is arbitrary. We may therefore pass to the limit $\theta\downarrow 0$, such that

$$\begin{aligned}
\int_0^T F(u(t))\mathrm{d}t\le&\int_0^T\bigg[\int_\Omega\partial_t v\cdot(v-u)\mathrm{d}x+F(v(t))\bigg]\mathrm{d}t\\
&+\tfrac12\|v(0)-u_o\|^2_{L^2(\Omega)}-\tfrac12\|(v-u)(T)\|^2_{L^2(\Omega)},
\end{aligned}$$

i.e., u is indeed a variational solution on Ω_T as in Definition 1.1. Since $T>0$ was arbitrary, this finishes the proof of Theorem 1.1.

Remark 2. If $p>\frac{2n}{n+2}$ the proof can be simplified due to the fact that in (5.12) we could conclude strong convergence $u_\varepsilon\to u$ as $\varepsilon\downarrow 0$ strongly in $L^2(\Omega_T,\mathbb{R}^N)$. This follows by an application of [18, Theorem 1].

The proof of Corollary 1.3 now follows from Theorem 1.1 on the existence of variational solutions on Ω_T, from Theorem 1.2 on the uniqueness of variational solutions on Ω_T and from the localization principle in §2.4. Thus, given $0<T_1<T_2<\infty$ and denoting by u_1 and u_2 the unique variational solutions on Ω_{T_1} and Ω_{T_2}, respectively, by means of the localizing principle in §2.4 it follows that u_2 is also a variational solution on Ω_{T_1} which then has to coincide with u_1 on Ω_{T_1}. Therefore, this allows for a construction of a unique global variational solution.

6. Regularity of variational solutions

In this chapter we are interested in the regularity properties of solutions to nonlocal parabolic problems.

6.1. $C^{1,\alpha}$**-regularity.** As a first model case we consider functionals of the type

$$F(v) := A\bigg(\int_\Omega |Dv|^p \mathrm{d}x\bigg) - \int_\Omega hv\mathrm{d}x, \text{ where } \quad A(s) := \int_0^s a(\sigma)\mathrm{d}\sigma,$$

where $p > \frac{2n}{n+2}$, $a\colon [0,\infty) \to [\mu, L]$ for some $0 < \mu < L < \infty$ is continuous and increasing and $h \in L^q(\Omega, \mathbb{R}^N)$ for some $q \geq \frac{p}{p-1}$. Moreover, we consider an initial datum u_o as in (1.7). Due to Theorems 1.1 – 1.4 there exists a unique variational solution $u \in L^p(0,T; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)) \cap C^0([0,T]; L^2(\Omega, \mathbb{R}^N))$ with $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ in the sense of Definition 1.1. Since the function A defined above is of class C^1, we deduce that the variational solution u satisfies the following Cauchy-Dirichlet problem

$$\left\{\begin{array}{ll} u_t - \operatorname{div}\big[\tilde{a}(t)\,|Du|^{p-2}\,Du\big] = h & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{array}\right. \tag{6.1}$$

where we have abbreviated

$$\tilde{a}(t) := a\big(\|Du(t)\|^p_{L^p(\Omega)}\big). \tag{6.2}$$

We note that $\tilde{a}$ is a measurable function on $(0,T)$ with values in $[\mu, L]$. Therefore, $(6.1)_1$ is a parabolic system of p-Laplacian type with coefficients $\tilde{a}\colon (0,T) \to [\mu, L]$ independent of x. From [10, Chapters IX.1 and IX.14] we infer that Du is locally Hölder continuous in Ω_T with some Hölder exponent $\alpha \in (0,1)$, provided that $q > n+2$.

6.2. **Global Calderón-Zygmund theory.** Our second regularity result is concerned with a global Calderón-Zygmund theory for variational solutions. Therefore, as a second model case we consider functionals of the type

$$F(v) := A\bigg(\int_\Omega |Dv|^p \mathrm{d}x\bigg) - \int_\Omega |F|^{p-2}F \cdot Dv\mathrm{d}x,$$

with $p > \frac{2n}{n+2}$, A as in §6.1 and $F \in L^q(\Omega, \mathbb{R}^{Nn})$ for some $q \geq p$. Moreover, we assume that the initial datum u_o satisfies $u_o \in W^{1,q}(\Omega, \mathbb{R}^N) \cap L^2(\Omega, \mathbb{R}^N)$. From Theorems 1.1 – 1.4 we infer the existence of a variational solution $u \in L^p(0,T; W^{1,p}_{u_o}(\Omega, \mathbb{R}^N)) \cap C^0([0,T]; L^2(\Omega, \mathbb{R}^N))$ with $\partial_t u \in L^2(\Omega_T, \mathbb{R}^N)$ in the sense of Definition 1.1, which solves the following Cauchy-Dirichlet problem

$$\left\{\begin{array}{ll} u_t - \operatorname{div}\big[\tilde{a}(t)\,|Du|^{p-2}\,Du\big] = \operatorname{div}(|F|^{p-2}F) & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{array}\right.$$

where $\tilde{a}$ is defined as in (6.2). Then, from [1, Theorem 2.3] (recall that $\tilde{a}$ is independent of x) we conclude that $Du \in L^q(\Omega \times (\delta, \tau), \mathbb{R}^{Nn})$ for any $\delta, \tau \in \mathbb{R}$ with $0 < \delta < \tau$.

Moreover, there exists $\rho_o > 0$ depending on $n, N, \mu, L, p, q, \partial\Omega$ such that for any $z_o = (x_o, t_o) \in \overline{\Omega} \times (0,T)$ and any parabolic cylinder $Q_\rho(z_o) := B_\rho(x_o) \times (t_o - \rho^2, t_o + \rho^2)$ with $Q_{2\rho}(z_o) \subset \mathbb{R}^n \times (0,T)$ and $\rho \in (0, \rho_o]$ there holds

$$⨍_{Q_\rho(z_o)\cap\Omega_T} |Du|^q \mathrm{d}z$$

$$\leq c\left[\left(\fint_{Q_{2\rho}(z_o)\cap\Omega_T}|Du|^p\mathrm{d}z\right)^{\frac{q}{p}}+\fint_{B_{2\rho}(x_o)\cap\Omega}(|Du_o|+|F|)^q\mathrm{d}x+1\right]^{d_{CZ}},$$

where we have abbreviated

$$d_{CZ}:=d-\tfrac{p}{q}(d-1)\quad\text{with}\quad d:=\begin{cases}\frac{p}{2}, & \text{if } p\geq 2,\\ \frac{2p}{p(n+2)-2n}, & \text{if } \frac{2n}{n+2}<p<2.\end{cases}$$

Note that the constant c depends on $n, N, \mu, L, p, q, \partial\Omega$.

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