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Classical solutions to the Fokker–Planck–BGK equation with infinite energy

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A B S T R A C T

We consider the BGK equation perturbed by Fokker–Planck operator, which is important in the kinetic theory of rarefied gases. This model equation, which we call the Fokker–Planck–BGK equation, has many physical features that the Fokker–Planck–Boltzmann equation possesses. By establishing a new moments lemma and L^∞ bounds for macroscopic quantities without the boundedness of energy, we get several global existence and uniqueness results to the Cauchy problem of the Fokker–Planck–BGK equation under various circumstances with infinite energy.

1. Introduction

We are concerned with global existence and uniqueness of solutions of the Fokker–Planck–BGK equation having infinite energy. Let $f(t,x,v) \in [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty]$ be the microscopic density of particles at time $t \geq 0$, position $x \in \mathbb{R}^3$ and moving with velocity $v \in \mathbb{R}^3$. In this model describing the evolution of rarefied gas, f is governed by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \beta \mathrm{div}_v(vf) - \sigma \Delta_v f = J(f), \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.1}$$

where $\beta \geq 0$ is the friction coefficient and $\sigma > 0$ is the diffusive coefficient. The BGK collision operator $J(f) = M[f] - f$ is a relaxation model of the Boltzmann collision operator, which contains most of the basic properties [1,9,10]. The nonlinear term $M[f]$ is the following local Maxwellian:

$$M[f](t,x,v) = \frac{\rho(t,x)}{(2\pi\theta(t,x))^{\frac{3}{2}}} \exp\left\{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}\right\}, \tag{1.2}$$

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where $\rho(t,x), u(t,x)$ and $\theta(t,x)$ respectively represent the mass density, bulk velocity and temperature of the gas at time t and position x, which are defined by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2 + 3\rho\theta \end{pmatrix}(t,x) = \int\limits_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)dv, \quad t \geq 0,\ x \in \mathbb{R}^3. \tag{1.3}$$

In this system, the collision term $J(f)$ models two-body interactions in the gas. It is simpler than the Boltzmann operator with binary collisions, however, it is still very complicated in the mathematical sense due to the occurrence of an occurrence of exponential nonlinearity in (1.2). The linear partial differential operator $-\beta \mathrm{div}_v(vf) - \sigma\Delta_v f$ is the so-called Fokker–Planck operator. It describes the fact that the gas molecules interact with background medium and their paths between two interactions obey Brownian motion.

For the Fokker–Planck–Boltzmann equation, by using renormalization and the regularizing effects of $\Delta_v f$, Diperna and Lions [11] established a global existence of renormalized solutions to the Cauchy problem, if the positive initial datum satisfies

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |v|^2 + |x|^2 + |\ln f_0|) f_0 dvdx < \infty. \tag{1.4}$$

The complete Fokker–Planck operator $-\beta \mathrm{div}_v(vf) - \sigma\Delta_v f$ also has similar mild regularizing effects, which were previously used by many authors, such as [2–7,17,25,26] for the Vlasov–Poisson–Fokker–Planck system and [14,20,27] for the Vlasov–Maxwell–Fokker–Planck system. Without Fokker–Planck operator, we pay attention to the existence and uniqueness results of the Cauchy problem of BGK equation (the equation (1.1) with $\beta = 0, \sigma = 0$). Assuming (1.4) holds, Perthame [21] established the global existence of solutions to the Cauchy problem of BGK equation in 1989. Then, Perthame and Pulvirenti [24] developed an weighted L^∞ method to get the uniqueness of polynomially decaying solutions for x in a periodic domain. Mischler generalized this result to the whole space by further assuming that the initial datum polynomially decays in x (not only in v) in [15]. By establishing weighted L^p estimates of the hydrodynamical quantities, the existence of L^p solutions and propagation properties of L^p moments were obtained in [29]. Then, combining the techniques used in the BGK equation with the regularizing effects of $\Delta_v f$, the existence of L^p solutions of the Cauchy problem (1.1) was established in [28], however, the friction term is not considered ($\beta = 0$) and an extra assumption of the velocity moment of the initial data is added ($\iint |v|^{2+0} f_0 dxdv < \infty$).

Notice that for the Vlasov–Poisson system, the Vlasov–Poisson–Fokker–Planck system, the Boltzmann equation, one can build solutions with infinite energy [7,8,13,16,18,19,22,30]. For example, Perthame [22] established the existence of infinite energy solution to the Vlasov–Poisson system by assuming $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x|^2)f_0 dvdx < \infty$ and $f_0 \in L^\infty$. Castella [7] built a solution with infinite energy to the Vlasov–Poisson–Fokker–Planck system by assuming $f_0 \in L^\infty$ and $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x|^2+|v|^\varepsilon)f_0 dvdx < \infty$ with $\varepsilon > 0$ (can be arbitrarily small). Mischler [16] established the existence of infinite energy solution to the Cauchy problem of the Bolzmann equation by assuming $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-v|^2+|x|^\varepsilon+|\ln f_0|)f_0 dvdx < \infty$ with $\varepsilon > 0$. Compared with (1.4), these initial datums don't decrease so rapidly or just decrease in "one" special direction. So we wonder whether there exist infinite energy solutions of the Cauchy problem (1.1) under similar assumptions. We say that a solution $f(t,x,v)$ of (1.1) has finite energy if for all $t \geq 0$

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v) dxdv < \infty.$$

In this paper, we get some special kinds of infinite energy solution of the Cauchy problem (1.1). Specifically speaking, we always assume that the initial datum satisfies $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\alpha v|^2)f_0dxdv<\infty$ for some $\alpha>0$, and construct a solution $f(t,x,v)$ verifying

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\omega(t)v|^2)f(t)dxdv=\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\alpha v|^2)f_0dxdv+6\sigma\int\limits_0^t\omega(s)^2ds\|f_0\|_1$$

for any $t\ge0$, where $\omega(t)=(e^{\beta t}-1)/\beta+\alpha e^{\beta t}$. This equality only yields local integrability of $\int_{\mathbb{R}^3}|v|^2f(t,x,v)dv$ for any $t\ge0$. So, generally speaking, this kind of solution has infinite energy. When establishing existence and uniqueness results in this situation, the critical issues are improving the integrability of microscopic density and estimating the macroscopic quantities without the boundedness of energy. We will deal with these problems in Proposition 2.2 and Proposition 3.1.

Our main existence results (including uniqueness) concerning the Fokker–Planck–BGK equation are the following:

(1) If the initial datum $f_0\ge\frac{\delta}{1+|x|^r}\chi_{\{|x-\alpha v|\le\epsilon\}}$ for some $\alpha>0$ and some $\delta,\epsilon>0$ (can be arbitrary small), and satisfies that for some $q>5, r>3$

$$\sup_{x,v}(1+|x-\alpha v|^q)(1+|x|^r+|x-\alpha v|^r)f_0(x,v)<\infty. \tag{1.5}$$

Then there exists an unique polynomially decaying solution to the Cauchy problem (1.1).

Note that (1.5) implies the boundedness of $\iint_{\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\alpha v|^2)f_0dxdv$, and the above result can be easily extended to the N-dimensional case by assuming $q>N+2, r>N$. The lower bound of f_0 will be used to deduce a lower bound of ρ and then θ, which is indispensable to prove the uniqueness. But this assumption implies that the gas cannot contain vacuum ($\rho=\int fdv=0$ in some $\Omega\subset\mathbb{R}^3$ with $|\Omega|>0$). However, we can use the above result to construct approximate solutions, and establish some much more general existence results in the three-dimensional case (also in the N-dimensional case), that is,

(2) If the initial datum $f_0\ge0$ satisfies that for some $\alpha>0$ and $1<p\le\infty$

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\big(1+|x-\alpha v|^2\big)f_0dvdx<\infty,\quad \|f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}<\infty, \tag{1.6}$$

or

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\big(1+|x-\alpha v|^2+\ln(1+|x|)+|\ln f_0|\big)f_0dvdx<\infty, \tag{1.7}$$

then one can build a global classical solution of the Cauchy problem (1.1).

More complete statements can be found in the main text. We emphasize that the case of $\beta=0$ is not specially discussed in the whole paper, since all the results in this case are in parallel with that in the case of $\beta>0$, and can be obtained by taking $\beta\to0$. And we do not require $\iint|x-\alpha v|^{2+0}f_0dxdv<\infty$ in the assumptions (1.6) and (1.7).

The rest of this paper is organized as follows: In Section 2, we give a new moments lemma to transport equation with Fokker–Planck operator. Section 3 is devoted to giving new estimates of the macroscopic quantities and then proving an existence and uniqueness result to the Cauchy problem (1.1), where the initial datums are polynomially decaying but could have infinite energy. In Section 4, we use the results in Section 2 and Section 3 to prove some general existence results. In this paper, the letter C denotes a generic positive constant which changes from line to line. $\|\cdot\|_p$ always denotes the norm of the space $L^p(\mathbb{R}^3_x\times\mathbb{R}^3_v)$ for $1\le p\le\infty$. For the sake of simplicity we will denote the integral $\iint_{\mathbb{R}^3\times\mathbb{R}^3}\cdots dvdx$ by $\iint\cdots dvdx$.

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2. A velocity-spatial moments lemma

Firstly, we consider the following linear evolution equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \beta \mathrm{div}_v(vf) - \sigma \Delta_v f = g, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{2.1}$$

where f_0 and g are assumed to be known. It is well-known that there exists a fundamental solution $G(t,x,v,y,\eta)$ of (2.1), which can be expressed by

$$G(t,x,v,y,\eta) = G_0(t, x - y - \eta(1-e^{-\beta t})/\beta, v - e^{-\beta t}\eta),$$

where $x,v,y,\eta \in \mathbb{R}^3, t \geq 0$ and

$$G_0(t,x,v) = \frac{1}{(4\pi\sigma)^3 D(t)^{3/2}} e^{-\Phi_0(t,x,v)/4\sigma},$$

$$\Phi_0(t,x,v) = \frac{1}{D(t)} \frac{1-e^{-2\beta t}}{2\beta} \left| x - \frac{1-e^{-\beta t}}{\beta(1+e^{-\beta t})} v \right|^2 + \frac{2}{1-e^{-2\beta t}} |v|^2,$$

$$D(t) = \frac{1}{\beta^2} \left[\frac{1-e^{-2\beta t}}{2\beta} t - \left(\frac{1-e^{-\beta t}}{\beta} \right)^2 \right].$$

The fundamental solution $G(t,x,v;y,\eta)$ has many important properties (see, for example, [2–5,7,11,28]), some of which are included in the following lemma.

Lemma 2.1. *The following properties hold for the fundamental solution $G(t,x,v;y,\eta)$:*

(1) For any $x,v,y,\eta \in \mathbb{R}^3$, $t \geq 0$

$$\iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} G(t,x,v,y,\eta) dx dv = 1, \quad \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} G(t,x,v,y,\eta) dy d\eta = e^{3\beta t}.$$

(2) For any positive integers n,k, any 6 dimensional multi-indices α,γ, and any $0 < h < T < \infty$, there exists a positive constant $C = C(n,k,\alpha,\gamma,\beta,\sigma,h,T)$ such that

$$\frac{(1+|y|+|\eta|)^n}{(1+|x|+|v|)^{n+2(|\alpha|+|\gamma|+2k)}} |\partial_t^k \partial_{x,v}^\alpha \partial_{y,\eta}^\gamma G(t,x,v,y,\eta)| \leq C, \ \ t \in [h,T], \ \ x,v,y,\eta \in \mathbb{R}^3.$$

(3) For any $T > 0$, there exists a positive constant $C = C(p,\beta,\sigma,T)$ such that

$$\sup_{t \in [0,T]} \left\| \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} G(t,x,v,y,\eta) f(y,\eta) dy d\eta \right\|_p \leq C \|f\|_p, \tag{2.2}$$

where $1 \leq p \leq \infty$. Furthermore, the inequality is sharp with $C = 1$ if $p = 1$ and $f \geq 0$.

With the fundamental solution $G(t,x,v;y,\eta)$, the solution of (2.1) can be represented by

$$f(t,x,v) = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} G(t,x,v,y,\eta) f_0(y,\eta) dy d\eta + \int_0^t ds \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} G(t-s,x,v,y,\eta) g(s,y,\eta) dy d\eta. \tag{2.3}$$

For simplicity, we define a linear operator $G(t)$ by

$$G(t)f(x,v) = \iint G(t,x,v,y,\eta)f(y,\eta)dyd\eta. \tag{2.4}$$

From the above lemma, we know that $G(t)$ maps $f \in L^p(\mathbb{R}^3\times\mathbb{R}^3)$ into $C([0,\infty), L^p(\mathbb{R}^3\times\mathbb{R}^3))\cap C^\infty((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$ for any $1 \le p < \infty$. And (2.3) is equivalent to

$$f(t,x,v) = G(t)f_0(x,v) + \int_0^t G(t-s)g(s,x,v)ds.$$

As we know, moments lemma is one of the most important tools in kinetic equations, which is used to improve the obvious integrability. Perthame [21] gave a classical "velocity moments lemma" for transport equation, that is, the solution of transport equation has three order of velocity moments locally in space under the boundedness of kinetic energy. Using the methods in [21,23], we are able to give a new moments lemma for the equation (2.1), which is not about velocity moments but about velocity-spatial moments.

Proposition 2.2. *Suppose that $f \in L^1_+([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$ is the unique distributional solution of the Cauchy problem (2.1) with $f_0, g \ge 0$. Assume there exists a positive constant $C(T)$ such that*

$$\iint (1+|x-\alpha v|^2)f_0(x,v)dxdv + \int_0^T \iint (1+|x-\omega(t)v|^2)g(t,x,v)dxdvdt \le C(T),$$

where $\omega(t) = (e^{\beta t}-1)/\beta + \alpha e^{\beta t}$ with $\alpha \ge 0$. Then for any $R > 0$ we have

$$\int_0^T \iint_{B_R\times\mathbb{R}^3_v} |x-\omega(t)v|^3 f(t,x,v)dxdvdt \le C(T,R).$$

Proof. Define a C^2 function

$$\phi(t,x,v) = \frac{x\cdot z}{(1+|x|^2)^{\frac{1}{2}}}(1+|z|^2)^{\frac{1}{2}}, \tag{2.5}$$

where $z = x - \omega(t)v$. Let $\varphi_n = (\chi_{|x|\le 2n} * \eta_n)(\chi_{|v|\le 2n} * \eta_n)\psi_n(t)$, where χ is the cutoff function and η is the mollifier. The function $\psi_n(t) \in C_c^\infty[0,T)$ satisfies that $\psi_n(t) = 1$ if $t \in [0, T-\frac{1}{n}]$, and $\psi_n(t) = 0$ if $t \in [T-\frac{1}{2n}, T)$ and $|\psi_n'(t)| \le Cn$ on $[T-\frac{1}{n}, T-\frac{1}{2n}]$. Note that $\omega(t)\phi\varphi_n \in C_c^2([0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$, by the definition of distributional solution we have

$$\begin{aligned}&\int_0^T \iint (\partial_t + v\cdot\nabla_x - \beta v\cdot\nabla_v + \sigma\Delta_v)[\omega(t)\phi\varphi_n]fdxdvdt \\ &+ \iint \alpha\phi(0,x,v)\varphi_n(0,x,v)f_0dxdv + \int_0^T \iint \omega(t)\phi\varphi_n gdxdvdt = 0.\end{aligned} \tag{2.6}$$

Define $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$, a basic computation gives

$$\partial_t\phi = -\omega'(t)\frac{x\cdot v}{\langle x\rangle}\langle z\rangle - \omega'(t)\frac{x\cdot z}{\langle x\rangle}\frac{v\cdot z}{\langle z\rangle} = -\frac{\omega'(t)}{\omega(t)}\frac{|x|^2 - x\cdot z}{\langle x\rangle}\langle z\rangle - \omega'(t)\frac{x\cdot z}{\langle x\rangle}\frac{v\cdot z}{\langle z\rangle}$$

and

$$\begin{aligned}
v\cdot\nabla_x\phi &= \frac{(x+z)\cdot v}{\langle x\rangle}\langle z\rangle + \frac{x\cdot z}{\langle x\rangle}\frac{v\cdot z}{\langle z\rangle} - \frac{(x\cdot z)(x\cdot v)}{\langle x\rangle^3}\langle z\rangle \\
&= -\frac{1}{\omega(t)}\frac{|z|^2-|x|^2}{\langle x\rangle}\langle z\rangle + \frac{x\cdot z}{\langle x\rangle}\frac{v\cdot z}{\langle z\rangle} - \frac{1}{\omega(t)}\frac{|x|^2(x\cdot z)-(x\cdot z)^2}{\langle x\rangle^3}\langle z\rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-\beta v\cdot\nabla_v\phi &= \beta\omega(t)\left(\frac{\langle z\rangle}{\langle x\rangle}x\cdot v + \frac{x\cdot z}{\langle x\rangle\langle z\rangle}z\cdot v\right) \\
&= \beta\frac{|x|^2-x\cdot z}{\langle x\rangle}\langle z\rangle + \beta\omega(t)\frac{x\cdot z}{\langle x\rangle}\frac{v\cdot z}{\langle z\rangle}
\end{aligned}$$

and

$$\begin{aligned}
\Delta_v\phi &= -\omega(t)\mathrm{div}_v\left(\frac{\langle z\rangle}{\langle x\rangle}x + \frac{x\cdot z}{\langle x\rangle\langle z\rangle}z\right) \\
&= -\omega(t)\sum_{i=1}^{3}\left[\frac{\partial_{v_i}\langle z\rangle}{\langle x\rangle}x_i + \partial_{v_i}\left(\frac{(x\cdot z)z_i}{\langle x\rangle\langle z\rangle}\right)\right] \\
&= 4\omega(t)^2\frac{x\cdot z}{\langle x\rangle\langle z\rangle} + \omega(t)^2\frac{x\cdot z}{\langle x\rangle\langle z\rangle^3}.
\end{aligned}$$

Adding the above equalities together, we have from $\omega'(t)=\beta\omega(t)+1$ that

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x - \beta v\cdot\nabla_v + \sigma\Delta_v)\,\phi \\
&= -\frac{1}{\omega(t)}\frac{|z|^2-(x\cdot z)+|x|^2|z|^2-(x\cdot z)^2}{\langle x\rangle^3}\langle z\rangle + 4\sigma\omega(t)^2\frac{x\cdot z}{\langle x\rangle\langle z\rangle} + \sigma\omega(t)^2\frac{x\cdot z}{\langle x\rangle\langle z\rangle^3} \\
&\le -\frac{1}{\omega(t)}\frac{|z|^3}{\langle x\rangle^3} + \frac{1}{\omega(t)}\langle z\rangle^2 + 5\sigma\omega(t)^2,
\end{aligned}$$

since $|x|^2|z|^2-(x\cdot z)^2\ge 0$. By the definition of φ_n, we obtain that $|v\cdot\nabla_x\varphi_n|, |v\cdot\nabla_v\varphi_n|, |\Delta_v\varphi_n|$ and $|\nabla_v\varphi_n|$ are uniformly bounded, and by (2.5) we have $|\nabla_v\phi|\le 2\langle z\rangle^2, |\phi|\le\langle z\rangle^2$. Thus, the above arguments give

$$\begin{aligned}
&(\partial_t + v\cdot\nabla_x - \beta v\cdot\nabla_v + \sigma\Delta_v)\,[\omega(t)\phi\varphi_n] \\
&= \omega(t)\varphi_n(\partial_t + v\cdot\nabla_x - \beta v\cdot\nabla_v + \sigma\Delta_v)\,\phi + \omega(t)\phi(v\cdot\nabla_x - \beta v\cdot\nabla_v + \sigma\Delta_v)\varphi_n \\
&\quad + 2\sigma\omega(t)\nabla_v\phi\cdot\nabla_v\varphi_n + \phi\varphi_n\omega'(t) + \omega(t)\phi\partial_t\varphi_n \\
&\le -\frac{|z|^3}{\langle x\rangle^3}\chi_{t\in[0,T-\frac{1}{n}]}\chi_{|x|\le n}\chi_{|v|\le n} + C(1+|\psi_n'(t)|)\langle z\rangle^2,
\end{aligned}$$

where C is a positive constant dependent upon α,β,σ,T. From (2.6) and the above inequality we have

$$\begin{aligned}
&\int_0^T\iint \chi_{t\in[0,T-\frac{1}{n}]}\chi_{|x|\le n}\chi_{|v|\le n}\frac{|z|^3}{\langle x\rangle^3}f\,dxdvdt \\
&\le \iint \alpha\phi(0,x,v)f_0\,dxdv + \int_0^T\iint\omega(t)\phi g\,dxdvdt + C\int_0^T\iint(1+|\psi_n'(t)|)\langle z\rangle^2 f\,dxdvdt
\end{aligned}$$

$$\leq C \sup_{t\in[0,T]} \iint \langle z\rangle^2 f dxdv + C\int_0^T \iint \langle z\rangle^2 g dxdvdt + Cn \int_{T-\frac{1}{n}}^{T-\frac{1}{2n}} \iint \langle z\rangle^2 f dxdvdt$$

$$\leq C \sup_{t\in[0,T]} \iint \langle z\rangle^2 f dxdv + C\int_0^T \iint \langle z\rangle^2 g dxdvdt, \tag{2.7}$$

since $|\psi_n'(t)| \leq Cn$ on $[T-\frac{1}{n}, T-\frac{1}{2n}]$. On the other hand, from (2.1) we can easily deduce

$$\sup_{t\in[0,T]} \|f(t)\|_1 \leq \|f_0\|_1 + \int_0^T \|g(s)\|_1 ds \tag{2.8}$$

and

$$\begin{aligned}
&\sup_{t\in[0,T]} \iint |x-\omega(t)v|^2 f(t,x,v)dxdv \\
&\leq \iint |x-\alpha v|^2 f_0(x,v)dxdv + 3\sigma\omega(T)^2 \sup_{t\in[0,T]} \|f(t)\|_1 \\
&\quad + \int_0^T \iint |x-\omega(t)v|^2 g(t,x,v)dxdvdt \\
&\leq \iint |x-\alpha v|^2 f_0(x,v)dxdv + 3\sigma\omega(T)^2 \left(\|f_0\|_1 + \int_0^T \|g(t)\|_1 dt \right) \\
&\quad + \int_0^T \iint |x-\omega(t)v|^2 g(t,x,v)dxdvdt.
\end{aligned} \tag{2.9}$$

Combining (2.7) with (2.8), (2.9) and letting $n\to\infty$ we can obtain

$$\begin{aligned}
\int_0^T \iint \frac{|z|^3}{(1+|x|^2)^{\frac{3}{2}}} f dxdvdt \leq C_{\alpha,\beta,\sigma,T} \Bigg(&\iint (1+|x-\alpha v|^2) f_0(x,v)dxdv \\
&+ \int_0^T \iint (1+|x-\omega(t)v|^2) g(t,x,v)dxdvdt \Bigg),
\end{aligned}$$

which gives our conclusion. □

Using the above method, we can give a velocity moments lemma to the equation (2.1).

Remark 2.1. If there exists a positive constant $C(T)$ such that

$$\iint (1+|v|^2) f_0(x,v)dxdv + \int_0^T \iint (1+|v|^2)|g(t,x,v)|dxdvdt \leq C(T),$$

then for any $R>0$, the positive solution of (2.1) satisfies

$$\int_0^T \iint_{B_R \times \mathbb{R}^3_v} |v|^3 f dx dv dt \le C(T,R).$$

Proof. Choosing

$$\phi(t,x,v) = \frac{x \cdot v}{(1+|x|^2)^{\frac{1}{2}}}(1+|v|^2)^{\frac{1}{2}},$$

and repeating the proof of the above lemma we can get this conclusion. □

Following from conservation laws and the above moments lemma we can give some *a priori* estimates permitting infiniteness of energy for the Fokker–Planck–BGK equation.

Lemma 2.3. *Let* $\alpha \ge 0$ *and let the initial datum* $f_0 \ge 0$ *be given with*

$$\iint \left(1+|x-\alpha v|^2\right) f_0 dv dx < +\infty.$$

Assume $f \in C([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ *is a solution of the Cauchy problem (1.1), then*

$$\sup_{t\in[0,T]} \iint \left(1+|x-\omega(t)v|^2\right) f(t,x,v) dv dx \le C(T) \tag{2.10}$$

for any $t \ge 0$, *and*

$$\int_0^T \iint_{B_R \times \mathbb{R}^3_v} |x-\omega(t)v|^3 f(t,x,v) dx dv dt \le C(T,R) \tag{2.11}$$

for any $R > 0$. *Moreover, the following macroscopic quantities are conserved:*

$$\frac{d}{dt}\iint \begin{pmatrix} 1 \\ x-\omega(t)v \\ |x-\omega(t)v|^2 \end{pmatrix} f(t,x,v) dv dx = \begin{pmatrix} 0 \\ 0 \\ 6\sigma\omega(t)^2 \|f_0\|_1 \end{pmatrix}. \tag{2.12}$$

Proof. Following from invariants of collision operator $J(f)$:

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ x-\omega(t)v \\ |x-\omega(t)v|^2 \end{pmatrix} f(t,x,v) dv = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ x-\omega(t)v \\ |x-\omega(t)v|^2 \end{pmatrix} M[f](t,x,v) dv.$$

Then we can easily deduce

$$\frac{d}{dt}\iint (1, x-\omega(t)v) f(t,x,v) dv dx = 0$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\iint |x-\omega(t)v|^2 f(t,x,v) dv dx\right) \\ &= \frac{1}{2}\iint |x-\omega(t)v|^2 \partial_t f dv dx - \omega'(t)\iint [x-\omega(t)v]\cdot v f dv dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \iint |x-\omega(t)v|^2[-\mathrm{div}_x(vf) - \beta \mathrm{div}_v(vf) + \sigma \Delta_v f] dvdx - \omega'(t) \iint (x-\omega(t)v)\cdot vf dvdx \\
&= \iint [x-\omega(t)v]\cdot vf dvdx + \beta\omega(t) \iint [x-\omega(t)v]\cdot vf dvdx \\
&\quad + 3\sigma\omega(t)^2 \iint f dvdx - \omega'(t) \iint (x-\omega(t)v)\cdot vf dvdx \\
&= 3\sigma\omega(t)^2 \iint f dvdx = 3\sigma\omega(t)^2 \|f_0\|_1,
\end{aligned}
$$

which yield (2.12) and then (2.10). By Proposition 2.2 we easily obtain (2.11). □

3. Existence and uniqueness in weighted L^∞ space

For establishing an existence and uniqueness result to the Cauchy problem (1.1), it is crucial to give some proper estimates of the macroscopic quantities and the local Maxwellian $M[f]$. A rather general L^∞ bounds for the macroscopic quantities had been established in [24], however, these estimates are based on the boundedness of weighted velocity L^∞ norms of the microscopic density, which is not satisfied under the assumption of infinite energy. The following proposition is devoted to establishing some similar estimates in terms of the weighted velocity-spatial L^p norms of $f \geq 0$.

Proposition 3.1. *Let $1 < p \leq +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha \geq 0$, the following estimates hold:*

$$
\frac{\rho(t,x)}{\theta(t,x)^{\frac{3}{2p'}}} \leq C\|f(t,x,v)\|_{L^p(\mathbb{R}^3_v)}. \tag{3.1}
$$

If $0 < q < \frac{3}{p'}$ or $q > \frac{3}{p'} + 2$, then

$$
\frac{\omega(t)^{\frac{3}{p'}}\rho(t,x)}{[|x-\omega(t)u(t,x)|^2 + 3\omega(t)^2\theta(t,x)]^{\frac{3-qp'}{2p'}}} \leq C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}. \tag{3.2}
$$

If $q \geq 1$, then

$$
\frac{\rho(t,x)|x-\omega(t)u(t,x)|^{\frac{3+qp'}{p'}}}{\theta(t,x)^{\frac{3}{2p'}}[|x-\omega(t)u(t,x)|^2 + 3\omega(t)^2\theta(t,x)]^{\frac{3}{2p'}}} \leq C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}. \tag{3.3}
$$

If $1 < q < \frac{3}{p'}$ or $q > \frac{3}{p'} + 2$, then

$$
\frac{\rho(t,x)|x-\omega(t)u(t,x)|^q}{\theta(t,x)^{\frac{3}{2p'}}} \leq C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}, \tag{3.4}
$$

and

$$
\frac{\omega(t)^q\rho(t,x)}{\theta(t,x)^{\frac{3-qp'}{2p'}}} \leq C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}. \tag{3.5}
$$

Moreover, if $q = 0$ or $1 < q < \frac{3}{p'}$ or $q > \frac{3}{p'} + 2$, then

$$
\big\||x-\omega(t)v|^q M[f]\big\|_{L^p(\mathbb{R}^3_v)} \leq C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}. \tag{3.6}
$$

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Proof. The proofs are similar to those stated in [24], we include them here for completeness. For any $R > 0$, we have

$$\begin{aligned}\rho(t,x) &= \int f(t,x,v)dv = \int_{|v-u(t,x)|\le R} f(t,x,v)dv + \int_{|v-u(t,x)|>R} f(t,x,v)dv\\ &\le CR^{\frac{3}{p'}}\|f(t,x,v)\|_{L^p(\mathbb{R}^3_v)} + \frac{1}{R^2}\int |v-u(t,x)|^2 f(t,x,v)dv\\ &\le CR^{\frac{3}{p'}}\|f(t,x,v)\|_{L^p(\mathbb{R}^3_v)} + \frac{3\rho(t,x)\theta(t,x)}{R^2}.\end{aligned}$$

By choosing $R^{2+\frac{3}{p'}} = \rho(t,x)\theta(t,x)\|f(t,x,v)\|^{-1}_{L^p(\mathbb{R}^3_v)}$, we can obtain (3.1).

For the proof of (3.2), we firstly consider the case $0 \le q < \frac{3}{p'}$.

$$\begin{aligned}\rho(t,x) &= \int f(t,x,v)dv = \int_{|x-\omega(t)v|\le R} f(t,x,v)dv + \int_{|x-\omega(t)v|>R} f(t,x,v)dv\\ &\le \big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}\left(\int_{|x-\omega(t)v|\le R} |x-\omega(t)v|^{-qp'}dv\right)^{\frac{1}{p'}}\\ &\quad + \frac{1}{R^2}\int |x-\omega(t)v|^2 f(t,x,v)dv\\ &\le C\omega(t)^{-\frac{3}{p'}}R^{\frac{3-qp'}{p'}}\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}\\ &\quad + R^{-2}\rho(t,x)\big(|x-\omega(t)u(t,x)|^2 + 3\omega(t)^2\theta(t,x)\big).\end{aligned}$$

Denote $z = |x-\omega(t)u(t,x)|^2 + 3\omega(t)^2\theta(t,x)$, then by choosing

$$R^{2+\frac{3}{p'}-q} = \omega(t)^{\frac{3}{p'}}\big\||x-\omega(t)v|^q f(t,x,v)\big\|^{-1}_{L^p(\mathbb{R}^3_v)}\rho(t,x)z,$$

we can obtain (3.2). For the case of $q > \frac{3}{p'} + 2$, we have

$$\begin{aligned}\rho(t,x)z &= \int |x-\omega(t)v|^2 f(t,x,v)dv\\ &\le \int_{|x-\omega(t)v|\le R} |x-\omega(t)v|^2 f(t,x,v)dv + \int_{|x-\omega(t)v|>R} |x-\omega(t)v|^2 f(t,x,v)dv\\ &\le R^2\rho(t,x) + \big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}\left(\int_{|x-\omega(t)v|>R} |x-\omega(t)v|^{p'(2-q)}dv\right)^{\frac{1}{p'}}\\ &\le R^2\rho(t,x) + C\omega(t)^{-\frac{3}{p'}}R^{\frac{3}{p'}+2-q}\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}.\end{aligned}$$

By choosing $R^{q-\frac{3}{p'}} = \omega(t)^{-\frac{3}{p'}}\rho(t,x)^{-1}\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}$ we also have (3.2).

Similarly,

$$
\begin{aligned}
&\rho(t,x)|x-\omega(t)u(t,x)| \le \int |x-\omega(t)v| f(t,x,v)dv \\
&\le \int_{|v-u(t,x)|\le R} |x-\omega(t)v| f(t,x,v)dv + \int_{|v-u(t,x)|>R} |x-\omega(t)v| f(t,x,v)dv \\
&\le \left(\int_{|v-u(t,x)|\le R} |x-\omega(t)v|^q f(t,x,v)dv \right)^{\frac{1}{q}} \rho(t,x)^{\frac{1}{q'}} \\
&\quad + \frac{1}{R}\int |v-u(t,x)||x-\omega(t)v| f(t,x,v)dv \\
&\le C\big\| |x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}^{\frac{1}{q}} R^{\frac{3}{qp'}} \rho(t,x)^{\frac{1}{q'}} \\
&\quad + \frac{1}{R}\left(\int |v-u(t,x)|^2 f(t,x,v)dv\right)^{\frac{1}{2}} \left(\int |x-\omega(t)v|^2 f(t,x,v)dv\right)^{\frac{1}{2}} \\
&\le C\big\| |x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}^{\frac{1}{q}} R^{\frac{3}{qp'}} \rho(t,x)^{\frac{1}{q'}} + \frac{1}{R}\theta(t,x)^{\frac{1}{2}} z^{\frac{1}{2}} \rho(t,x).
\end{aligned}
$$

Choosing

$$
R^{\frac{3}{qp'}+1} = \rho(t,x)^{\frac{1}{q}} \theta(t,x)^{\frac{1}{2}} z^{\frac{1}{2}} \big\| |x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}^{-\frac{1}{q}},
$$

we have (3.3).

Now, we treat (3.4) and (3.5). When $1 \le q < \frac{3}{p'}$, if $|x-\omega(t)u(t,x)|^2 > 3\omega(t)^2\theta(t,x)$, it follows from (3.3) that

$$
\begin{aligned}
\frac{\big(3\omega(t)^2\theta(t,x)\big)^{\frac{q}{2}}\rho(t,x)}{\theta(t,x)^{\frac{3}{2p'}}} &\le \frac{\rho(t,x)|x-\omega(t)u(t,x)|^q}{\theta(t,x)^{\frac{3}{2p'}}} \\
&\le \frac{\rho(t,x)|x-\omega(t)u(t,x)|^{\frac{3+qp'}{p'}}}{(\theta(t,x)z/2)^{\frac{3}{2p'}}} \\
&\le C\big\| |x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}.
\end{aligned} \tag{3.7}
$$

If $|x-\omega(t)u(t,x)|^2 \le 3\omega(t)^2\theta(t,x)$, it follows from (3.2) that

$$
\begin{aligned}
\frac{\rho(t,x)|x-\omega(t)u(t,x)|^q}{\theta(t,x)^{\frac{3}{2p'}}} &\le \frac{\big(3\omega(t)^2\theta(t,x)\big)^{\frac{q}{2}}\rho(t,x)}{\theta(t,x)^{\frac{3}{2p'}}} \\
&\le C\frac{\omega(t)^{\frac{3}{p'}}\rho(t,x)}{z^{\frac{3-qp'}{2p'}}} \\
&\le C\big\| |x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}.
\end{aligned}
$$

When $q > \frac{3}{p'}+2$, if $|x-\omega(t)u(t,x)|^2 > 3\omega(t)^2\theta(t,x)$, a similar computation as (3.7) gives the conclusion. If $|x-\omega(t)u(t,x)|^2 \le 3\omega(t)^2\theta(t,x)$, by using (3.2) again we have

$$\begin{aligned}\frac{\rho(t,x)|x-\omega(t)u(t,x)|^q}{\theta(t,x)^{\frac{3}{2p'}}} &\le \frac{\left(3\omega(t)^2\theta(t,x)\right)^{\frac{q}{2}}\rho(t,x)}{\theta(t,x)^{\frac{3}{2p'}}}\\ &\le (\sqrt{3}\omega(t))^{\frac{3}{p'}}\rho(t,x)z^{\frac{qp'-3}{2p'}}\\ &\le C\big\||x-\omega(t)v|^q f(t,x,v)\big\|_{L^p(\mathbb{R}^3_v)}.\end{aligned}$$

Consequently, we obtain (3.4) and (3.5).

At last, we prove (3.6).

$$\begin{aligned}&\big\||x-\omega(t)v|^q M[f]\big\|^p_{L^p(\mathbb{R}^3_v)}\\ &\le 2^{qp}\int(|x-\omega(t)u(t,x)|^{qp}+\omega(t)^{qp}|v-u(t,x)|^{qp})\frac{\rho(t,x)^p}{(2\pi\theta(t,x))^{\frac{3}{2}p}}\exp\left\{-\frac{p|v-u(t,x)|^2}{2\theta(t,x)}\right\}dv\\ &= 2^{qp}\left(|x-\omega(t)u(t,x)|^q\frac{\rho(t,x)}{\theta(t,x)^{\frac{3}{2p'}}}\right)^p+2^{qp}\left(\omega(t)^q\frac{\rho(t,x)}{\theta(t,x)^{\frac{3-qp'}{2p'}}}\right)^p.\end{aligned}$$

By (3.1) with $q=0$ and (3.4), (3.5) with $1<q<\frac{3}{p'}$ or $q>\frac{3}{p'}+2$, we get (3.6). □

From the above lemma we can get the following corollary. Since the proof is only an interpolation of the above estimates, we omit it.

Corollary 3.2. *Let $1<p\le+\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$ and $\alpha\ge 0$. If $q>\frac{3}{p'}+2$ and $a\in[-\frac{3}{p'},q-\frac{3}{p'}]$, then*

$$\omega(t)^{a+3/p'}\rho(t,x)\theta(t,x)^{\frac{a}{2}}\le C\|(1+|x-\omega(t)v|^q)f(t,x,v)\|_{L^p(\mathbb{R}^3_v)}. \tag{3.8}$$

If $q>\frac{3}{p'}+2,\beta\in[1-\frac{p'q}{3},1]$ and $\gamma\in[0,q-\frac{3}{p'}(1-\beta)]$, then

$$\omega(t)^{\frac{3(1-\beta)}{p'}}\rho(t,x)|x-\omega(t)u(t,x)|^\gamma\theta(t,x)^{-\frac{3\beta}{2p'}}\le C\|(1+|x-\omega(t)v|^q)f(t,x,v)\|_{L^p(\mathbb{R}^3_v)}. \tag{3.9}$$

Inspired by the thoughts in [15,24,28], we will use Proposition 3.1 and its corollary to show that there exists a unique solution of the Cauchy problem (1.1) with bounded velocity-spatial weighted L^∞ norms. To this end, we firstly define precisely those weighted L^∞ norms and give some estimates of the operator $G(t)$ with such norms.

Definition 3.1. Let $q,r>0$, define

$$\begin{aligned}H_q(f(t)) &= \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3}(1+|x-\omega(t)v|^q)|f(t,x,v)|,\\ H_{q,r}(f(t)) &= \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3}(1+|x|^r)(1+|x-\omega(t)v|^q)|f(t,x,v)|.\end{aligned}$$

Lemma 3.3. *There exist two positive continuous functions $C_{q+r,\alpha,\beta,\sigma}(t)$ and $C_{q,r,\alpha,\beta,\sigma}(t)$ such that for any $0\le s<t$ and any nonnegative and measurable function $g(s,x,v)$,*

$$H_q(G(t-s)g(s,x,v))\le C_{q,\alpha,\beta,\sigma}(t)H_q(g(s)), \tag{3.10}$$

$$H_{q,r}(G(t-s)g(s,x,v))\le C_{q,r,\alpha,\beta,\sigma}(t)[H_{q+r}(g(s))+H_{q,r}(g(s))]. \tag{3.11}$$

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Proof. Let $\bar{y} = x - y - \frac{1-e^{-\beta(t-s)}}{\beta}\eta - \frac{1-e^{-\beta(t-s)}}{\beta(1+e^{-\beta(t-s)})}(v - e^{-\beta(t-s)}\eta)$, $\bar{\eta} = v - e^{-\beta(t-s)}\eta$, we have

$$\begin{aligned} 1 + |x - \omega(t)v|^q &\le 1 + \left[|\bar{y}| + \left(\frac{e^{-\beta(t-s)} - 1}{\beta(1+e^{-\beta(t-s)})} + \omega(t)\right)|\bar{\eta}| + |y - \omega(s)\eta|\right]^q \\ &\le C_{q,\alpha,\beta,\sigma,t}(1 + |y - \omega(s)\eta|^q)(1 + |\bar{y}|^q)(1 + |\bar{\eta}|^q). \end{aligned} \tag{3.12}$$

Then,

$$\begin{aligned} &\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |x - \omega(t)v|^q)G(t - s, x, v, y, \eta)g(s, y, \eta)dyd\eta \\ &\le C_{q,\alpha,\beta,\sigma}(t) \sup_{(y,\eta)\in\mathbb{R}^3\times\mathbb{R}^3} (1 + |y - \omega(s)\eta|^q)g(s, y, \eta) \\ &\quad \cdot \iint (1 + |\bar{y}|^q)(1 + |\bar{\eta}|^q)G_0(t - s, \bar{y} + \frac{1 - e^{-\beta(t-s)}}{\beta(1 + e^{-\beta(t-s)})}\bar{\eta}, \bar{\eta})d\bar{y}d\bar{\eta} \\ &\le C_{q,\alpha,\beta,\sigma}(t)H_q(g(s)), \end{aligned}$$

which implies (3.10). On the other hand,

$$\begin{aligned} 1 + |x|^r &\le 1 + \left(|\bar{y}| + \frac{1 - e^{-\beta(t-s)}}{\beta(1 + e^{-\beta(t-s)})}|\bar{\eta}| + \frac{1 - e^{-\beta(t-s)}}{\beta\omega(s)}|y - \omega(s)\eta| + \frac{1 - e^{-\beta(t-s)} + \beta\omega(s)}{\beta\omega(s)}|y|\right)^r \\ &\le C_{r,\alpha,\beta,\sigma}(t)(1 + |\bar{y}|^r)(1 + |\bar{\eta}|^r)(1 + |y - \omega(s)\eta|^r + |y|^r). \end{aligned}$$

Combining the above estimate with (3.12), we have

$$\begin{aligned} &(1 + |x|^r)(1 + |x - \omega(t)v|^q) \\ &\le C_{r,\alpha,\beta,\sigma}(t)(1 + |\bar{y}|^r)(1 + |\bar{\eta}|^r)(1 + |y - \omega(s)\eta|^r + |y|^r) \\ &\quad \cdot C_{q,\alpha,\beta,\sigma,t}(1 + |y - \omega(s)\eta|^q)(1 + |\bar{y}|^q)(1 + |\bar{\eta}|^q) \\ &\le C_{q,r,\alpha,\beta,\sigma}(t)\left[1 + |y - \omega(s)\eta|^{q+r} + (1 + |y - \omega(s)\eta|^q)|y|^r\right](1 + |\bar{y}|^{q+r})(1 + |\bar{\eta}|^{q+r}), \end{aligned}$$

and then

$$\begin{aligned} &\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |x|^r)(1 + |x - \omega(t)v|^q)G(t - s, x, v, y, \eta)g(s, y, \eta)dyd\eta \\ &= C_{q,r,\alpha,\beta,\sigma}(t) \sup_{(y,\eta)\in\mathbb{R}^3\times\mathbb{R}^3} \left[1 + |y - \omega(s)\eta|^{q+r} + (1 + |y - \omega(s)\eta|^q)|y|^r\right]g(s, y, \eta) \\ &\quad \cdot \iint (1 + |\bar{y}|^{q+r})(1 + |\bar{\eta}|^{q+r})G_0(t - s, \bar{y} + \frac{1 - e^{-\beta(t-s)}}{\beta(1 + e^{-\beta(t-s)})}\bar{\eta}, \bar{\eta})d\bar{y}d\bar{\eta} \\ &\le C_{q,r,\alpha,\beta,\sigma}(t)[H_{q+r}(g(s)) + H_{q,r}(g(s))], \end{aligned}$$

which yields (3.11). □

At last, we give the existence and uniqueness result, that is,

Theorem 3.4. *Let the initial datum $f_0(x, v) \ge 0$ satisfies*

$$H_{q+r}(f_0) < \infty, \qquad H_{q,r}(f_0) < \infty$$

for some $q > 5, r > 3$ *and* $\alpha > 0$. *Assume further* $(1+|x|^r)f_0(x,v) \geq \phi(x-\alpha v)$ *for almost all* $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$, *where* $\phi(z) \in L^1(\mathbb{R}^3)$ *and* $\phi(z) \geq \delta$ *with some positive constant* δ *when* $|z| \leq \epsilon$. *Then there exists an unique solution* $0 \leq f(t,x,v) \in C([0,\infty); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^\infty((0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ *to the Cauchy problem (1.1). Moreover, there are positive functions* $A(t), B(t)$ *such that for almost all* $(t,x) \in [0,\infty) \times \mathbb{R}^3$

$$\begin{aligned}
&H_{q,r}(f(t)), \quad H_{q+r}(f(t)) \leq A(t) < \infty,\\
&\rho(t,x), \quad |u(t,x)|, \quad \theta(t,x) \leq A(t) < \infty,\\
&(1+|x|^r)\rho(t,x), \quad \theta(t,x) \geq B(t) > 0.
\end{aligned}$$

Proof. A mild solution of the Cauchy problem (1.1) can be written as

$$f(t,x,v) = e^{-t}G(t)f_0(x,v) + \int_0^t e^{s-t}G(t-s)M[f](s,x,v)ds. \tag{3.13}$$

Let $\mathcal{T}f$ be the right hand side of (3.13). So we need to show that the operator $\mathcal{T}$ has a positive fixed point in $L^\infty([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3, (1+|x-\omega(t)v|^2)dvdx))$ for any $T > 0$, and this fixed point is unique and satisfies all the estimates. Set

$$\begin{aligned}
X = \big\{f \geq 0:\ & H_{q+r}(f(t)) \leq C_1 e^{C_1 t} H_{q+r}(f_0), && (3.14)\\
& H_{q,r}(f(t)) \leq C_2 e^{C_2 t}\big(H_{q+r}(f_0) + H_{q,r}(f_0)\big), && (3.15)\\
& (1+|x|^r)\rho(t,x) \geq C_3\big\}, && (3.16)
\end{aligned}$$

where the positive constants C_1, C_2, C_3 will be given later on.

Step 1. We firstly prove that $\mathcal{T}f$ maps X into itself. It is obvious that $\mathcal{T}f \geq 0$ if $f \geq 0$. Using (3.10), we have

$$\begin{aligned}
&(1+|x-\omega(t)v|^{q+r})(\mathcal{T}f)(t,x,v)\\
&= e^{-t}(1+|x-\omega(t)v|^{q+r})G(t)f_0(x,v) + \int_0^t e^{s-t}(1+|x-\omega(t)v|^{q+r})G(t-s)M[f](s,x,v)ds\\
&\leq C_{q+r,\alpha,\beta,\sigma}(t)\left[e^{-t}H_{q+r}(f_0) + \int_0^t e^{s-t}H_{q+r}M[f](s)ds\right].
\end{aligned}$$

By (3.6) with $p = \infty$ we know that for any $T > 0$ there exists a positive constant $C_1 \geq 1$ dependent of $q+r, \sigma, \alpha, \beta, T$ such that

$$H_{q+r}(\mathcal{T}f)(t) \leq C_1\left[H_{q+r}(f_0) + \int_0^t H_{q+r}f(s)ds\right], \quad \forall\ 0 \leq t \leq T.$$

So we have that for any $t \in [0,T]$

$$H_{q+r}(\mathcal{T}f(t)) \leq C_1 e^{C_1 t} H_{q+r}(f_0) \quad \text{if} \quad H_{q+r}(f(t)) \leq C_1 e^{C_1 t} H_{q+r}(f_0). \tag{3.17}$$

Similarly, using (3.11) and (3.6) with $p = \infty$, we have

$$\begin{aligned}
&(1+|x|^r)(1+|x-\omega(t)v|^q)(\mathcal{T}f)(t,x,v)\\
&\le C_{q,r,\alpha,\beta,\sigma}(t)\left[H_{q+r}(f_0)+H_{q,r}(f_0)+\int_0^t e^s\big(H_{q+r}M[f](s)+H_{q,r}M[f](s)\big)ds\right]\\
&\le C_{q,r,\alpha,\beta,\sigma}(t)\left[H_{q+r}(f_0)+H_{q,r}(f_0)+\int_0^t \big(H_{q+r}f(s)+H_{q,r}f(s)\big)ds\right].
\end{aligned}$$

Combining this estimate with (3.17), there exists a positive constant $C_2 \ge 1$ dependent of $q, r, \sigma, \alpha, \beta, T$ such that

$$H_{q,r}(\mathcal{T}f)(t) \le C_2\left[H_{q+r}(f_0)+H_{q,r}(f_0)+\int_0^t H_{q,r}f(s)ds\right], \quad \forall\, 0\le t\le T.$$

So we have that for any $t \in [0,T]$

$$H_{q+r}(\mathcal{T}f(t)) \le C_2 e^{C_2 t}[H_{q+r}(f_0)+H_{q,r}(f_0)] \quad \text{if} \quad f\in X. \tag{3.18}$$

Now we prove (3.16). Note that

$$\int_{\mathbb{R}^3} G(t,x,v,y,\eta)dv = \frac{1}{(4\pi\sigma\int_0^t \omega_0(\tau)^2 d\tau)^{3/2}} e^{-\frac{|x-y-\omega_0(t)\eta|^2}{4\sigma\int_0^t \omega_0(\tau)^2 d\tau}},$$

where $\omega_0(t) = \frac{1-e^{-\beta t}}{\beta}$. Let $\tilde{y} = x - y - \omega_0(t)\eta, \tilde{\eta} = y/\alpha - \eta$, then $y = (1+\omega_0(t)/\alpha)^{-1}(x-\tilde{y}+\omega_0(t)\tilde{\eta})$ and

$$\begin{aligned}
\rho(\mathcal{T}f)(t,x) &\ge e^{-t}\int_{\mathbb{R}^3}\iint_{\mathbb{R}^3\times\mathbb{R}^3} G(t,x,v,y,\eta)f_0(y,\eta)dyd\eta dv\\
&\ge e^{-t}\int_{\mathbb{R}^3}\iint_{\mathbb{R}^3\times\mathbb{R}^3} G(t,x,v,y,\eta)\frac{\phi(y-\alpha\eta)}{1+|y|^r}dyd\eta dv\\
&\ge e^{-t}\int_{\mathbb{R}^3}\iint_{|y-\alpha\eta|\le\epsilon} G(t,x,v,y,\eta)\frac{\delta}{1+|y|^r}dyd\eta dv\\
&\ge e^{-t}\iint_{|\tilde{\eta}|\le\epsilon/\alpha} \frac{1}{(4\pi\sigma\int_0^t\omega_0(\tau)^2d\tau)^{3/2}}e^{-\frac{|\tilde{y}|^2}{4\sigma\int_0^t\omega_0(\tau)^2d\tau}}\frac{\delta}{1+|x-\tilde{y}+\omega_0(t)\tilde{\eta}|^r}d\tilde{y}d\tilde{\eta}\\
&\ge e^{-t}\int_{\mathbb{R}^3}\frac{1}{(4\pi\sigma\int_0^t\omega_0(\tau)^2d\tau)^{3/2}}e^{-\frac{|\tilde{y}|^2}{4\sigma\int_0^t\omega_0(\tau)^2d\tau}}\frac{\delta}{(1+|\tilde{y}|^r)}d\tilde{y}\int_{|\tilde{\eta}|\le\epsilon/\alpha}\frac{1}{1+|x+\omega_0(t)\tilde{\eta}|^r}d\tilde{\eta}.
\end{aligned}$$

Thus, there exists a positive constant C_3 dependent of $r, \alpha, \beta, \delta, \epsilon, T$ such that

$$\rho(\mathcal{T}f)(t,x) \ge \frac{C_3}{1+|x|^r}, \quad \forall\, t\in[0,T]. \tag{3.19}$$

Step 2. We now prove that $M[f] : X \mapsto L^\infty([0,T]; L^1(\mathbb{R}^3\times\mathbb{R}^3, (1+|x-\omega(t)v|^2)dxdv))$ is Lipschitz. Let $g_0, g_1 \in X$ and $g_\lambda = (1-\lambda)g_0 + \lambda g_1$, where $\lambda\in[0,1]$, and then $g_\lambda \in X$. For convenience, the macroscopic density, bulk velocity and temperature of g_λ are denoted by ρ_λ, u_λ and θ_λ respectively, and let $M_\lambda := M[g_\lambda]$. Thus,

$$
\begin{aligned}
|M_1 - M_0| &= \left| \int_0^1 \frac{d}{d\lambda} M_\lambda d\lambda \right| \leq \int_0^1 |\frac{d}{d\lambda} M_\lambda| d\lambda \\
&\leq C \int_0^1 \left(\frac{|\rho'_\lambda|}{\rho_\lambda} + \frac{|\theta'_\lambda|}{\theta_\lambda} + \frac{|v-u_\lambda||u'_\lambda|}{\theta_\lambda} + \frac{|v-u_\lambda|^2|\theta'_\lambda|}{\theta_\lambda^2} \right) M_\lambda d\lambda.
\end{aligned}
$$

Multiplying by $(1 + |x - \omega(t)v|^2)$ and integrating against velocity variable v, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |M_1 - M_0|(1 + |x - \omega(t)v|^2) dv \\
&\leq C \int_0^1 \left(\frac{|\rho'_\lambda|}{\rho_\lambda} + \frac{|\theta'_\lambda|}{\theta_\lambda} + \frac{|u'_\lambda|}{\theta_\lambda^{1/2}} \right) [1 + |x - \omega(t)u_\lambda|^2 + \omega(t)^2\theta_\lambda] d\lambda
\end{aligned}
\tag{3.20}
$$

by a tedious computation. Taking the derivative of λ on both sides of the following identities

$$
\begin{aligned}
\int_{\mathbb{R}^3} g_\lambda dv &= \rho_\lambda, \\
\int_{\mathbb{R}^3} (x - \omega(t)v) g_\lambda dv &= \rho_\lambda [x - \omega(t)u_\lambda], \\
\int_{\mathbb{R}^3} |x - \omega(t)v|^2 g_\lambda dv &= \rho_\lambda [|x - \omega(t)u_\lambda|^2 + 3\omega(t)^2\theta_\lambda],
\end{aligned}
$$

and using $g_\lambda = (1 - \lambda)g_0 + \lambda g_1$, we can obtain

$$
\begin{aligned}
|\rho'_\lambda| &= |\int_{\mathbb{R}^3} (g_1 - g_0) dv| \leq \int_{\mathbb{R}^3} (1 + |x - \omega(t)v|^2)|g_1 - g_0| dv, \\
|u'_\lambda| &= \left| \frac{(x - \omega(t)u_\lambda)(\rho_1 - \rho_0) - \int_{\mathbb{R}^3} (x - \omega(t)v)(g_1 - g_0) dv}{\omega(t)\rho_\lambda} \right| \\
&\leq \frac{1 + |x - \omega(t)u_\lambda|}{\omega(t)\rho_\lambda} \int_{\mathbb{R}^3} (1 + |x - \omega(t)v|^2)|g_1 - g_0| dv, \\
|\theta'_\lambda| &= 3^{-1}\omega(t)^{-2}\rho_\lambda^{-1} \Bigg| \int_{\mathbb{R}^3} |x - \omega(t)v|^2 (g_1 - g_0) dv - \rho'_\lambda [|x - \omega(t)u_\lambda|^2 \\
&\quad + 3\omega(t)^2\theta_\lambda] + 2\omega(t)\rho_\lambda (x - \omega(t)u_\lambda) u'_\lambda \Bigg| \\
&\leq \frac{1 + |x - \omega(t)u_\lambda|^2 + 2|x - \omega(t)u_\lambda| + 3\omega(t)^2\theta_\lambda}{3\omega(t)^2\rho_\lambda} \int_{\mathbb{R}^3} (1 + |x - \omega(t)v|^2)|g_1 - g_0| dv.
\end{aligned}
$$

Combining the above inequalities with (3.20) we have

$$\int_{\mathbb{R}^3} |M_1 - M_0|(1+|x-\omega(t)v|^2)dv$$
$$\le C\int_0^1 \frac{1+|x-\omega(t)u_\lambda|^4+\omega(t)^4\theta_\lambda^2}{\omega(t)^2\theta_\lambda}d\lambda \int_{\mathbb{R}^3}(1+|x-\omega(t)v|^2)|g_1-g_0|dv. \tag{3.21}$$

Note that $g_\lambda \in X$, by (3.16) we have

$$\begin{aligned}\theta_\lambda &= (1+|x|^r)\rho_\lambda\theta_\lambda \cdot \frac{1}{(1+|x|^r)\rho_\lambda}\\ &\le C(1+|x|^r)\rho_\lambda\theta_\lambda, \qquad (3.22)\\ |x-\omega(t)u_\lambda| &= (1+|x|^r)\rho_\lambda|x-\omega(t)u_\lambda| \cdot \frac{1}{(1+|x|^r)\rho_\lambda}\\ &\le C(1+|x|^r)\rho_\lambda|x-\omega(t)u_\lambda| \qquad (3.23)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\theta_\lambda} &= (1+|x|^r)^{2/3}\frac{\rho_\lambda^{2/3}}{\theta_\lambda}\cdot\frac{1}{(1+|x|^r)^{2/3}\rho_\lambda^{2/3}}\\ &\le C(1+|x|^r)^{2/3}\frac{\rho_\lambda^{2/3}}{\theta_\lambda}. \qquad (3.24)\end{aligned}$$

For the bound of (3.22), we use (3.8) with $p=\infty$, $a=2$ and (3.15) to obtain

$$\omega(t)^5(1+|x|^r)\rho_\lambda\theta_\lambda \le CH_{q,r}(g_\lambda(t)) \le Ce^{Ct}\big(H_{q+r}(f_0)+H_{q,r}(f_0)\big),$$

which means

$$\omega(t)^2\theta_\lambda(t,x) \le C_T \tag{3.25}$$

for any $t\in[0,T]$, since $\alpha>0$. For the estimate (3.23), from (3.9) with $p=\infty$, $\beta=0$, $\gamma=1$ and (3.15) we get

$$\omega(t)^3(1+|x|^r)\rho_\lambda|x-\omega(t)u_\lambda| \le CH_{q,r}(g_\lambda(t)) \le C_Te^{C_Tt}\big(H_{q+r}(f_0)+H_{q,r}(f_0)\big)$$

for any $t\in[0,T]$, which implies

$$|x-\omega(t)u_\lambda| \le C_T \tag{3.26}$$

for any $t\in[0,T]$. For the estimate of (3.24), by (3.1) with $p=\infty$ and (3.15) we have

$$(1+|x|^r)\rho_\lambda\theta_\lambda^{-\frac{3}{2}} \le CH_{q,r}(g_\lambda(t)) \le C_Te^{C_Tt}\big(H_{q+r}(f_0)+H_{q,r}(f_0)\big)$$

for any $t\in[0,T]$, so

$$\frac{1}{\theta_\lambda} \le C_T\omega(t)^2 \tag{3.27}$$

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for any $t \in [0, T]$. Combining (3.21) with (3.25)–(3.27), we have

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} |M[g_1] - M[g_0]|(1 + |x - \omega(t)v|^2)dvdx \\
&\le C_T \iint_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |x - \omega(t)v|^2)|g_1 - g_0|dvdx
\end{aligned} \tag{3.28}$$

for any $t \in [0, T]$.

Step 3. Lastly, we prove that $\mathcal{T}$ has a unique positive fixed point. Let $\bar{x} = x - y - \frac{1-e^{-\beta(t-s)}}{\beta}\eta - \frac{1-e^{-\beta(t-s)}}{\beta(1+e^{-\beta(t-s)})}(v - e^{-\beta(t-s)}\eta)$, $\bar{v} = v - e^{-\beta(t-s)}\eta$, we have

$$\begin{aligned}
\frac{1 + |x - \omega(t)v|^2}{1 + |y - \omega(s)\eta|^2} &\le \frac{1 + \left[|\bar{x}| + \left(\frac{1-e^{-\beta(t-s)}}{\beta(1+e^{-\beta(t-s)})} + \omega(t)\right)|\bar{v}| + |y - \omega(s)\eta|\right]^2}{1 + |y - \omega(s)\eta|^2} \\
&\le C_{\alpha,\beta,\sigma,t}(1 + |\bar{x}|^2 + |\bar{v}|^2)
\end{aligned}$$

for any $0 \le s \le t$. Combining it with (3.28) we obtain

$$\begin{aligned}
&\iint (1 + |x - \omega(t)v|^2)|G(t - s)M[g_1] - G(t - s)M[g_0]|(s, x, v)dxdv \\
&= \iint \frac{1 + |x - \omega(t)v|^2}{1 + |y - \omega(s)\eta|^2} G(t - s, x, v, y, \eta) \\
&\quad \cdot \iint |M[g_1] - M[g_0]|(s, y, \eta)(1 + |y - \omega(s)\eta|^2)dyd\eta dxdv \\
&\le C_{\alpha,\beta,\sigma,T} \iint (1 + |\bar{x}|^2 + |\bar{v}|^2)G_0(t - s, \bar{x} + \frac{1 - e^{-\beta(t-s)}}{\beta(1 + e^{-\beta(t-s)})}\bar{v}, \bar{v})d\bar{x}d\bar{v} \\
&\quad \cdot \iint |g_1 - g_0|(s, x, v)(1 + |x - \omega(s)v|^2)dxdv
\end{aligned}$$

for any $0 \le s \le t \le T$. Thus, there exists a positive constant L dependent of α, β, σ, T such that

$$\iint (1 + |x - \omega(t)v|^2)|\mathcal{T}g_1 - \mathcal{T}g_0|(t)dxdv \le L \int_0^t \iint (1 + |x - \omega(s)v|^2)|g_1 - g_0|(s)dxdvds,$$

which yields

$$\iint (1 + |x - \omega(t)v|^2)|\mathcal{T}^n g_1 - \mathcal{T}^n g_0|(t)dxdv \le \frac{L^n t^n}{n!} \sup_{s\in[0,t]} \iint (1 + |x - \omega(s)v|^2)|g_1 - g_0|(s)dxdv.$$

Choosing N large enough such that $\frac{L^N T^N}{N!} < 1$, we can obtain from the Banach fixed point theorem that the operator $\mathcal{T}^N$ has a positive and unique fixed point f in $L^\infty([0, T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3, (1 + |x - \omega(t)v|^2)dvdx))$. Note that $\mathcal{T}f$ is also a fix point of $\mathcal{T}^N$, we get from the uniqueness of $\mathcal{T}^N$ that f is the unique fixed point of $\mathcal{T}$. That is, f is the unique nonnegative solution to (1.1). Since $G(t)$ maps $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ into $C([0, \infty), L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^\infty((0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$, so f belongs to $C([0, \infty); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^\infty((0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ and then it is easy to verify the desired estimates. □

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Remark 3.2. If $r > 5$, it is obvious that the solution has finite energy. However, if we further assume $3 < r \le 5$, the above theorem can deal with some kinds of initial datum with infinite energy. For example, $f_0 = (1 + |x|^r)^{-1}(1 + |x - \alpha v|^{q+r})^{-1}$ satisfies all conditions of the theorem, but its energy is infinite. Consequently, the solution launched by the initial datum has infinite energy due to $\iint |v|^2 f(t)dxdv = e^{-2\beta t} \iint |v|^2 f_0 dxdv + 3\sigma \|f_0\|_1 \frac{1-e^{-2\beta t}}{\beta}$.

4. Some general existence results

In order to give our general existence results, we firstly use Theorem 3.4 to construct approximate solutions, and then use compactness arguments to take the limits. The key is how to get strong compactness of the macroscopic quantities. We will need not only Proposition 2.2 but also the following compactness of the Fokker–Planck operator, which was established by Diperna and Lions in [11].

Lemma 4.1. *For $n \in \mathbb{N}$, let $g^n \in L^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $f_0^n \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ such that*

$$\lim_{R\to\infty} \sup_n \int_0^T \iint_{|x|^2+|v|^2>R^2} |g^n(t,x,v)|dxdvdt = 0$$

and

$$\lim_{R\to\infty} \sup_n \iint_{|x|^2+|v|^2>R^2} |f_0^n(x,v)|dxdv = 0.$$

Suppose that f^n are solutions to

$$\begin{cases} \partial_t f^n + v \cdot \nabla_x f^n - \beta \mathrm{div}_v(v f^n) - \sigma \Delta_v f^n = g^n, \\ f^n(0,x,v) = f_0^n(x,v). \end{cases}$$

Then the sequence $\{f^n : n \in \mathbb{N}\}$ is relatively compact in $L^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Then, we can give the following existence results:

Theorem 4.2. *Let $1 < p \le \infty, \alpha > 0$ and let the initial datum $0 \le f_0 \in L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ be given with*

$$\iint \left(1 + |x - \alpha v|^2\right) f_0 dvdx < +\infty.$$

Then there exists a solution $0 \le f(t,x,v) \in C([0,\infty); L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^\infty((0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$ to the Cauchy problem (1.1) such that

$$\iint (1, x - \omega(t)v) f(t)dxdv = \iint (1, x - \omega(t)v) f_0 dxdv, \tag{4.1}$$

$$\iint |x - \omega(t)v|^2 f(t)dxdv = \iint |x - \omega(t)v|^2 f_0 dxdv + 6\sigma \int_0^t \omega(s)^2 ds \|f_0\|_1 \tag{4.2}$$

for any $t \ge 0$, and $\|f(t)\|_p < C_T$ for any $0 \le t \le T < \infty$.

Proof. Similar to those regulation in [29], we regularize the initial datum f_0 as follows:

$$f_0^n = \chi_{|x|^2+|v|^2\le n^2}\min\{f_0, n\} + \frac{1}{n}\frac{e^{-|x-\alpha v|}}{1+|x|^r},$$

where $r > 3$ and χ is the cutoff function. Then we immediately obtain

$$\|f_0^n\|_{p_1} \le \|f_0\|_{p_1} + C, \quad \forall\, p_1 \in [1, p], \tag{4.3}$$

$$\iint |x-\alpha v|^2 f_0^n(x, v)dvdx \le \iint |x-\alpha v|^2 f_0(x, v)dvdx + C, \tag{4.4}$$

and

$$\iint (1 + |x-\alpha v|^2)|f_0^n - f_0|dvdx + \|f_0^n - f_0\|_p \to 0, \quad n \to 0.$$

Note that $H_{q+r}(f_0^n), H_{q,r}(f_0^n) < \infty$ and $(1 + |x|^r)f_0^n \ge \frac{1}{ne}$ for $|x-\alpha v| \le 1$, so by Theorem 3.4 there is an unique global solution $f^n(t, x, v)$ to the following Cauchy problem

$$\begin{cases} \partial_t f^n + v\cdot\nabla_x f^n - \beta \mathrm{div}_v(vf^n) - \sigma\Delta_v f^n = M[f^n] - f^n, \\ f^n(0, x, v) = f_0^n(x, v), \end{cases} \tag{4.5}$$

where

$$M[f^n](t, x, v) = \frac{\rho^n(t, x)}{(2\pi\theta^n(t, x))^{\frac{3}{2}}}\exp\left\{-\frac{|v-u^n(t, x)|^2}{2\theta^n(t, x)}\right\} \tag{4.6}$$

and

$$\begin{pmatrix} \rho^n \\ \rho^n u^n \\ \rho^n|u^n|^2 + 3\rho^n\theta^n \end{pmatrix}(t, x) = \int\limits_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f^n(t, x, v)dv, \quad t \ge 0,\ x \in \mathbb{R}^3. \tag{4.7}$$

By Lemma 2.3 and (4.4), we have

$$\iint (1 + |x-\omega(t)v|^2)f^n(t, x, v)dvdx \le C_T, \quad \forall\, 0 \le t \le T < \infty. \tag{4.8}$$

Now we show

$$\|f^n(t)\|_p \le C_T, \qquad \forall\, 0 \le t \le T < \infty. \tag{4.9}$$

A mild solution of the Cauchy problem (4.5) can be represented by

$$f^n(t, x, v) = e^{-t}G(t)f_0^n(x, v) + \int\limits_0^t e^{s-t}G(t-s)M[f^n](s, x, v)ds.$$

Using (2.2) in Lemma 2.1 and (3.6) with $q = 0$ we can obtain

$$
\begin{aligned}
\|f^n(t)\|_p &\le e^{-t}\|G(t)f_0^n\|_p + \int_0^t e^{s-t}\|G(t-s)M[f^n](s)\|_p ds \\
&\le C_T e^{-t}\|f_0^n\|_p + C_T \int_0^t e^{s-t}\|f^n(s)\|_p ds.
\end{aligned}
$$

From Gronwall's inequality and (4.3) we obtain (4.9). Then using (3.6) with $q = 0$ we can deduce

$$
\|M[f^n(t)]\|_p \le C\|f^n(t)\|_p \le C_T, \quad \forall\ 0 \le t \le T < \infty, \tag{4.10}
$$

and from (4.6), (4.7), (4.8) we have

$$
\iint (1 + |x - \omega(t)v|^2) M[f^n](t) dvdx = \iint (1 + |x - \omega(t)v|^2) f^n(t) dvdx \le C_T \tag{4.11}
$$

for any $0 \le t \le T < \infty$. Combining (4.10) with (4.11), we get that the sequence $M[f^n]$ is weak compact in $L^1([0,T] \times B_R \times \mathbb{R}_v^3)$ for any $R > 0$.

On the one hand, by (4.8), (4.11) and a sight changed version of Lemma 4.1 as stated in [28] we can get that the sequence f^n is compact in $L^1([0,T] \times B_R \times \mathbb{R}_v^3)$. On the other hand, from (4.8), (4.11) and Lemma 2.3 we have

$$
\int_0^T \iint_{B_R \times \mathbb{R}_v^3} |x - \omega(t)v|^3 f^n(t,x,v) dxdvdt \le C(T,R).
$$

The above arguments give

$$
\int_{\mathbb{R}^3} \big(1, x - \omega(t)v, |x - \omega(t)v|^2\big) f^n dv \to \int_{\mathbb{R}^3} \big(1, x - \omega(t)v, |x - \omega(t)v|^2\big) f dv \quad \text{in} \quad L^1([0,T] \times B_R),
$$

where f is the limit of f^n in $L^1([0,T] \times B_R \times \mathbb{R}_v^3)$ (choosing a subsequence if necessary). As a consequence,

$$
\begin{aligned}
\rho^n &\to \rho, &&\text{in } L^1([0,T] \times B_R); \\
\rho^n u^n &\to \rho u, &&\text{in } L^1([0,T] \times B_R); \\
\rho^n |u^n|^2 + 3\theta^n &\to \rho |u|^2 + 3\theta, &&\text{in } L^1([0,T] \times B_R).
\end{aligned}
$$

Combining the above arguments with the weak compactness of $M[f^n]$ in $L^1([0,T] \times B_R \times \mathbb{R}_v^3)$, we can use the standard procedure developed in [21] to show

$$
M[f^n] \to M[f], \quad \text{in } L^1([0,T] \times B_R \times \mathbb{R}_v^3),
$$

which yields that the sequence $M[f^n]$ converges to $M[f]$ in the distributional sense.

Finally, passing to the limits in the approximate equation (4.5) for $n \to \infty$, we know that $f(t,x,v)$ is a distributional solution of the Cauchy problem (1.1). It is easy to show the smoothness of f and verify the desired conservation laws. □

Theorem 4.3. *Let the initial datum $f_0 \ge 0$ be given with*

$$
\iint \big(1 + |x - \alpha v|^2 + \ln(1 + |x|) + |\ln f_0|\big) f_0 dvdx < +\infty,
$$

where $\alpha > 0$. *Then there exists a solution* $0 \le f(t,x,v) \in C(\mathbb{R}^+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ *to the Cauchy problem (1.1) such that for any* $T > 0$

$$\sup_{t\in[0,T]} \iint \big(1 + |x - \omega(t)v|^2 + \ln(1+|x|) + |\ln f(t,x,v)|\big) f(t,x,v)dvdx \le C_T.$$

Furthermore, (4.1), (4.2) hold and

$$\iint f(t)\ln f(t)dxdv + 3\beta\|f_0\|_1 t + 2\sigma \int_0^t \iint |\nabla_v \sqrt{f(s)}|^2 dxdvds \le \iint f_0 \ln f_0 dxdv.$$

Proof. Similar to those regulation in [11,12], we regularize the initial datum f_0 as follows:

$$f_0^n = \eta_{\delta_n} * (\chi_{|x|^2+|v|^2\le n^2} f_0) + \frac{1}{n}\frac{e^{-|x-\alpha v|}}{1+|x|^r},$$

where $r > 3$ and η is the mollifier. We can choose δ_n such that

$$\iint (1 + \ln(1+|x|) + |x-\alpha v|^2)|f_0^n - f_0|dvdx \to 0, \quad n \to 0 \tag{4.12}$$

and

$$\iint f_0^n |\ln f_0^n| dxdv \le C. \tag{4.13}$$

Furthermore, by classical arguments we can get

$$\lim_{n\to\infty} \iint f_0^n \ln f_0^n dxdv = \iint f_0 \ln f_0 dxdv.$$

Note that $H_{q+r}(f_0^n), H_{q,r}(f_0^n) < \infty$ and $(1+|x|^r)f_0^n \ge \frac{1}{ne}$ for $|x-\alpha v| \le 1$, so by Theorem 3.4 there is an unique global solution $f^n(t,x,v) \in C([0,T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ to (4.5)–(4.7). Now, we try to show that for any $T > 0$, there exists a positive constant C_T such that

$$\iint (1 + \ln(1+|x|) + |x-\omega(t)v|^2 + |\ln f^n(t)|) f^n(t) dxdv \le C_T, \quad \forall\, t \in [0,T]. \tag{4.14}$$

Firstly, by Lemma 2.3 and (4.12), we have

$$\iint (1 + |x-\omega(t)v|^2) f^n(t,x,v)dvdx \le C_T, \qquad \forall\ 0 \le t \le T. \tag{4.15}$$

Secondly, we show

$$\iint \ln(1+|x|) f^n(t,x,v)dxdv \le C_T, \qquad \forall\ 0 \le t \le T. \tag{4.16}$$

From (4.5)–(4.7) we have

$$\frac{d}{dt}\iint \ln(1+|x|) f^n(t,x,v)dxdv = \iint (1+|x|)^{-1} \frac{x\cdot v}{|x|} f^n dxdv$$

$$\leq \omega(t)^{-1} \iint (1+|x|)^{-1}(|x| + |x - \omega(t)v|) f^n dxdv$$
$$\leq \omega(t)^{-1} \left(\|f^n(t)\|_1 + \iint |x - \omega(t)v| f^n dxdv \right),$$

then by (4.12) we get (4.16).

Lastly, we prove the uniform boundedness of entropy of f^n. A classical computation gives

$$\frac{d}{dt} \iint f^n(t) \ln f^n(t) dxdv + 3\beta \|f^n(t)\|_1 + 2\sigma \iint |\nabla_v \sqrt{f^n(t)}|^2 dxdv$$
$$= -\iint (M[f^n] - f^n)(\ln M[f^n] - \ln f^n) dxdv \leq 0,$$

which implies

$$\iint f^n(t) \ln f^n(t) dxdv \leq \iint f_0^n \ln f_0^n dxdv \leq C.$$

Following from (4.15) and (4.16), we can get the uniformly boundedness of $\iint f^n(t) \ln^- f^n(t) dxdv$ for $0 \leq t \leq T$, since

$$\begin{aligned}
&\iint f^n(t) \ln^- f^n(t) dxdv \\
&= \iint_{f^n(t) \leq e^{-8\ln(1+|x|) - |x-\omega(t)v|^2}} f^n(t) \ln \frac{1}{f^n(t)} dxdv \\
&\quad + \iint_{e^{-8\ln(1+|x|) - |x-\omega(t)v|^2} < f^n(t) < 1} f^n(t) \ln \frac{1}{f^n(t)} dxdv \\
&\leq \iint_{f^n(t) \leq e^{-8\ln(1+|x|) - |x-\omega(t)v|^2}} \sqrt{f^n(t)} dxdv \\
&\quad + \iint_{e^{-8\ln(1+|x|) - |x-\omega(t)v|^2} < f^n(t) < 1} (8\ln(1+|x|) + |x - \omega(t)v|^2) f^n(t) dxdv \\
&\leq \iint \frac{e^{-\frac{1}{2}|x-\omega(t)v|^2}}{1+|x|^4} dxdv + \iint (8\ln(1+|x|) + |x-\omega(t)v|^2) f^n(t) dxdv.
\end{aligned} \tag{4.17}$$

Thus, there exists a positive constant C_T such that

$$\begin{aligned}
&\iint f^n(t) |\ln f^n(t)| dxdv \\
&\leq \iint f^n(t) \ln f^n(t) dxdv + 2 \iint f^n(t) \ln^- f^n(t) dxdv \leq C_T
\end{aligned} \tag{4.18}$$

for any $t \in [0, T]$. As a consequence, using (4.15)–(4.18) we obtain (4.14).

For proving that the sequence $M[f^n]$ is weakly compact in $L^1([0,T] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$, we show

$$\iint (1 + \ln(1+|x|) + |x - \omega(t)v|^2 + |\ln M[f^n](t)|) M[f^n](t) dxdv \leq C_T, \quad \forall\, t \in [0,T]. \tag{4.19}$$

By (4.6), (4.7) and (4.14) we can easily get

$$\begin{aligned}&\iint (1+\ln(1+|x|)+|x-\omega(t)v|^2)M[f^n](t)dxdv\\ &=\iint (1+\ln(1+|x|)+|x-\omega(t)v|^2)f^n(t)dxdv \le C_T\end{aligned} \tag{4.20}$$

for any $t \in [0,T]$. And from Gibbs's lemma we have

$$\begin{aligned}&\iint |\ln M[f^n](t)|M[f^n](t)dxdv\\ &=\iint \ln M[f^n](t)M[f^n](t)dxdv + 2\iint \ln^- M[f^n](t)M[f^n](t)dxdv\\ &\le \iint f^n(t)\ln f^n(t)dxdv + 2\iint \ln^- M[f^n](t)M[f^n](t)dxdv\\ &\le C_T + 2\iint \ln^- M[f^n](t)M[f^n](t)dxdv\end{aligned}$$

for any $t \in [0,T]$. Following from (4.20) and the proof of (4.17), we can obtain

$$\iint \ln^- M[f^n](t)M[f^n](t)dxdv \le C_T, \quad \forall\, t \in [0,T].$$

Thus, we have the uniform boundedness of $\iint |\ln M[f^n](t)|M[f^n](t)dxdv$ on $[0,T]$. Combining it with (4.20) we obtain (4.19).

Using (4.14), (4.19) and Lemma 4.1, we can get that the sequence f^n is compact in $L^1([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$. According to the weak compactness of the sequence $M[f^n]$ in $L^1([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$ and (2.11) in Lemma 2.3, we can totally repeat the remaining steps in the above theorem and deduce the desired results. □

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