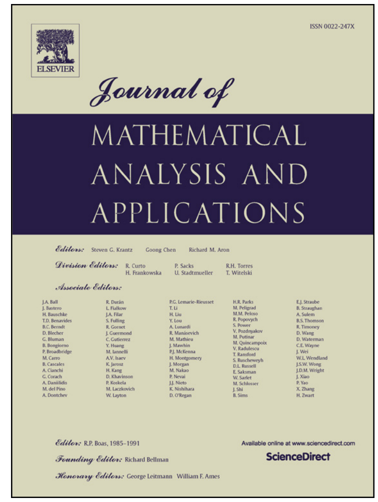
Accepted Manuscript



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PII: S0022-247X(16)30189-5
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.05.036
Reference: YJMAA 20454

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 29 January 2016

Please cite this article in press as: X. Zhao, S. Zheng, Global boundedness of solutions in a parabolic-parabolic chemotaxis system with singular sensitivity, *J. Math. Anal. Appl.* (2016), http://dx.doi.org/10.1016/j.jmaa.2016.05.036

Global boundedness of solutions in a parabolic-parabolic chemotaxis system with singular sensitivity*

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May 19, 2016

Abstract

We consider a parabolic-parabolic Keller-Segel system of chemotaxis model with singular sensitivity: $u_t = \Delta u - \chi\nabla \cdot (\frac{u}{v}\nabla v)$, $v_t = k\Delta v - v + u$ under the homogeneous Neumann boundary condition in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, with $\chi, k > 0$. It is proved that for any $k > 0$, the problem admits global classical solutions, whenever $\chi \in \big(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}}\big)$. The global solutions are moreover globally bounded if $n \leq 8$. This shows a way the size of the diffusion constant k of the chemicals v effects the behavior of solutions.

2010MSC: 35B40; 92C17; 35K55

Keywords: Keller-Segel system; Chemotaxis; Singular sensitivity; Boundedness

1 Introduction

In this paper, we consider the parabolic-parabolic chemotaxis system with singular sensitivity

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (\dfrac{u}{v}\nabla v), & x \in \Omega, \quad t \in (0,T), \\ v_t = k\Delta v - v + u, & x \in \Omega, \quad t \in (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\chi, k > 0$, Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $\frac{\partial}{\partial \nu}$ denotes the derivation with respect to the outer normal of $\partial\Omega$, and the initial data $u_0 \in C^0(\overline{\Omega})$, $u_0(x) \geq 0$ on $\overline{\Omega}$, $v_0 \in W^{1,q}(\Omega)$ $(q > n)$, $v_0(x) > 0$ on $\overline{\Omega}$.

The classical Keller-Segel system of chemotaxis model was introduced by Keller and Segel [6] in 1970 to describe the cells (with density u) move towards the concentration gradient

*Supported by the National Natural Science Foundation of China (11171048).

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of a chemical substance v produced by the cells themselves. Various forms of sensitivity functions can be chosen to model different types of chemotaxis mechanisms. Among them $\phi(v) = \frac{\chi}{v}$ was selected in (1.1) largely due to the Weber-Fechner's law for cellular behaviors, where the subjective sensation is proportional to the logarithm of the stimulus intensity [7]. With $\phi(v) = \frac{\chi}{v}$, the cellular movements are governed by the taxis flux $\frac{\chi \nabla v}{v}$, which may be unbounded when $v \approx 0$. In the model (1.1), the values of the chemotactic sensitivity coefficient χ and the chemical diffusion constant k play significant roles to determine the behavior of solutions.

Recall the known results in the field with $k = 1$. At first consider the parabolic-elliptic analogue of (1.1), where the second parabolic equation in (1.1) is replaced by the elliptic equation $0 = \Delta v - v + u$. It was known that all radial classical solutions are global-in-time if either $n \geq 3$ with $\chi < \frac{2}{n-2}$, or $n = 2$ with $\chi > 0$ arbitrary [9]. When $0 < \phi(v) < \frac{\chi}{v^l}$ with $l \geq 1$, $\chi > 0$, there is a unique and globally bounded classical solution if $\chi < \frac{2}{n}$ $(l = 1)$ or $\chi < \frac{2}{n} \cdot \frac{l^l}{(l-1)^{(l-1)}} \gamma^{(l-1)}$ $(l > 1)$, with $\gamma > 0$ depending on Ω and u_0 [3]. Next consider the parabolic-parabolic case. All solutions of (1.1) are global in time when either $n = 1$ [11], or $n = 2$ and $\chi < \frac{5}{2}$ under the radial assumption, while $\chi < 1$ under the non-radial assumption [1, 10]. For $n \geq 2$, (1.1) possesses global classical solutions if $0 < \chi < \sqrt{\frac{2}{n}}$, and moreover, $\chi < \sqrt{\frac{n+2}{3n-4}}$ ensures the global existence of weak solutions [14]. In addition, the global solutions are globally bounded with $\chi < \sqrt{\frac{2}{n}}$ [2]. Refer to [8, 12, 16] for more results on chemotaxis models with singular sensitivities.

Recently, under somewhat complicated conditions, Wang [13] established classical global solutions to the problem, a similar model to (1.1),

$$
\begin{cases}
u_t = \nabla \cdot (\nabla u - \dfrac{\chi u}{v+c} \nabla v), & x \in \Omega, \quad t \in (0,T), \\
v_t = k\Delta v - \alpha v + \beta u, & x \in \Omega, \quad t \in (0,T), \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0,T), \\
(u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \Omega,
\end{cases}
$$

with $\chi, c, k, \alpha, \beta > 0$. In the present paper, motivated by Winkler [14] and Fujie [2], we will prove the global existence-boundedness of classical solutions to (1.1), with simplified conditions. That is the following theorem.

Theorem 1 *Let $n \geq 2$, $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,q}(\Omega)$ $(q > n)$ with $u_0 \geq 0$, $v_0 > 0$ on $\overline{\Omega}$. Then, for any $k > 0$, there exists a global classical solution to (1.1), provided $\chi \in \left(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}}\right)$. Moreover, the solution is globally bounded under $n \leq 8$.*

Remark 1 Theorem 1 shows in what way the size of $k > 0$ (the diffusion strength of the chemicals v) effects the behavior of solutions to (1.1). It is interesting to observe that when $n = 2$ the global existence-boundedness of solutions is independent of the size of $k > 0$, since $-\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}} \equiv 1$ with $n = 2$. Quite differently, when $n \geq 3$ the contribution of

$k > 0$ is significant that the range of χ for global existence-boundedness of solutions will be enlarged (shrunk) as $k > 0$ is decreasing (increasing). The arbitrariness of $k > 0$ yields the "maximal" range with $\chi \in (0, 1)$ or the "minimal" range with $\chi \in (0, \frac{2}{n})$. That is to say for any $\chi \in (0, 1)$ (close to 1), there is $k > 0$ (small) such that the classical solution of (1.1) is globally bounded. On the other hand, for any $k > 0$ (large), there is $\chi \in (0, \frac{2}{n})$ to ensure the global boundedness. Finally, it is pointed out that if $k = 1$, the required range of χ in Theorem 1 becomes $0 < \chi < \sqrt{\frac{2}{n}}$, which coincides with those in [2, 14].

Remark 2 Now compare our results for the parabolic-parabolic chemotaxis model (1.1) with those for the corresponding parabolic-elliptic model, which can be considered as a special case of (1.1) with the diffusion constant of the chemicals v sufficiently large [5]. Just as mentioned in Remark 1, letting $k > 0$ be arbitrarily large results in the "minimal" permitted range with $0 < \chi < \frac{2}{n}$. This does agree with those obtained for the parabolic-elliptic model in [3].

2 Preliminaries

In this section we introduce the local existence of classical solutions to (1.1) with required estimates involving χ and k, as well as some technical lemmas for the global boundedness as preliminaries.

Lemma 2.1 *Let $n \geq 2$, $u_0 \in C^0(\overline{\Omega})$, $v_0 \in W^{1,q}(\Omega)$ $(q > n)$ with $u_0 \geq 0$, $v_0 > 0$ on $\overline{\Omega}$. Then, for any $k, \chi > 0$, there exists $T_{\max} \in (0, \infty]$, such that (1.1) has a unique nonnegative solution $u \in C^0([0, T_{\max}); C^0(\overline{\Omega})) \times C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ and $v \in C^0([0, T_{\max}); C^0(\overline{\Omega})) \times C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \times L^\infty_{\text{loc}}\big([0, T);\ W^{1,q}(\Omega)\big)$, where either $T_{\max} = \infty$, or $T_{\max} < \infty$ with $\lim_{t\to T_{\max}}(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)}) = \infty$.*

Proof. For $k > 0$, it is known from Lemma 2.2 in [2] that there is $\eta > 0$, such that $\inf_{x\in\Omega} v(x, t) \geq \eta > 0$ for all $t > 0$. Consequently, the local existence lemma can be obtained by the classical parabolic theory, refer to [4, Theorem 3.1]. $\square$

The following lemma is crucial to establish the global existence-boundedness conclusions of the paper. Denote $r_\pm(p) := (p-1)\big[\frac{p\chi(1-k)+2k}{p(k-1)^2+4k} \pm \frac{2\sqrt{k^2-p\chi k(k-1)-p\chi^2 k}}{p(k-1)^2+4k}\big]$.

Lemma 2.2 *Let (u, v) solve (1.1) with $k > 0$ and $\chi \in \big(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}}\big)$. If*

$$\|v(\cdot, t)\|_{L^{p-r}(\Omega)} \leq c, \quad t \in (0, T_{\max}) \tag{2.1}$$

with $p < \frac{k}{[\chi^2-\chi(1-k)]_+}$, $r \in (r_-(p), r_+(p))$, and $c > 0$, then

$$\int_\Omega u^p v^{-r} dx \leq \tilde{c}, \quad t \in (0, T_{\max}) \tag{2.2}$$

with some $\tilde{c} > 0$.

Proof. It is known via a simple computation with (1.1) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p v^{-r}dx &= p\int_\Omega u^{p-1}v^{-r}[\Delta u - \chi\nabla(\frac{u}{v}\nabla v)]dx - r\int_\Omega u^p v^{-r-1}(k\Delta v - v + u)dx \\
&= -p\int_\Omega \nabla(u^{p-1}v^{-r})\cdot(\nabla u - \chi\frac{u}{v}\nabla v)dx + rk\int_\Omega \nabla(u^p v^{-r-1})\cdot\nabla v dx \\
&\quad + r\int_\Omega u^p v^{-r}dx - r\int_\Omega u^{p+1}v^{-r-1}dx \\
&= -p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2dx + [pr + prk + p(p-1)\chi]\int_\Omega u^{p-1}v^{-r-1}\nabla u\cdot\nabla v dx \\
&\quad - [r(r+1)k + pr\chi]\int_\Omega u^p v^{-r-2}|\nabla v|^2 dx + r\int_\Omega u^p v^{-r}dx - r\int_\Omega u^{p+1}v^{-r-1}dx \\
&\le \int_\Omega \Big[\frac{p[(p-1)\chi + r + rk]^2}{4(p-1)} - pr\chi - r(r+1)k\Big]u^p v^{-r-2}|\nabla v|^2dx \\
&\quad + r\int_\Omega u^p v^{-r}dx - r\int_\Omega u^{p+1}v^{-r-1}dx
\end{aligned}$$

by Young's inequality. Denote

$$f(r;p,\chi,k) := \frac{p[(p-1)\chi + r + rk]^2}{4(p-1)} - pr\chi - r(r+1)k,$$

and rewrite it as the quadric expression in r

$$4(p-1)f(r;p,\chi,k) = [p(k-1)^2 + 4k]r^2 + [2p(p-1)\chi(k-1) - 4(p-1)k]r + p(p-1)^2\chi^2.$$

We know

$$\begin{aligned}
\Delta_r : &= 4(p-1)^2[p\chi(k-1) - 2k]^2 - 4(p-1)^2p\chi^2[p(k-1)^2 + 4k] \\
&= 16(p-1)^2[k^2 - p\chi k(k-1) - p\chi^2 k] > 0
\end{aligned}$$

whenever $p < \frac{k}{[\chi^2 + \chi(k-1)]_+}$. Consequently, $f(r;p,\chi,k) < 0$ for any $r \in \big(r_-(p), r_+(p)\big)$. This yields

$$\frac{d}{dt}\int_\Omega u^p v^{-r}dx \le r\int_\Omega u^p v^{-r}dx - r\int_\Omega u^{p+1}v^{-r-1}dx, \quad t \in (0, T_{\max}). \tag{2.3}$$

Due to $\int_\Omega u^p v^{-r} \le \big(\int_\Omega u^{p+1}v^{-r-1}\big)^{\frac{p}{p+1}}\big(\int_\Omega v^{p-r}\big)^{\frac{1}{p+1}}$, we obtain (2.2) from (2.3) and (2.1). □

Lemma 2.3 *Let v solve the second equation of (1.1) with $u \in L^\infty((0,T); L^q(\Omega))$, $k, T > 0$, $1 \le q \le p \le \infty$ with $\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) < 1$. Then*

$$\|v(\cdot,t)\|_{L^p(\Omega)} \le C(1 + \sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^q(\Omega)}), \quad t \in (0,T) \tag{2.4}$$

with some $C = C(v_0, k, p, q, \Omega) > 0$.

Proof. Noticing $v(\cdot,t) = \mathrm{e}^{t(k\Delta-1)}v_0 + \int_0^t \mathrm{e}^{(t-s)(k\Delta-1)}u(\cdot,s)ds$ for $t>0$, by the standard smoothing estimates for the heat semigroup under the homogeneous Neumann boundary conditions [15], we can obtain for $q \le p$ that

$$\begin{aligned}
\|v(\cdot,t)\|_{L^p(\Omega)} &\le \|\mathrm{e}^{t(k\Delta-1)}v_0\|_{L^p(\Omega)} + \int_0^t \|\mathrm{e}^{(t-s)(k\Delta-1)}u(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le C_1\|v_0\|_{L^\infty(\Omega)} + C_2 \sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^q(\Omega)} \int_0^t (1+[k(t-s)])^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\mathrm{e}^{-(\lambda_1+\frac{1}{k})[k(t-s)]}ds \\
&\le C_1\|v_0\|_{L^\infty(\Omega)} + \frac{C_2}{k} \sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^q(\Omega)} \int_0^\infty (1+\alpha)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\mathrm{e}^{-\lambda_1\alpha}d\alpha, \quad t\in(0,T),
\end{aligned}$$

where $C_1, C_2 > 0$ depend only on Ω, and λ_1 is the first nonzero eigenvalue of $-\Delta$ under the Neumann boundary condition. This proves (2.4). $\square$

Instead of taking $r = \frac{p-1}{2}$ in [2], we deal with the more complicated $r = (p-1)\frac{[p\chi(1-k)+2k]}{p(1-k)^2+4k}$ to describe the effect of the chemical diffusion rate k, and denote $h(p;\chi,k) := \frac{p\chi(1-k)+2k}{p(1-k)^2+4k}$. With $h(p) := h(p;\chi,k)$ for simplicity, we have the following lemma.

Lemma 2.4 *Let $k>0$, $\chi \in \big(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2+\frac{8k}{n}}\big)$, and $p \in \big(1, \frac{k}{[\chi^2+\chi(k-1)]_+}\big)$. Then $h(p)\in(0,1)$.*

Proof. We have $h'(p) = \frac{2k(1-k)[2\chi-(1-k)]}{[p(1-k)^2+4k]^2}$.

If $k=1$, then $h(p)\equiv\frac{1}{2}$.

If $k>1$, then $h'(p)<0$, and so
$0 < \frac{\chi}{2\chi+k-1} = h(\frac{k}{\chi^2-\chi(1-k)}-0) = h(\frac{k}{[\chi^2-\chi(1-k)]_+}-0) < h(p) < h(1+0) = \frac{2k+\chi(1-k)}{(1-k)^2+4k} < \frac{2k}{(1+k)^2} < 1$.

Now suppose $0<k<1$.

If $0<\chi<\frac{1-k}{2}$, then $h'(p)<0$, and so
$0 < \frac{\chi}{1-k} = \lim_{p\to\infty} \frac{p\chi(1-k)+2k}{p(1-k)^2+4k} < h(p) < h(1+0) = \frac{2k+\chi(1-k)}{(1-k)^2+4k} < \frac{1}{2}$;

If $\chi=\frac{1-k}{2}$, then $h(p)\equiv\frac{1}{2}$;

If $\frac{1-k}{2}<\chi\le 1-k$, then $h'(p)>0$, and so
$\frac{1}{2} < \frac{2k+\chi(1-k)}{(1+k)^2} = h(1+0) < h(p) < \lim_{p\to\infty}\frac{p\chi(1-k)+2k}{p(1-k)^2+4k} = \frac{\chi}{1-k} \le 1$;

If $1-k<\chi<\frac{-(k-1)+\sqrt{(k-1)^2+\frac{8k}{n}}}{2}$, then $h'(p)>0$, and so
$0 < \frac{1-k^2}{(1+k)^2} < \frac{2k+\chi(1-k)}{(1+k)^2} = h(1+0) < h(p) < h(\frac{k}{\chi^2-\chi(1-k)}-0) = \frac{\chi}{2\chi+k-1} < 1$.

The proof is complete. $\square$

Denote $c_0 := \inf_{p\in(1,\frac{n}{2}]} h(p)$ and $c^0 := \sup_{p\in(1,\frac{n}{2}]} h(p)$ with $n\ge3$. By Lemma 2.4 and its proof, $c_0, c^0 \in (0,1)$. The following lemma with c_0 and c^0 will play an important role for estimating the bound of u by the involved iteration in the next section.

Lemma 2.5 *Let $k>0$, $\chi \in \left(0, -\frac{k-1}{2}+\frac{1}{2}\sqrt{(k-1)^2+\frac{8k}{n}}\right)$. Then $\frac{n[(1-c^0)x+c^0-c_0]+2c_0x}{(n-2x)(1-c_0)}-x>0$ for all $x\in(1,\frac{n}{2}]$, provided $n\le 8$.*

Proof. Denote

$$f(x) := \frac{n[(1-c^0)x+c^0-c_0]+2c_0x}{(n-2x)(1-c_0)} - x,\ \ x\in(1,\frac{n}{2}].$$

It suffices to show that $g(x) := 2(1-c_0)x^2+[2c_0-n(c^0-c_0]x+n(c^0-c_0)>0$ in $(1,\frac{n}{2}]$.

The case of $c_0=c^0\in(0,1)$ is trivial.

Now suppose $0<c_0<c^0<1$. We have $\Delta_g=[2c_0-n(c^0-c_0)]^2-8n(1-c_0)(c^0-c_0)=[n(c^0-c_0)]^2+(4c_0-8)n(c^0-c_0)+4c_0^2<0$, and hence $g(x)>0$, whenever $4-2c_0-4\sqrt{1-c_0}<n(c^0-c_0)<4-2c_0+4\sqrt{1-c_0}$.

If $n(c^0-c_0)\le 4-2c_0-4\sqrt{1-c_0}$, then $\Delta_g>0$. Due $n(c^0-c_0)-2c_0<4(1-c_0)-4\sqrt{1-c_0}<0$, we know that both the two roots of $g(x)$ must be negative. With $g(1+0)=2$, we obtain $g(x)>0$.

If $n(c^0-c_0)\ge 4-2c_0+4\sqrt{1-c_0}$, then $\Delta_g>0$, and the two roots of $g(x)$ satisfy $x_2\ge x_1>0$. Together with $g(1+0)=2$ and $g(\frac{n}{2})=\frac{n^2}{2}(1-c^0)+nc_0>0$, the positivity of $g(x)$ for $x\in(1,\frac{n}{2}]$ requires that the minimal point of $g(x)$ satisfies $\frac{n(c^0-c_0)-2c_0}{4(1-c_0)}<1$, i.e., $n(c^0-c_0)+2c_0<4$, by Vieta's formulas. This contradicts the case $n(c^0-c_0)\ge 4-2c_0+4\sqrt{1-c_0}$. So, the case itself should be excluded to ensure $g(x)>0$ in $(1,\frac{n}{2}]$. Rewrite the case as $c^0\ge\frac{(n-2)c_0+4+4\sqrt{1-c_0}}{n}\triangleq\alpha(c_0)$ with $c_0\in(0,1)$. We get a contradiction that $c^0\ge\alpha(c_0)>\min\{\alpha(+0),\alpha(1-0)\}=\min\{\frac{n+2}{n},\frac{8}{n}\}\ge 1$, whenever $n\le 8$. $\square$

3 Proof of main result

We deal with the proof of the main result of the paper in this section.

Proof of Theorem 1.

We at first show that the local solutions ensured by Lemma 2.1 should be global. For simplicity, denote $T=T_{\max}$.

Assume $p>q$. By the Hölder inequality, we have

$$\int_\Omega u^q dx = \int_\Omega (u^p v^{-r})^{\frac{q}{p}} v^{\frac{rq}{p}}\,dx \le \Big(\int_\Omega u^p v^{-r} dx\Big)^{\frac{q}{p}}\Big(\int_\Omega v^{\frac{rq}{p-q}}\,dx\Big)^{\frac{p-q}{p}},\ \ 0<t<T. \tag{3.1}$$

Let $p<\frac{k}{[\chi^2+\chi(k-1)]_+}$. We know from (2.3) in the proof of Lemma 2.2 with $r\in(r_-(p),r_+(p))$ that

$$\frac{d}{dt}\int_\Omega u^p v^{-r}dx \le r\int_\Omega u^p v^{-r}dx,\ \ t\in(0,T),$$

and hence

$$\int_\Omega u^p v^{-r}dx\le C,\ \ t\in(0,T) \tag{3.2}$$

with $C = C(t) > 0$. Notice that $\chi \in \left(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}}\right)$ with $p < \frac{k}{[\chi^2+\chi(k-1)]_+}$ admits $p > \frac{n}{2}$.

Take $q \in (\frac{n}{2}, p) \subset \left(1, \min\{p, \frac{n(p-r)}{[n-2r]_+}\}\right)$. Then $\frac{n}{2}(\frac{1}{q} - \frac{p-q}{rq}) < 1$. By Lemma 2.3, we have with $C_1 > 0$ that

$$\|v\|_{L^{\frac{rq}{p-q}}(\Omega)} \le C_1(1 + \sup_{s\in(0,t)} \|u\|_{L^q(\Omega)}), \quad 0 < t < T. \tag{3.3}$$

Combining (3.1)–(3.3) yields

$$\int_\Omega u^q dx \le C_2\Big(1 + (\sup_{s\in(0,t)} \int_\Omega u^q dx)^{\frac{r}{p}}\Big), \quad t \in (0,T),$$

and hence $\int_\Omega u^q dx \le C_3$ for $t \in (0,T)$, where $C_2 = C_2(t) = \tilde{C}_2 e^{rt}$ with $\tilde{C}_2 > 0$, $C_3 = C_3(t) > 0$, and $\frac{r}{p} < \frac{p-1}{p} < 1$ by Lemma 2.4. Thus, we can follow the proof of [14, Lemma 3.4] to obtain the global existence of solutions for (1.1).

Next, we prove the solutions established above are also globally bounded if $n \le 8$. We should verify that $\int_\Omega u^q dx < C$ with some $q > \frac{n}{2}$ and $C > 0$ [14].

The case of $n = 2$ is simple. In fact, take

$$\begin{cases} p \in (1, \dfrac{k}{[\chi^2 + \chi(k-1)]_+}), \\ r = (p-1)h(p). \end{cases}$$

Then $1 - \frac{1}{p-r} < 1$. By Lemma 2.3 with the L^1-conservation of u, there exists $C_4 > 0$ such that

$$\|v\|_{L^{p-r}(\Omega)} \le C_4, \quad t \in (0,T).$$

By Lemma 2.2,

$$\int_\Omega u^p v^{-r} dx \le C_5, \quad t \in (0,T) \tag{3.4}$$

with some $C_5 > 0$. We deduce from (3.1), (3.3) and (3.4) that there exists $q > 1$, such that $\|u\|_{L^q(\Omega)} \le C_6$ for $t \in (0,T)$ with $C_6 > 0$.

Now consider the case of $3 \le n \le 8$. We should prove (2.1) for some $p > \frac{n}{2}$ with the constant independent of t there. We will do it via an iteration procedure based of Lemma 2.5.

1° Take

$$\begin{cases} p_0 \in \Big(1, \min\big\{\dfrac{k}{[\chi^2 + \chi(k-1)]_+}, \dfrac{n(1-c_0) + 2c_0}{(n-2)(1-c_0)}\big\}\Big), \\ r_0 = (p_0 - 1)h(p_0). \end{cases} \tag{3.5}$$

Then $p_0 - r_0 \le p_0 - (p_0 - 1)c_0 = (1 - c_0)p_0 + c_0 < \frac{n}{n-2}$, i.e., $\frac{n}{2}(1 - \frac{1}{p_0 - r_0}) < 1$. By Lemma 2.3 with the L^1-conservation of u, we have

$$\|v\|_{L^{p_0-r_0}(\Omega)} \le C_7, \quad t \in (0,T),$$

and hence

$$\int_\Omega u^{p_0} v^{-r_0} dx \le C_8, \quad t \in (0,T) \tag{3.6}$$

by Lemma 2.2, with some $C_7, C_8 > 0$. We know from (3.1), (3.3) and (3.6) that $\|u\|_{L^{q_0}(\Omega)} \le C_9$ for $t \in (0,T)$ with $C_9 > 0$. If $p_0 > \frac{n}{2}$, take $q_0 \in (\frac{n}{2}, p_0) \subset \big(1, \min\{p_0, \frac{n(p_0-r_0)}{[n-2r_0]_+}\}\big)$.

2° Assume $p_0 \le \frac{n}{2}$. Take

$$\begin{cases} p_1 \in \Big(p_0, \min\big\{\dfrac{k}{[\chi^2+\chi(k-1)]_+}, \dfrac{n[(1-c^0)p_0+c^0-c_0]+2c_0p_0}{(n-2p_0)(1-c_0)}\big\}\Big), \\ r_1 = (p_1-1)h(p_1), \end{cases}$$

where the well-definedness of the interval for p_1 is ensured by Lemma 2.5. A simple calculation shows

$$p_1 - r_1 \le (1-c_0)p_1 + c_0 < \frac{n[(1-c^0)p_0+c^0]}{n-2p_0} \le \frac{n(p_0-r_0)}{n-2p_0} = \frac{n\frac{n(p_0-r_0)}{n-2r_0}}{n-\frac{2n(p_0-r_0)}{n-2r_0}},$$

i.e., $\frac{n}{2}\big(\frac{1}{\frac{n(p_0-r_0)}{n-2r_0}} - \frac{1}{p_1-r_1}\big) < 1$. By Lemma 2.3 with $p_0 \le \frac{n}{2}$, there is $q_0 \in (1, \frac{n(p_0-r_0)}{n-2r_0})$ such that

$$\|v\|_{L^{p_1-r_1}(\Omega)} < C_{10}, \quad t \in (0,T),$$

and thus

$$\int_\Omega u^{p_1} v^{-r_1} dx \le C_{11}, \quad t \in (0,T) \tag{3.7}$$

by Lemma 2.2, with some $C_{10}, C_{11} > 0$. It follows from (3.1), (3.3) and (3.7) that $\|u\|_{L^{q_1}(\Omega)} \le C_{12}$ for $t \in (0,T)$ with $C_{12} > 0$. If $p_1 > \frac{n}{2}$, take $q_1 \in (\frac{n}{2}, p_1) \subset \big(1, \min\{p_1, \frac{n(p_1-r_1)}{[n-2r_1]_+}\}\big)$.

3° Assume $p_{l-1} \le \frac{n}{2}$ for some $l \in \{1,2,3,\dots\}$. Take

$$\begin{cases} p_l \in \Big(p_{l-1}, \min\big\{\dfrac{k}{[\chi^2+\chi(k-1)]_+}, \dfrac{n[(1-c^0)p_{l-1}+c^0-c_0]+2c_0p_{l-1}}{(n-2p_{l-1})(1-c_0)}\big\}\Big), \\ r_l = (p_{l-1}-1)h(p_{l-1}), \end{cases} \tag{3.8}$$

where the interval for p_l is well defined due to Lemma 2.5. Repeat the procedure in 2°, we deduce with $q_l \in \big(1, \min\{p_l, \frac{n(p_l-r_l)}{[n-2r_l]_+}\}\big)$ that $\|u\|_{L^{q_l}(\Omega)} \le C_{13}$, $0 < t < T$, with some $C_{13} > 0$. Noticing $\frac{n[(1-c^0)p_{l-1}+c^0-c_0]+2c_0p_{l-1}}{(n-2p_{l-1})(1-c_0)} \to \infty$ as $l \to \infty$ by Lemma 2.5, we can realize $p_l > \frac{n}{2}$ after finite steps. Let $q = q_l \in (\frac{n}{2}, p_l)$ to get $\int_\Omega u^q dx \le C_{14}$ for all $t \in (0,T)$ with $C_{14} > 0$. It is mentioned that for any fixed $k > 0$, the involved constants C_i, $i = 4, \dots, 14$, are all independent of $t \in (0,T)$ here.

Based on the above estimate $\|u(\cdot,t)\|_{L^q(\Omega)}$ with $q > \frac{n}{2}$, uniform for $t \in (0,T)$, we conclude the global boundedness of u by repeating the related arguments in [14] for the case of $k = 1$. $\square$

Remark 3 Notice that $p_0 < \frac{k}{[\chi^2+\chi(k-1)]_+}$ in (3.5) with $\chi \in \left(0, -\frac{k-1}{2} + \frac{1}{2}\sqrt{(k-1)^2 + \frac{8k}{n}}\right)$ admits $p_0 > \frac{n}{2}$. Moreover, a simple computation shows $p_0 < \frac{n(1-c_0)+2c_0}{(n-2)(1-c_0)}$ in (3.5) ensures $\frac{n(1-c_0)+2c_0}{(n-2)(1-c_0)} > \frac{n}{2}$ whenever $n = 3, 4$. Therefore, the Steps 2° and 3° in the proof of Theorem 1 are unnecessary for $n = 3, 4$ there. In addition, it should be pointed out that if $k = 1$, then $c^0 = c_0 = \frac{1}{2}$, and thus the requirement $n \leq 8$ itself can be removed away for the global boundedness of solutions.

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