J. Math. Anal. Appl. 438 (2016) 759–771

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Property T of group homomorphisms ☆

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ARTICLE INFO

Article history:
Received 16 September 2014
Available online 12 February 2016
Submitted by K. Jarosz

Keywords:
Locally compact groups
Property T
Strong property T

ABSTRACT

In this article, we define and study property T for continuous homomorphisms between topological groups. If G is a locally compact group, we show that $\lambda_G : G \to U(L^2(G))$ has property T if and only if either G is compact or G is non-amenable. Moreover, the abelianization $G^{\mathrm{ab}} := G/\overline{[G,G]}$ is compact if and only if every continuous homomorphism from G to any abelian topological group has property T. Moreover, we show that G has property (T, FD) if and only if any continuous homomorphism from G to any compact group has property T. In the case when G is almost connected, the above is also equivalent to the canonical map from G to its Bohr compactification being a quotient map. We also give some new equivalent forms of the strong property T of a locally compact group. As a consequence, if G is a second countable and has strong property T and H is a closed subgroup of G, there exist at most one G-invariant mean on G/H.

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1. Introduction and notations

Motivated by the notion of *strong property* T of a topological group G (namely, there exists a Kazhdan pair for G with the compact set being finite), we consider in this article property T of a continuous homomorphism $\varphi : H \to G$ between topological groups. This notion is slight stronger than the relative property T of $(G, \overline{\varphi(H)})$, and coincides with the strong property T of G when H is the group G equipped with the discrete topology. We show in Proposition 2.4 that if G is a locally compact group and λ_G is its left regular representation, then $\lambda_G : G \to U(L^2(G))$ has property T (when $U(L^2(G))$ is equipped with the SOT) if and only if either G is compact or G is non-amenable. In the case when H and G are locally compact and $\varphi(H)$ is dense in G, we give in Theorem 2.9 several equivalent forms for the property T of φ, which are analogues of the property T of a locally compact group. Moreover, we will consider three special cases.

We first study the situation when the range group is abelian. In this case, we have the following result (see Theorem 3.2).

☆ This work is supported by the National Natural Science Foundation of China (11071126 and 11471168).
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http://dx.doi.org/10.1016/j.jmaa.2016.02.026
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Theorem 1.1. *Let G be a locally compact group.*

(a) *The abelianization $G^{\mathrm{ab}} := G/\overline{[G,G]}$ is compact if and only if the canonical homomorphism $\beta_G : G \to G^{\mathrm{ab}}$ has property T.*
(b) *If G is second countable and connected, the above are also equivalent to G having property (T_{ℓ_p}) (in the sense of [3]) for some (and equivalently, for all) $p \in (1,\infty) \setminus \{2\}$.*

Secondly, we consider the case when the range group is compact, and obtain the following result (see Theorem 3.6 and Corollary 3.7).

Theorem 1.2. *Let G be a locally compact group and $(\mathrm{b}G, \sigma_G)$ be its Bohr compactification.*

(a) *The following statements are equivalent.*
   1) *G has property (T, FD) (in the sense of [10]).*
   2) *$\sigma_G : G \to \mathrm{b}G$ has property T.*
   3) *Any continuous homomorphism from G to any compact group has property T.*
(b) *If G is almost connected, the above are also equivalent to the following two statements.*
   4) *σ_G is a quotient map from G onto $\mathrm{b}G$.*
   5) *For any continuous homomorphism φ from G to any compact group, one has $\varphi(H)$ being compact and $\varphi : H \to \varphi(H)$ being a quotient map.*
(c) *Suppose that G is almost connected. Then G has property T if and only if G has property (T, FD) and $\ker \sigma_G$ has property T.*

Finally, we consider the case when the domain group is discrete. This situation is closely relation to the strong property T of G, and we obtain the following equivalent formulations for strong property T (see Corollary 3.12).

Theorem 1.3. *The following statement are equivalent for a locally compact group G.*

1) *G has strong property T (in the sense of [13]).*
2) *G has property (T, FD), and for any continuous action α of G on a von Neumann algebra N without α-invariant normal state, there is no α-invariant state.*
3) *G has property (T, FD) and for any $\epsilon > 0$, one can find a finite subset $Q \subseteq G$ and $\delta > 0$ such that if α is a continuous action of G on a von Neumann algebra N and ω is a normal state on N satisfying $\|\omega \circ \alpha_t - \omega\| < \delta$ for any $t \in Q$, there exists an α-invariant normal state τ with $\|\omega - \tau\| < \epsilon$.*

Note that (Q, δ) in the above can be viewed as a kind of Kazhdan pair for actions (on von Neumann algebras). This theorem produces the following corollary (see Corollary 3.13).

Corollary 1.4. *Let G be a σ-compact locally compact group with strong property T, and $H \subseteq G$ be a closed subgroup. There exists at most one G-invariant mean on G/H.*

Let us fix some notations. Throughout this article, G and H are Hausdorff topological groups and $\varphi : H \to G$ is a continuous group homomorphism. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a Hilbert space $\mathcal{H}$, we denote by $\mathfrak{S}_1(\mathcal{H})$ and $U(\mathcal{H})$ the set of unit vectors in $\mathcal{H}$ and the group of unitary operators on $\mathcal{H}$ respectively. We let $\widehat{G}$ (respectively, $\mathrm{Rep}(G)$) be the collection of unitary equivalence classes of continuous irreducible (respectively, continuous) unitary representations of G and use the notation $\widehat{G}_{\mathrm{FD}}$ to denote the set of all finite dimensional irreducible representations. We equip subsets of $\mathrm{Rep}(G)$ with the Fell topology. For any

$(\pi, \mathcal{H}), (\mu, \mathcal{K}) \in \mathrm{Rep}(G)$, we denote $\mu \subseteq \pi$ if there exists an isometry $\Psi : \mathcal{K} \to \mathcal{H}$ such that $\Psi \circ \mu_t = \pi_t \circ \Psi$ for any $t \in G$. On the other hand, we denote $\mu \prec \pi$ if for any $\xi \in \mathcal{K}$, compact subset $K \subseteq G$ and $\epsilon > 0$, there exists a finite subset $F \subseteq \mathcal{H}$ such that $\sup_{t\in K} \left|\langle \mu_t(\xi), \xi\rangle - \sum_{\eta\in F}\langle \pi_t(\eta), \eta\rangle\right| < \epsilon$. Moreover, we write $\mu \sim \pi$ if $\mu \prec \pi$ and $\pi \prec \mu$. Furthermore, we put $\operatorname{supp}\pi := \{\mu \in \widehat{G} : \mu \prec \pi\}$. We recall that if $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$, then a net $\{\xi_i\}_{i\in I}$ in $\mathfrak{S}_1(\mathcal{H})$ is an *almost π-invariant unit vector* if $\sup_{t\in K} \|\pi_t(\xi_i) - \xi_i\| \to 0$, for any compact subset $K \subseteq G$.

2. Property T of continuous group homomorphism

Definition 2.1. A continuous group homomorphism $\varphi : H \to G$ (or simply φ, when G and H are understood) is said to have *property T* if there exist a compact subset $C \subseteq H$ and $\kappa > 0$ such that for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$, whenever

$$V_\pi(\varphi(C), \kappa) \;:=\; \{\xi \in \mathfrak{S}_1(\mathcal{H}) : \|\pi_{\varphi(t)}\xi - \xi\| < \kappa, \forall t \in C\} \;\neq\; \emptyset,$$

one has $\mathcal{H}^{\pi\circ\varphi} := \{\xi \in \mathcal{H} : \pi_{\varphi(s)}\xi = \xi, \forall s \in H\} \neq (0)$. In this case, (C, κ) is called a *Kazhdan pair* for φ.

The following are some strict-forward facts.

Lemma 2.2. *Let F, G and H be topological groups and let $\varphi : H \to G$ and $\psi : G \to F$ be continuous homomorphism.*

(a) *If φ has property T, then so does $\psi \circ \varphi$.*
(b) *If $\overline{\varphi(H)} = G$ and $\psi \circ \varphi$ has property T, then so does ψ.*
(c) *If H is locally compact, $\varphi : H \to G$ is a quotient map and G has property T, then φ has property T.*
(d) *If H has property T, then φ has property T.*
(e) *If φ has property T, then $(G, \overline{\varphi(H)})$ has relative property T.*
(f) *φ have property T if and only if for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$, the existence of an almost $\pi \circ \varphi$-invariant unit vector in $\mathcal{H}$ will imply $\mathcal{H}^{\pi\circ\varphi} \neq \{0\}$; in other words, $1_H \prec \pi \circ \varphi$ will imply $1_H \subseteq \pi \circ \varphi$.*
(g) *If H is discrete, $\overline{\varphi(H)} = G$ and φ have property T, then G have strong property T (in the sense of [13, Definition 5.2]). Furthermore, if $H = G_d$ (i.e. H is the group G equipped with the discrete topology) and φ is the identity map, then φ have property T if and only if G have strong property T.*

We recall that $(\phi, \mathcal{H}) \in \mathrm{Rep}(H)$ is said to have a *spectral gap* if there is no almost ϕ-invariant unit vector in $(\mathcal{H}^\phi)^\perp$, where $\mathcal{H}^\phi$ is the subspace of all ϕ-invariant vectors. Notice that Lemma 2.2(f) is weaker than the property that $\pi \circ \varphi$ has a spectral gap for every $\pi \in \mathrm{Rep}(G)$, since $\mathcal{H}^{\pi\circ\varphi}$ needs not be a π-invariant subspace of $\mathcal{H}$ (see Theorem 2.9 below).

The following are some other easy facts, which are not needed in this article.

- If G and H are locally compact with H being amenable, $\overline{\varphi(H)} = G$ and φ has property T, then G is compact.
- If G is a locally compact amenable group and φ has property T, then $m(\varphi(H)) < \infty$, where m is the Haar measure on G. In fact, since $1_H \prec \lambda_G \circ \varphi$ (where λ_G is the left regular representation), the T-property of φ produces a unit vector $\xi \in L^2(G)$ such that for any $t \in H$, one has $\xi(\varphi(t)^{-1}s) = \xi(s)$ for almost all $s \in G$, which gives the conclusion.

From now on, $\widehat{\varphi} : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ is the correspondence given by $\pi \mapsto \pi \circ \varphi$. For any set $X \subseteq \mathrm{Rep}(G)$, we put

$$X^\varphi \;:=\; \{\pi \in X : 1_H \nsubseteq \pi \circ \varphi\}.$$

Lemma 2.3.

(a) *φ has property T if and only if 1_H is an isolated point in $\widehat{\varphi}(X^\varphi) \cup \{1_H\}$ for any subset $X \subseteq \mathrm{Rep}(G)$.*
(b) *Suppose that H is locally compact and every object in $\mathrm{Rep}(G)$ is a direct sum of elements in $\widehat{G}$. Then φ has property T if and only if 1_H is an isolated point in $\widehat{\varphi}(\widehat{G}^\varphi) \cup \{1_H\}$.*

Proof. (a) $\Rightarrow$). Suppose on the contrary that 1_H is not isolated in $\widehat{\varphi}(X^\varphi) \cup \{1_H\}$ for a subset $X \subseteq \mathrm{Rep}(G)$. Let $(\pi, \mathcal{H}) := \bigoplus_{(\mu,\mathcal{H}_\mu)\in X^\varphi}(\mu, \mathcal{H}_\mu)$. Then there is a net $\{\xi_i\}_{i\in I}$ in $\mathfrak{S}_1(\mathcal{H})$ such that $\sup_{s\in K} \|\pi_{\varphi(s)}\xi_i - \xi_i\| \to 0$ for any compact subset $K \subseteq H$. Hence Lemma 2.2(f) produces an element $\eta \in \mathfrak{S}_1(\mathcal{H})$ satisfying $\pi_{\varphi(s)}\eta = \eta$ for any $s \in H$. This gives the contradiction that $1_H \subseteq \mu \circ \varphi$ for some $\mu \in X^\varphi$.
$\Leftarrow$). Suppose that $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ satisfying $1_H \nsubseteq \pi \circ \varphi$. By the hypothesis, 1_H is an isolated point in $\{1_H, \pi\circ\varphi\}$, which means that there does not exist an almost $\pi\circ\varphi$-invariant unit vector in $\mathcal{H}$. Consequently, φ has property T because of Lemma 2.2(f).

(b) By part (a), we only need to show the sufficiency. Suppose that $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ such that $\mathcal{H}^{\pi\circ\varphi} = (0)$. Let $S(\pi) := \{\mu \in \widehat{G} : \mu \subseteq \pi\}$. By the hypothesis, $\pi \sim \bigoplus_{\mu\in S(\pi)} \mu$. Moreover, as $S(\pi) \subseteq \widehat{G}^\varphi$, we know that $1_H \notin \overline{\widehat{\varphi}(S(\pi))}$. These imply that $1_H \not\prec \bigoplus_{\mu\in S(\pi)} \mu \circ \varphi \sim \pi \circ \varphi$ (note that H is locally compact), and Lemma 2.2(f) tells us that φ has property T. $\square$

Part (b) above applies to the case when either G is a compact group or $G = U(\ell^2)$ equipped with the SOT (notice that we do not assume that φ has dense range).

Let us recall some well-known facts on $U(\ell^2)$ (see e.g. [11]). As $U(\ell^2)$ (equipped with SOT) is separable and metrizable, it is second countable. Any representation of $U(\ell^2)$ is a direct sum of separable representations and hence is a direct sum of irreducible representations. Any irreducible representation of $U(\ell^2)$ is a subrepresentation of some $\rho^{\otimes k} \otimes \bar{\rho}^{\otimes l}$, where $k, l \in \mathbb{N}_0$ and $\rho : U(\ell^2) \to \mathcal{B}(\ell^2)$ is the inclusion.

On the other hand, if λ_H is the left regular representation of H and $(\pi, \mathcal{H}) \in \mathrm{Rep}(H)$, then by considering $u \in U(L^2(H; \mathcal{H}))$ given by $u\xi(t) := \pi(t)\xi(t)$, one can show that $\lambda_H \otimes \pi$ is unitary equivalent to $\lambda_H \otimes \mathrm{id}_\mathcal{H}$. This fact is called the Fell absorption principle in some literature (e.g. [4]). As a consequence,

$$\bigoplus_{\substack{k,l\in\mathbb{N}_0 \\ k+l\geq 1}} \lambda_H^{\otimes k} \otimes \bar{\lambda}_H^{\otimes l} \;\cong\; \lambda_H^\alpha \tag{2.1}$$

for some non-zero cardinal α (note that $\bar{\lambda}_H \cong \lambda_H$).

Proposition 2.4. *Let H be a second countable locally compact group. The following statements are equivalence.*

L1) *$\lambda_H : H \to U(L^2(G))$ has property T.*
L2) *$\lambda_H \otimes \bar{\lambda}_H : H \to U(L^2(G) \otimes \bar{L}^2(G))$ has property T.*
L3) *Either H is compact or H is non-amenable.*

Proof. L1) $\Rightarrow$ L2). Consider the homomorphism $\Psi : U(L^2(G)) \to U(L^2(G)\otimes\bar{L}^2(G))$ given by $\Psi(u) = u\otimes\bar{u}^*$. It is not hard to check that Ψ is SOT-continuous. As $\lambda_G \otimes \bar{\lambda}_G = \Psi \circ \lambda_G$, the implication follows from Lemma 2.2(a).

L2) $\Rightarrow$ L3). Suppose that H is amenable. Then $1_H \prec \lambda_H \otimes \bar{\lambda}_H = \rho \circ (\lambda_H \otimes \bar{\lambda}_H)$ by [1, Theorem 2.2] (where ρ is the canonical injection as above). Now, Lemma 2.2(d) and the absorption principle implies that $1_H \subseteq \lambda_H \otimes \mathrm{id}$ and hence $1_H \subseteq \lambda_H$ which means that H is compact.

L3) $\Rightarrow$ L1). If H is compact, then Lemma 2.2(d) implies that λ_H has property T. On the other hand, suppose that on the contrary that H is non-amenable but λ_H does not have property T. By Lemma 2.2(f), there exists $(\pi, \mathcal{H}) \in \mathrm{Rep}(U(L^2(H)))$ satisfying $1_H \prec \pi \circ \lambda_H$ and $1_H \not\subseteq \pi \circ \lambda_H$. Let $\pi = \bigoplus_{i\in I} \mu_i$ with $\mu_i \in U(\widehat{L^2(H)})$ $(i \in I)$. For every $i \in I$, since $\mu_i \circ \lambda_H \neq 1_H$, one can find $k_i, l_i \in \mathbb{N}_0$ with $k_i + l_i \geq 1$ such that $\mu_i \subseteq \rho^{\otimes k_i} \otimes \bar{\rho}^{\otimes l_i}$. Thus, one can find a cardinal β with $\pi \subseteq \left(\bigoplus_{\substack{k,l\in\mathbb{N}_0\\ k+l\geq 1}} \rho^{\otimes k} \otimes \bar{\rho}^{\otimes l}\right)^{\beta}$. Now, (2.1) tells us that $1_H \prec \lambda_H^{\alpha\beta}$ and hence $1_H \prec \lambda_H$, which is a contradiction. □

Proposition 2.5. *Let $(\psi, \mathcal{H}) \in \mathrm{Rep}(H)$.*

(a) *$\psi : H \to U(\mathcal{H}^\psi) \oplus U((\mathcal{H}^\psi)^\perp)$ has property T if and only if the induced homomorphism from $\phi : H \to U((\mathcal{H}^\psi)^\perp)$ has property T. In this case, ψ has a spectral gap.*

(b) *$\varphi : H \to G$ has property T if and only if for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ with $\mathcal{H}^{\pi\circ\varphi} = (0)$, the homomorphism $\pi \circ \varphi : H \to U(\mathcal{H})$ has property T.*

(c) *Suppose that $\psi : H \to U(\mathcal{H})$ has property T. Then ψ is an amenable representation (in the sense of [1]) if and only if ψ contains a finite dimensional subrepresentation.*

Proof. (a) Let $\mathcal{H}_1 := (\mathcal{H}^\psi)^\perp$. Consider $P : U(\mathcal{H}^\psi) \oplus U(\mathcal{H}_1) \to U(\mathcal{H}_1)$ to be the projection onto the second coordinate and $J : U(\mathcal{H}_1) \to U(\mathcal{H}^\psi) \oplus U(\mathcal{H}_1)$ to be the injection that sends $u \in U(\mathcal{H}_1)$ to $u \oplus 1$. Since $\phi = P \circ \psi$ and $\psi = J \circ \phi$, Lemma 2.2(a) gives the first statement. The second statement follows from the first one and Lemma 2.2(f).

(b) Suppose that φ does not have property T. Then Lemma 2.2(f) produces a representation $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ with $1_H \prec \pi\circ\varphi$ and $\mathcal{H}^{\pi\circ\varphi} = (0)$. As $\pi\circ\varphi$ does not have a spectral gap, part (a) tells us that $\pi\circ\varphi : H \to U(\mathcal{H})$ does not have property T. The converse follows from Lemma 2.2(a).

(c) Clearly, ψ is amenable if it contains a finite dimensional subrepresentation. Conversely, suppose that $1_H \prec \psi \otimes \bar{\psi}$. If $\pi : U(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \bar{\mathcal{H}})$ is the unitary representation given by $\pi(u) := u \otimes \bar{u}^*$, then $1_H \prec \pi \circ \psi$ which implies that $1_H \subseteq \psi \otimes \bar{\psi}$ (by Lemma 2.2(f)) and ψ contains a finite dimensional subrepresentation (see e.g. [2, Proposition A.1.12]). □

Note, however, that even one dimensional representation with dense range need not have property T (see part (b) of the following).

Example 2.6. For any $\theta \in \mathbb{R}_+$, we define $(\phi_\theta, \mathbb{C}) \in \mathrm{Rep}(\mathbb{Z})$ by $\phi_\theta(k) := e^{k\theta\pi \mathrm{i}}$ $(k \in \mathbb{Z})$. Clearly, $\widehat{\phi_\theta}(\widehat{U(1)}) \subseteq \widehat{\mathbb{Z}}$. In fact, $\widehat{\phi_\theta}(\widehat{U(1)})$ can be identified with $\{e^{n\theta\pi\mathrm{i}} : n \in \mathbb{Z}\} \subseteq U(1)$.

(a) If θ is rational, then $\widehat{\phi_\theta}(\widehat{U(1)})$ is a finite set and hence $\phi_\theta : \mathbb{Z} \to U(1)$ has property T because of Lemma 2.3(b) (one may also obtain this fact by a more direct argument).

(b) If θ is irrational, then $\widehat{\phi_\theta}(\widehat{U(1)})$ is dense in $U(1)$ and $\phi_\theta : \mathbb{Z} \to U(1)$ does not have property T because of Lemma 2.3(b).

(c) Suppose that θ is irrational. Consider $(\psi, \mathbb{C}^2) \in \mathrm{Rep}(\mathbb{Z})$ defined by

$$\psi(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{k\theta\pi\mathrm{i}} \end{pmatrix} \qquad (k \in \mathbb{Z}).$$

Part (b) and Proposition 2.5(a) tell us that $\psi : \mathbb{Z} \to U(1) \oplus U(1)$ does not have property T.

In the remainder of this section, we mainly concern with the more manageable situation when $\overline{\varphi(H)} = G$. Let us start with the following result, which follows from the argument of [2, Proposition 1.1.9] (observe that $\mathcal{H}^{\pi\circ\varphi} = \mathcal{H}^\pi$ for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$, when $\overline{\varphi(H)} = G$).

Proposition 2.7. *Suppose that $\overline{\varphi(H)} = G$. If (C, κ) is a Kazhdan pair for φ, then for any $\alpha > 0$, $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ and $\xi \in V_\pi(\varphi(C), \alpha\kappa)$, one has $\|\xi - P^\pi\xi\| \le \alpha$, where $P^\pi : \mathcal{H} \to \mathcal{H}^\pi$ is the orthogonal projection. Consequently, if φ has property T, then $\|\xi_i - P^\pi(\xi_i)\| \to 0$ for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ and any almost $\pi \circ \varphi$-invariant unit vector $\{\xi_i\}_{i\in I}$ in $\mathcal{H}$.*

Notice also that if $\overline{\varphi(H)} = G$, then $\widehat{\varphi}^{-1}(1_H) = \{1_G\}$ and $\widehat{\varphi}$ restricts to a continuous injection from $\widehat{G}$ to $\widehat{H}$. Thus, $\widehat{G}^\varphi = \widehat{G} \setminus \{1_G\}$.

Lemma 2.8. *Suppose that G is locally compact and $\overline{\varphi(H)} = G$.*

(a) *For any countable subset $X \subseteq \widehat{G}_{\mathrm{FD}}$, the set $\widehat{\varphi}(X)$ is open in $\widehat{\varphi}(\widehat{G})$ if and only if for any $\pi \in \mathrm{Rep}(G)$ with $\widehat{\varphi}(X) \cap \overline{\widehat{\varphi}(\operatorname{supp}\pi)} \neq \emptyset$, there exists $\mu \in X$ with $\mu \subseteq \pi$.*
(b) *Suppose that H is locally compact as well. If $\mu \in \widehat{G}_{\mathrm{FD}}$, then $\widehat{\varphi}(\mu)$ is an isolated point in $\widehat{\varphi}(\widehat{G})$ if and only if for any $\pi \in \mathrm{Rep}(G)$ with $\widehat{\varphi}(\mu) \prec \widehat{\varphi}(\pi)$, one has $\mu \subseteq \pi$.*

Proof. (a) $\Rightarrow$). Since $\widehat{\varphi}(X)$ is open in $\widehat{\varphi}(\widehat{G})$, the assumption on π implies that $\widehat{\varphi}(X) \cap \widehat{\varphi}(\operatorname{supp}\pi) \neq \emptyset$. As $\widehat{\varphi}$ is a continuous injection, we know that X is open in $\widehat{G}$ and $X \cap \operatorname{supp}\pi \neq \emptyset$. Now, we can apply [15, Theorem 1.8] to obtain $\mu \in X$ satisfying $\mu \subseteq \pi$.
$\Leftarrow$). Suppose on the contrary that $\widehat{\varphi}(X)$ is not open in $\widehat{\varphi}(\widehat{G})$. Then $\widehat{\varphi}(X) \cap \overline{\widehat{\varphi}(\widehat{G} \setminus X)} \neq \emptyset$. Let $\pi := \bigoplus_{\nu \in \widehat{G}\setminus X} \nu$. As $\widehat{G} \setminus X \subseteq \operatorname{supp}\pi$, we know that $\widehat{\varphi}(X) \cap \overline{\widehat{\varphi}(\operatorname{supp}\pi)} \neq \emptyset$ and the hypothesis produces $\mu \in X$ with $\mu \subseteq \pi$, which is a contradiction.

(b) By [2, Proposition F.2.7], we know that $\widehat{\varphi}(\mu) \prec \widehat{\varphi}(\pi)$ if and only if $\widehat{\varphi}(\mu) \prec \bigoplus_{\nu\in\operatorname{supp}\pi} \widehat{\varphi}(\nu)$, which is the same as $\widehat{\varphi}(\mu) \in \overline{\widehat{\varphi}(\operatorname{supp}\pi)}$ (as H is locally compact). One may apply part (a) to obtain the conclusion. □

Theorem 2.9. *Let G and H be locally compact groups. If $\varphi : H \to G$ is a continuous group homomorphism with $\overline{\varphi(H)} = G$, the following statements are equivalent.*

(S1) *φ has property T.*
(S2) *1_H is an isolated point in $\widehat{\varphi}(\widehat{G})$.*
(S3) *$\widehat{\varphi}(\mu)$ is an isolated point in $\widehat{\varphi}(\widehat{G})$, for any $\mu \in \widehat{G}_{\mathrm{FD}}$.*
(S4) *For any subset $X \subseteq \widehat{G}_{\mathrm{FD}}$ and $\pi \in \mathrm{Rep}(G)$ with $\widehat{\varphi}(X) \cap \overline{\widehat{\varphi}(\operatorname{supp}\pi)} \neq \emptyset$, there exists $\mu \in X$ satisfying $\mu \subseteq \pi$.*
(S5) *There exists $\mu \in \widehat{G}_{\mathrm{FD}}$ such that for any $\pi \in \mathrm{Rep}(G)$ with $\widehat{\varphi}(\mu) \prec \widehat{\varphi}(\pi)$, one has $\mu \subseteq \pi$.*
(S6) *$\pi \circ \varphi$ has a spectral gap for any $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$.*

Proof. (S1) $\Rightarrow$ (S2). As $\widehat{G}^\varphi = \widehat{G} \setminus \{1_G\}$, the implication follows directly from Lemma 2.3(a).

(S2) $\Rightarrow$ (S3). Suppose that $\pi \in \mathrm{Rep}(G)$ with $\widehat{\varphi}(\mu) \in \overline{\widehat{\varphi}(\operatorname{supp}\pi)}$. Then $\widehat{\varphi}(\mu) \prec \bigoplus_{\nu\in\operatorname{supp}\pi} \widehat{\varphi}(\nu)$. By [2, Proposition A.1.12], we have $1_H \subseteq \widehat{\varphi}(\mu) \otimes \overline{\widehat{\varphi}(\mu)} \prec \bigoplus_{\nu\in\operatorname{supp}\pi} \widehat{\varphi}(\nu \otimes \overline{\mu})$. Thus, Lemma 2.8(b) and Statement (S2) gives $1_G \subseteq \bigoplus_{\nu\in\operatorname{supp}\pi} \nu \otimes \overline{\mu} \sim \pi \otimes \overline{\mu}$. Therefore, [2, Proposition A.1.12] tells us that $\mu \subseteq \pi$. Now, Lemma 2.8(b) implies that μ is an isolated point in $\widehat{\varphi}(\widehat{G})$.

(S3) $\Rightarrow$ (S1). By Statement (S3), 1_H is an isolated point in $\widehat{\varphi}(\widehat{G})$. Since $\widehat{\varphi}$ is a continuous injection, 1_G is an isolated point in $\widehat{G}$ and G has property T. Let $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$ such that $\mathcal{H}$ contains an almost $\pi\circ\varphi$-invariant unit vector. Thus, [2, Proposition F.2.7] tells us that $1_H \prec \bigoplus_{\sigma\in\operatorname{supp}\pi} \sigma \circ \varphi$, or equivalently, $1_H \in \overline{\widehat{\varphi}(\operatorname{supp}\pi)}$. As 1_H is an isolated point of $\widehat{\varphi}(\widehat{G})$, we see that $1_H \in \widehat{\varphi}(\operatorname{supp}\pi)$ and $1_G \in \operatorname{supp}\pi$. Consequently, $1_G \subseteq \pi$ (because G has property T) and we have $1_H \subseteq \pi \circ \varphi$. Now, Lemma 2.2(f) tells us that φ has property T.

(S3) $\Rightarrow$ (S4). Since $\widehat{\varphi}(X) \cap \overline{\widehat{\varphi}(\operatorname{supp} \pi)} \neq \emptyset$, there exists a non-empty countable subset $Y \subseteq X$ with $\widehat{\varphi}(Y) \subseteq \overline{\widehat{\varphi}(\operatorname{supp} \pi)}$. Moreover, Statement (S3) implies that $\widehat{\varphi}(Y)$ is open in $\widehat{\varphi}(\widehat{G})$ and we may apply Lemma 2.8(a) to obtain Statement (S4).

(S4) $\Rightarrow$ (S5). This follows from Lemma 2.8.

(S5) $\Rightarrow$ (S2). This implication follows from the argument of (iv) $\Rightarrow$ (ii) of [2, Theorem 1.2.5]. More precisely, suppose on the contrary that 1_H is not isolated in $\widehat{\varphi}(\widehat{G})$. If $\mu \otimes \bar{\mu} = \mu_1 \oplus \cdots \oplus \mu_n$ with $\mu_k \in \widehat{G}$, there exists a net $\{\sigma_i\}_{i\in I}$ in $\widehat{G} \setminus \{\mu_1, \ldots, \mu_n\}$ with $\mu \circ \varphi \prec \bigoplus_{i\in I} \sigma_i \circ \varphi \otimes \mu \circ \varphi$. Statement (S5) tells us that $\mu \otimes \bar{\mu} \subseteq \bigoplus_{i\in I} \sigma_i \otimes \mu \otimes \bar{\mu}$. Thus, $1_G \subseteq \sigma_{i_0} \otimes \mu \otimes \bar{\mu}$ for some $i_0 \in I$, and [2, Proposition A.1.12] produces the contradiction that $\sigma_{i_0} \subseteq \mu \otimes \bar{\mu}$.

(S1) $\Leftrightarrow$ (S6). Notice that $\mathcal{H}^{\pi\circ\varphi} = \mathcal{H}^{\pi}$ and the equivalent follows from Lemma 2.2(f). $\square$

3. Some specials cases

Let G and H be locally compact groups and $\varphi : H \to G$ be a continuous homomorphism. In the following, we will consider three special situations.

Special case 1: G is an abelian group

Our first special case is the situation when G is abelian. Let us start with the following result, which shows in particular that if G is abelian, then one may replace G by $\overline{\varphi(H)}$ when considering the T property of a morphism.

Proposition 3.1. *Let G and H be locally compact groups, and $Z(G)$ be the center of G.*

(a) *If $\varphi : H \to G$ has property T and $\varphi(H) \subseteq Z(G)$, then $\varphi : H \to N := \overline{\varphi(H)}$ has property T (and hence N is compact).*
(b) *If N is a closed subgroup of $Z(G)$, the inclusion map $j : N \to G$ has property T if and only if N is compact.*

Proof. (a) Suppose on the contrary that $\varphi : H \to N$ does not have property T. By Theorem 2.9, we have $1_H \in \overline{\widehat{\varphi}(\widehat{N}) \setminus \{1_H\}}$. Thus, for any compact subset $C \subseteq H$ and $\epsilon > 0$, there exists $\nu_{C,\epsilon} \in \widehat{N} \setminus \{1_N\}$ such that

$$|1 - \nu_{C,\epsilon}(\varphi(t))| \ < \ \epsilon \qquad (t \in C). \tag{3.1}$$

On the other hand, for any $\nu \in \widehat{N} \setminus \{1_N\}$ and $t \in H$, we have

$$(\operatorname{Ind}_N^G \nu)_{\varphi(t)} \xi \ = \ \nu(\varphi(t))\xi \qquad (\xi \in L^2(G/N)),$$

which shows that $1_H \nsubseteq \pi$ for any $\pi \in X := \big\{\operatorname{Ind}_N^G \nu : \nu \in \widehat{N} \setminus \{1_N\}\big\}$. By choosing any unit vector $\xi \in L^2(G/N)$, we know from (3.1) that $1_H \in \overline{\widehat{\varphi}(X)}$, which contradicts Lemma 2.3(a).

(b) This follows from part (a) and Lemma 2.2(d). $\square$

In the following, we will give an alternative description of the compactness of $H^{\mathrm{ab}} := H/\overline{[H,H]}$ through the T-property of the canonical map $\beta_H : H \to H^{\mathrm{ab}}$ (which is a generalization of [10, Corollary 2.7]). Let us first set some more notation.

In the spirit of [10], we say that a topological group H has *property* $(T, 1)$ if 1_H is an isolated point in the subset $\widehat{H}_1$ of all one dimensional irreducible representations.

Theorem 3.2. *Let H be a locally compact group.*

(a) *The following statements are equivalent.*
  (O1) *H^{ab} is compact.*
  (O2) *H has property $(T, 1)$.*
  (O3) *β_H has property T.*
  (O4) *Continuous homomorphisms from H to abelian topological groups have property T.*
(b) *If H has property $(T, 1)$, then H is unimodular.*
(c) *If H is connected and second countable, then* (O1)–(O4) *are also equivalent to the following.*
  (O5) *H having property (T_{ℓ_p}) (in the sense of [3]) for one (and equivalently, for all) $p \in (1, \infty) \setminus \{2\}$.*

Proof. (a) (O1) $\Leftrightarrow$ (O2). Notice that $\beta_H : H \to H^{\mathrm{ab}}$ induces a homeomorphism $\widehat{\beta_H} : \widehat{H^{\mathrm{ab}}} \to \widehat{H}_1$. Thus, H have property $(T, 1)$ if and only if H^{ab} has property T.

(O2) $\Leftrightarrow$ (O3). This follows from Theorem 2.9.

(O3) $\Leftrightarrow$ (O4). This follows from Lemma 2.2(a).

(b) By part (a), the image of the modular function is a compact subgroup of $\mathbb{R}_+ \setminus \{0\}$ and hence is trivial.

(c) This follows directly from [3, Corollary 3]. □

Our next result follows from Theorem 3.2, [2, Theorem 1.7.1] and [2, Proposition 1.7.6].

Corollary 3.3.

(a) *Suppose that H is either σ-compact or discrete. Then H has property T if and only if H has property $(T, 1)$ and the topological subgroup $[H, H]$ has property T.*
(b) *If H has property $(T, 1)$ and $\varphi : H \to G$ has dense range, then G has property $(T, 1)$.*

Example 3.4. (a) As $\mathbb{F}_2^{\mathrm{ab}} = \mathbb{Z}^2$, we know that $\mathbb{F}_2$ does not have property $(T, 1)$.

(b) As in [2, Example 1.3.7], if $n \geq 1$, the fundamental group of an orientable closed surface of genus greater than n does not have property $(T, 1)$.

(b) Let $H := SL(2, \mathbb{Z})$. Since $H^{\mathrm{ab}} = \mathbb{Z}_{12}$, we know that H has property $(T, 1)$. On the other hand, $[H, H] \cong \mathbb{F}_2$ does not have property $(T, 1)$ (see part (a)). These tells us that the corresponding result of [2, Theorem 1.7.1] does not holds for property $(T, 1)$.

Special case 2: G is a compact group

Secondly, we consider the case when G is compact. We recall from [10] that a topological group H has *property* (T, FD) if 1_H is an isolated point in $\widehat{H}_{\mathrm{FD}}$. Clearly, if H has property (T, FD), then it has property $(T, 1)$.

We denote by $\mathrm{b}H$ the Bohr compactification of H and by $\sigma_H : H \to \mathrm{b}H$ the canonical map. Recall that $\ker \sigma_H = \{t \in H : \mu(t) = 1, \text{ for any } \mu \in \widehat{H}_{\mathrm{FD}}\}$. Moreover, any continuous group homomorphism $\varphi : H \to G$ induces a continuous group homomorphism $\mathrm{b}\varphi : \mathrm{b}H \to \mathrm{b}G$ satisfying $\mathrm{b}\varphi \circ \sigma_H = \sigma_G \circ \varphi$.

Lemma 3.5.

(a) *If σ_H has property T and $\varphi : H \to G$ is a continuous homomorphism with dense range, then σ_G has property T.*
(b) *If G is a Moore group, then it has property (T, FD) if and only if it is compact.*

Proof. (a) This part follows from Lemma 2.2(a)&(b) as well as the fact that $\mathrm{b}\varphi \circ \sigma_H = \sigma_G \circ \varphi$.

(c) Since G is a Moore group, $\widehat{G}_{\mathrm{FD}} = \widehat{G}$. Thus, if G has property (T, FD), then it has property T and hence is compact (as Moore groups are amenable). □

Theorem 3.6. *Let H be a locally compact group and consider the following statements.*

(F1) *σ_H is a quotient map from H onto $\mathrm{b}H$.*
(F2) *For any continuous homomorphism φ from H to any compact group G, the subgroup $\varphi(H)$ is compact and $\varphi : H \to \varphi(H)$ is a quotient map.*
(F3) *σ_H has property T.*
(F4) *Any continuous homomorphism from H to any compact group G has property T.*
(F5) *H has property (T, FD).*

(a) (F1) ⇔ (F2) ⇒ (F3) ⇔ (F4) ⇔ (F5).
(b) *If, in addition, H is almost connected, then* (F5) ⇒ (F1).

Proof. (a) (F1) ⇔ (F2). Suppose that Statement (F1) holds and set $G_0 := \overline{\varphi(H)}$. Then $\mathrm{b}\varphi : \mathrm{b}H \to \mathrm{b}G_0$ is surjective (because $\varphi(H)$ is dense in G_0) and hence is a quotient map. Since $\sigma_{G_0} \circ \varphi = \mathrm{b}\varphi \circ \sigma_H$ and σ_{G_0} is a homeomorphism, we know that $\varphi : H \to G_0$ is a quotient map. The converse is clear.

(F1) ⇒ (F3). This follows from Lemma 2.2(c).

(F3) ⇔ (F4). This follows from Lemma 2.2(a).

(F3) ⇔ (F5). As $\widehat{\sigma_H}(\widehat{\mathrm{b}H}) = \widehat{H}_{\mathrm{FD}}$, the equivalence follows from Theorem 2.9.

(b) Let $F := H/\ker \sigma_H$ and $q : H \to F$ be the quotient map. Notice that $\mathrm{b}q$ is surjective because $\mathrm{b}q(\mathrm{b}H)$ is compact and contains a dense subset of $\mathrm{b}F$. If $\check{\sigma}_H : F \to \mathrm{b}H$ is the canonical injection, then $\mathrm{b}q \circ \check{\sigma}_H = \sigma_F$. Furthermore, if $\psi : \mathrm{b}F \to \mathrm{b}H$ is the continuous homomorphism with $\psi \circ \sigma_F = \check{\sigma}_H$, then $\psi \circ \mathrm{b}q \circ \check{\sigma}_H = \check{\sigma}_H$, which shows that $\mathrm{b}q$ is injective (since $\check{\sigma}_H$ has dense range).

On the other hand, suppose $s \in H$ satisfying $q(s) \in \ker \sigma_F$. For any $(\pi, \mathcal{H}) \in \widehat{H}_{\mathrm{FD}}$, there exists $(\tilde{\pi}, \mathcal{H}) \in \widehat{F}_{\mathrm{FD}}$ with $\pi = \tilde{\pi} \circ q$, which implies that $\pi(s) = \tilde{\pi}(q(s)) = 1$ (as $\tilde{\pi}$ is finite dimensional). Hence, $s \in \ker \sigma_H$ and $\ker \sigma_F$ is trivial. This means F is an almost connected maximally almost periodic group and hence is a Moore group (see e.g. [12, p. 698]). By part (a) and Lemma 3.5(a), we see that F has property (T, FD) and hence is compact by Lemma 3.5(b). Consequently, σ_F is a homeomorphism, and Statement (F1) follows from $\mathrm{b}q \circ \sigma_H = \sigma_F \circ q$. □

The two results above show, in particular, that if there exist a non-compact Moore group G and a continuous homomorphism from H to G with dense range, then H cannot have property (T, FD). Moreover, the following corollary follows from Theorem 3.6 and [2, Theorem 1.7.1].

Corollary 3.7. *Let H be an almost connected locally compact group. Then H has property T if and only if H has property (T, FD) and $\ker \sigma_H$ has property T.*

The argument of Theorem 3.6(b) also implies that if H is an almost connected maximally almost periodic group with property (T, FD), then it is compact. Part (b) of the following tells us that this is not true without the almost connectedness assumption.

Example 3.8. (a) Let A be an abelian discrete group and F is a finite group acting on A by group automorphisms. Then the semi-direct product $H := A \cdot F$ is a Moore group. Thus, by Lemma 3.5(b), H has property (T, FD) if and only if A is finite.

(b) It is well-known that $SL_3(\mathbb{Z})$ has property T and is maximally almost periodic. Consequently, $\mathrm{b}SL_3(\mathbb{Z})$ is not a finite group. Thus, the almost connectedness assumption in Theorem 3.6(b) is essential.

(c) Let $H := SL(2,\mathbb{Z})$, $G := [H,H]$ and $F := [G,G]$. As $G \cong \mathbb{F}_2$, we have $G/F \cong \mathbb{Z}^2$. Moreover, since $H/G \cong \mathbb{Z}_{12}$, we see that H/F is an infinite virtually abelian group. Therefore, Lemma 3.5 and Theorem 3.6(a) tells us that H does not have property (T,FD) (see the discussion following Theorem 3.6).

Special case 3: H is a discrete group

Finally, we consider the case when H is discrete. Let us first study the situation when $H = G_d$, i.e. when G has strong property T. In the following, $\mathfrak{F}(G)$ is the set of non-empty finite subsets of G. Moreover, $C^*(G)$ is the (full) group C^*-algebra of G with $u^G : G \to M(C^*(G))$ being the canonical map. We consider $\tilde{\varphi} : C^*(H) \to M(C^*(G))$ to be the *-homomorphism induced by φ, and $\tilde{\pi} : C^*(H) \to \mathcal{B}(\mathcal{H})$ to be the *-representation induced by $\pi \in \mathrm{Rep}(G)$. We denote by $p_G \in C^*(G)^{**}$ the support projection of $\tilde{1}_G$.

Lemma 3.9. *Concerning the following statements, one has* (sT1) $\Rightarrow$ (sT2) $\Rightarrow$ (sT3) *and* (sT1) $\Rightarrow$ (sT4) $\Rightarrow$ (sT5) $\Rightarrow$ (sT3).

(sT1) *G has strong property T.*

(sT2) *For any continuous action α of G on a von Neumann algebra N, any α-invariant state is a weak-*-limit of a net of α-invariant normal states.*

(sT3) *For any continuous action α of G on a von Neumann algebra N without any α-invariant normal state, there is no α-invariant state.*

(sT4) *For any $\epsilon > 0$, one can find $(Q,\delta) \in \mathfrak{F}(G) \times \mathbb{R}_+$ such that if α is a continuous action of G on a von Neumann algebra N and ω is a normal state on N which is (Q,δ)-invariant (in the sense that $\|\omega \circ \alpha_t - \omega\| < \delta$ for any $t \in Q$), there is an α-invariant normal state τ satisfying $\|\omega - \tau\| < \epsilon$.*

(sT5) *There exists $(Q,\delta) \in \mathfrak{F}(G) \times \mathbb{R}_+$ such that for any continuous action α of G on a von Neumann algebra N, the existence of a (Q,δ)-invariant normal state on N will imply the existence of a α-invariant normal state.*

Proof. (sT1) $\Rightarrow$ (sT2). Let $(N,\mathfrak{H},\mathfrak{J},\mathfrak{P})$ be the standard form of N and $\mathfrak{u}_{\alpha_t} \in U(\mathfrak{H})$ be the unitary implementation of the automorphism α_t ($t \in G$) as in [7, Theorem 3.2]. Then $\mathfrak{u}_\alpha$ is a continuous representation of G. As G has strong property T, we know from Theorem 2.9 that the representation $\mathfrak{u}_\alpha$ of G_d has a spectral gap. Thus, α has a spectral gap in the sense of [9], and [9, Theorem 2.2] gives the required conclusion.

(sT2) $\Rightarrow$ (sT3). This implication is obvious.

(sT1) $\Rightarrow$ (sT4). Let $(Q,\delta_0) \in \mathfrak{F}(G) \times \mathbb{R}_+$ be a Kazhdan pair for G and set $\delta := \epsilon^2\delta_0^2/4$. Suppose that $(N,\mathfrak{H},\mathfrak{J},\mathfrak{P})$ is the standard form of N and $\mathfrak{u}_\alpha$ is the unitary implementation of α. Consider $\xi \in \mathfrak{P}$ to be the unique element with $\omega = \omega_{\xi,\xi}$. By the Power–Stormer type inequality as in [7, Lemma 2.10], we know that

$$\|\mathfrak{u}_{\alpha_t}(\xi) - \xi\| \ < \ \epsilon\delta_0/2 \qquad (t \in Q).$$

Thus, [2, Proposition 1.1.9] tells us that $\|\xi - P(\xi)\| < \epsilon/2$, where $P : \mathfrak{H} \to \mathfrak{H}^{\mathfrak{u}_\alpha}$ is the orthogonal projection. Now, if we set $\tau := \omega_{P(\xi),P(\xi)} \in \mathfrak{M}_{N,\alpha} \cap N_*$, then $\|\omega - \tau\| < \epsilon$.

(sT4) $\Rightarrow$ (sT5). This implication is clear.

(sT5) $\Rightarrow$ (sT3). Suppose on the contrary that one can find a von Neumann algebra N and a continuous action α of G on N without α-invariant normal state but N has an α-invariant state f. Let $\{\omega_i\}_{i\in I}$ be a net of normal states that weak-*-converges to f. Since $\omega_i(\alpha_t(x)) - \omega_i(x) \to 0$ for any $x \in N$ and $t \in G$, one can find, by using the "convergence to invariance" type argument, a net $\{\tau_j\}_{j\in J}$ in the convex hull of $\{\omega_i\}_{i\in I}$ such that $\|\tau_j \circ \alpha_t - \tau_j\| \to 0$ for any $t \in G$. Thus, τ_i is (Q, δ)-invariant for large enough i, and Statement (sT5) gives the contradiction of the existence of an α-invariant normal state. □

Remark 3.10. In the case when G is countable and discrete, [9, Theorem 5.1] tells us that (sT1) ⇔ (sT3) and hence (sT1) ⇔ (sT4) ⇔ (sT5). This yield new characterizations of property T of countable discrete groups. In this case, one may regard (Q, δ) in (sT5) as a Kazhdan pair for group actions on von Neumann algebras (instead of unitary representations).

Moreover, we have the following operator algebraic characterizations for the T-property of φ in the case when H is discrete. For the notation of (D2), we refer the readers to [8].

Theorem 3.11. *Suppose that H is discrete and G is locally compact.*

(a) *The following statements are equivalent.*
 (D1) *φ has property T.*
 (D2) *There exist a non-empty finite subset $F \subseteq C^*(H)$ and $\kappa > 0$ such that for any unital *-bimodule $\mathcal{H}$ of $C^*(G)$ with $V_{\mathcal{H}}(\tilde{\varphi}(F), \kappa) \neq \emptyset$, one has $\mathcal{H}^{\tilde{\varphi}(C^*(H))} \neq (0)$.*
(b) *If $\overline{\varphi(H)} = G$, then (D1) and (D2) are also equivalent to the following.*
 (D3) *$p_G \in \tilde{\varphi}(C^*(H))$.*
 (D4) *$\sigma_G \circ \varphi$ has property T and G has strong property T.*
 (D5) *$\sigma_G \circ \varphi$ has property T and G satisfies one (and equivalently, all) of the conditions* (sT2)–(sT5).

Proof. (a) (D1) ⇒ (D2). If (Q, κ) is a Kazhdan pair for φ and F is the image of Q in $C^*(H)$, then a standard argument will show that (D2) holds.

(D2) ⇒ (D1). Note first of all that, by a simple approximate argument and suitably reducing κ, one may assume that $K = \{x_1, \ldots, x_m\}$ with $x_k = \sum_{i=1}^{N_k} \lambda_{i,k} u^H_{s_{i,k}}$ for some $\lambda_{i,k} \in \mathbb{C}$ and $s_{i,k} \in H$. Let $(\pi, \mathcal{H}) \in \mathrm{Rep}(G)$. We regard $\mathcal{H}$ as a *-bimodule of $C^*(G)$ by setting

$$y\xi = \tilde{\pi}(y)\xi \quad \text{and} \quad \xi y = \tilde{1}_G(y)\xi \qquad (y \in C^*(G); \xi \in \mathcal{H}).$$

Suppose that $\{\xi_\alpha\}_{\alpha\in I}$ is an almost $\pi \circ \varphi$-invariant unit vector in $\mathcal{H}$. Then

$$\|\tilde{\varphi}(x_k)\xi_\alpha - \xi_\alpha\tilde{\varphi}(x_k)\| \;=\; \Big\|\sum\nolimits_{i=1}^{N_k} \lambda_{i,k}\big(\pi_{\varphi(s_{i,k})}\xi_\alpha - \xi_\alpha\big)\Big\| \to 0 \qquad (k = 1, \ldots, m).$$

Thus, $\xi_\alpha \in V_{\mathcal{H}}(\tilde{\varphi}(K), \kappa)$ when α is large enough. This shows that $\mathcal{H}^{\pi\circ\varphi} = \mathcal{H}^{\tilde{\varphi}(C^*(H))} \neq (0)$.

(b) (D1) ⇒ (D3). This implication basically follows from the argument of [14, Theorem 3.2]. More precisely, if $\pi := \bigoplus_{\mu\in\widehat{G}\setminus\{1_G\}} \tilde{\mu}$, then $\tilde{\pi} \oplus \tilde{1}_G$ induces a faithful representation of $M(C^*(G))$. Let (F, κ) be the Kazhdan pair for φ and $a := \frac{1}{2|F|}\sum_{t\in F} u^H_t + (u^H_t)^* \in C^*(H)$. Clearly, $\|\tilde{\varphi}(a)\| \leq 1$. The argument of [14, Theorem 3.2] implies that $\|\tilde{\pi}(\tilde{\varphi}(a)) - 1\| > \frac{\kappa^2}{2|F|}$. Moreover, as $\tilde{1}_G(\tilde{\varphi}(a)) = \tilde{1}_H(a) = 1$, we know that 1 is an isolated point in $\sigma_{\tilde{\varphi}(C^*(H))}(\tilde{\varphi}(a))$ and hence $p := \chi_{\{1\}}(\tilde{\varphi}(a)) \in \tilde{\varphi}(C^*(H))$. Since $pxp = 1_G(x)p$ $(x \in C^*(G))$, one deduces from [14, Lemma 3.1] that $p = p_G$ (note that G has property T because of Lemma 2.2(e)).

(D3) ⇒ (D1). Let $x \in C^*(H)$ such that $p_G = \tilde{\varphi}(x)$. It is clear that $\widehat{H}_x := \{\nu \in \widehat{H} : \tilde{\nu}(x) = 0\}$ is closed in $\widehat{H}$. Thus,

$$\widehat{\varphi}(\widehat{G}) \setminus \{1_H\} \;=\; \widehat{\varphi}\big(\{\mu \in \widehat{G} : \tilde{\mu}^{**}(p_G) = 0\}\big) \;=\; \widehat{\varphi}\big(\{\mu \in \widehat{G} : (\mu \circ \varphi)\tilde{}\,(x) = 0\}\big) \;=\; \widehat{\varphi}(\widehat{G}) \cap \widehat{H}_x,$$

which is closed in $\widehat{\varphi}(\widehat{G})$. Consequently, 1_H is isolated in $\widehat{\varphi}(\widehat{G})$ and one can apply Theorem 2.9 to conclude that φ has property T.

(D1) $\Rightarrow$ (D4). This follows from Lemma 2.2(a) and Lemma 2.2(g).

(D4) $\Rightarrow$ (D5). This follows from Lemma 3.9.

(D5) $\Rightarrow$ (D1). By Lemma 3.9, it suffices to show that if $\sigma_G \circ \varphi$ has property T and G satisfies (sT3), then φ has strong property T. It is well-known that the image of $\widehat{\sigma_G} : \widehat{\mathrm{b}G} \to \widehat{G}$ is precisely $\widehat{G}_{\mathrm{FD}}$. Thus, Theorem 2.9 tells us that 1_H is an isolated point of $\widehat{\varphi}(\widehat{G}_{\mathrm{FD}})$. Now, the implication follows from the argument of [9, Proposition 4.3]. To be a bit more precise, suppose on contrary that 1_H is not an isolated point of $\widehat{\varphi}(\widehat{G})$ (see Theorem 2.9). As $\widehat{\varphi}$ is injective, the above tells us that one can find a net $\{(\pi^i, \mathcal{K}^i)\}_{i\in I}$ in $\widehat{G} \setminus \widehat{G}_{\mathrm{FD}}$ and a unit vector $\xi_i \in \mathcal{K}^i$ for each $i \in I$ satisfying

$$\|\pi^i_{\varphi(t)}\xi_i - \xi_i\| \ \to \ 0 \qquad (t \in H).$$

Let $N := \bigoplus_{i\in I} \mathcal{B}(\mathcal{K}^i)$ and set $\pi := \bigoplus_{i\in I} \pi^i$ as well as $\alpha := \mathrm{Ad}\,\pi$. Then the Hilbert space $\mathfrak{H}$ for the standard form of N is $\bigoplus_{i\in I} \mathcal{K}^i \otimes \overline{\mathcal{K}^i}$ with the representation of N on $\mathfrak{H}$ being the canonical one. Now, the argument of [9, Proposition 4.3] gives $\mathfrak{H}^{\mathfrak{u}_\alpha} = (0)$, which means that there is no α-invariant normal state on N (see, e.g., [9, Lemma 2.3(a)]; note that one may regard α as an action of G_d on N). However, the $\sigma(N^*, N)$-limit of a subnet of $\{\omega_{\xi_i}\}_{i\in I}$ will produce a state f on N such that $f \circ \alpha_{\varphi(t)} = f$ for any $t \in H$, which implies that f is α-invariant (since $\varphi(H)$ is dense in G). This contradicts (sT3). $\square$

Consequently, if G has strong property T and φ is a group homomorphism from a discrete group H to G with dense range, then φ has property T if and only if 1_H is an isolated point of $\widehat{\varphi}(\widehat{G}_{\mathrm{FD}})$.

Moreover, part (b) of the above and Theorem 3.6 give the following equivalent forms of the strong property T, which seems to be new (except that those concerning (sT2) and (sT3) are extensions of [9, Proposition 4.3]).

Corollary 3.12. *A locally compact G has strong property T if and only if G has property (T, FD) and G satisfies one (and hence, all) of the statements* (sT2)–(sT5).

Let G be a locally compact group, $F \subseteq G$ be a closed subgroup. There exists a strongly quasi-invariant measure μ on G/F, which is unique up to equivalence (see, e.g., [6, p. 58]). In the following, we write $L^\infty(G/F)$ for $L^\infty(\mu)$.

Corollary 3.13. *Let G be either a discrete group with property T or a σ-compact locally compact group with strong property T, and let $F \subseteq G$ be a closed subgroup. Then there exists at most one G-invariant state on $L^\infty(G/F)$.*

Proof. Let ρ be the rho-function corresponding to μ with $\rho(e) = 1$. Suppose that α is the canonical action of G on $L^\infty(\mu)$ and β is the action of G on $L^1(\mu)$ induced by α. For each $g \in G$ and $\omega \in L^1(\mu)$, one has $\beta_g(\omega)(yF) = \omega(g^{-1}yF)\frac{\rho(g^{-1}y)}{\rho(y)}$ μ-a.e. Thus, if $\omega \in L^1(\mu)^\beta$ and $g \in G$, then

$$\rho(gy)\omega(gyF) \ = \ \omega(yF)\rho(y) \tag{3.2}$$

for μ-almost-all $yF \in G/F$. Using the argument of [2, Theorem E.3.1] (note that we assume G is σ-compact if it is non-discrete), one may assume that (3.2) holds for all $y \in G$. Hence, $\omega(gF) = \frac{\omega(F)}{\rho(g)}$ for every $g \in G$ and $\dim L^1(\mu)^\beta \leq 1$. Now, Statement (sT2) gives the conclusion. $\square$

Note that if G is second countable group, the existence of a G-invariant mean on $L^\infty(G/F)$ is one equivalent form of the amenability of the homogeneous space G/F (see the main theorem in Section 4 of Chapter 2 in [5]). Since the amenability of G/F is equivalent to $1_G \prec \lambda_{G/F}$, we know from [2, Theorem E.3.1] that if G has property T and G/F is amenable, then there exists a finite G-invariant regular measure on G/F. Corollary 3.13 tells us that if we assume that G has strong property T, then the only G-invariant mean on $L^\infty(G/F)$ (if exists) is given by this finite G-invariant measure.

On the other hand, it is well-known that there are more than one invariant means on the circle. Therefore, one cannot relax the assumption of strong property T in Corollary 3.13 above to property T.

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