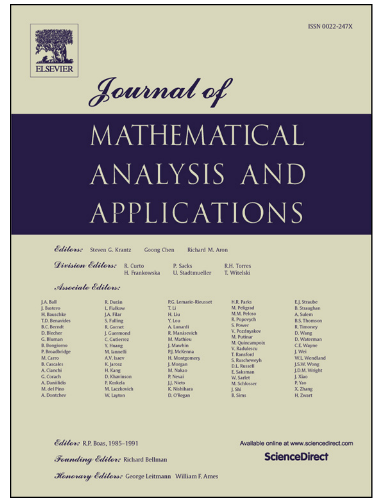
# Accepted Manuscript

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PII: S0022-247X(16)30402-4
DOI: http://dx.doi.org/10.1016/j.jmaa.2016.08.004
Reference: YJMAA 20640

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 28 October 2015

# Asymptotically autonomous multivalued Cauchy problems with spatially variable exponents

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**Abstract**

We study the asymptotic behavior of a non-autonomous multivalued Cauchy problem of the form

$$\frac{\partial u}{\partial t}(t) - \operatorname{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) + F(t, u(t)) \ni 0$$

on a bounded smooth domain $\Omega$ in $\mathbb{R}^n$, $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\overline{\Omega})$ satisfies $p^- := \min p(x) > 2$. We prove the existence of a pullback attractor and study the asymptotic upper semicontinuity of the elements of the pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ as $t \to \infty$ for the non-autonomous evolution inclusion in a Hilbert space $H$ under the assumptions, amongst others, that $F$ is a measurable multifunction and $D \in L^\infty([\tau, T] \times \Omega)$ is bounded above and below and is monotonically nonincreasing in time. The global existence of solutions is obtained through results of Papageorgiou and Papalini.

*Keywords:* Multivalued Cauchy problem, variable exponents, pullback attractors, time-dependent operator, asymptotically autonomous inclusion
*2010 MSC:* 35K55, 37L30, 35B40

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*Preprint submitted to Journal of Mathematical Analysis and Applications August 8, 2016*

## 1. Introduction

In this paper we study a multivalued Cauchy problem of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t) - \operatorname{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) + F(t,u(t)) \ni 0 \\ u(\tau) = u_0 \end{cases} \tag{1}$$

on a bounded smooth domain $\Omega$ in $\mathbb{R}^n$, $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\overline{\Omega})$ satisfies

$$p^+ := \max_{x\in\overline{\Omega}} p(x) \geq p^- := \min_{x\in\overline{\Omega}} p(x) > 2,$$

and the initial condition $u(\tau) \in H := L^2(\Omega)$. The terms $D$ and $F$ are assumed to satisfy:

**Assumption D.** $D : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ satisfying:
(D1) There are positive constants, $\beta$ and $M$ such that $0 < \beta \leq D(t,x) \leq M$ for almost all $(t,x) \in [\tau, T] \times \Omega$.
(D2) $D(t,x) \geq D(s,x)$ for each $x \in \Omega$ and $t \leq s$ in $[\tau, T]$.

**Assumption F.** $F : [\tau, T] \times H \to P_f(H)$, where

$$P_f(H) := \{A \subset H : A \text{ nonempty and closed}\},$$

is a multifunction satisfying:
(F1) For all $x \in H$, $t \mapsto F(t,x)$ is measurable, that is, for all $y \in H$, the function

$$[\tau, T] \ni t \mapsto d(y, F(t,x)) := \inf\{\|y - z\|_H : z \in F(t,x)\} \in \mathbb{R}$$

is measurable.
(F2) There exists $k \in L^1([\tau, T], \mathbb{R}^+)$ such that

$$h(F(t,x), F(t,y)) \leq k(t)\|x - y\|$$

a.e. on $[\tau, T]$, for all $x, y \in H$. Here $h$ denotes the Hausdorff metric on $P_f(H)$ given by: for $A, B \in P_f(H)$,

$$h(A,B) := \max\{\operatorname{dist}(A,B), \operatorname{dist}(B,A)\},$$

where $\mathrm{dist}(A,B) := \sup\{d(a,B) : a \in A\}$, $d(a,B) := \inf\{\|a-b\|_H : b \in B\}$ and similarly for $\mathrm{dist}(B,A)$.
(F3) There exist $a, c \in L^2([\tau,T],\mathbb{R}^+)$ :

$$\|F(t,x)\| := \sup\{\|z\|_H : z \in F(t,x)\} \leq a(t) + c(t)\|x\|_H,$$

a.e. in $[\tau,T]$, for all $x \in H$.

In [9] the authors proved the existence of a pullback attractor for the following non-autonomous evolution equation

$$\frac{\partial u}{\partial t}(t) - \mathrm{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t) = B(t,u(t)) \qquad (2)$$

on a bounded smooth domain $\Omega$ in $\mathbb{R}^n$, $B$ was globally Lipschitz in its second variable and $D$ was assumed to satisfy **Assumption D** above. Moreover, they proved upper semicontinuity of pullback attractors when the diffusion parameters vary. A similar problem was studied in [15] for a constant exponent $p$ and stronger conditions on the diffusion coefficients $D$. In [10], the authors considered $B(t,u(t)) \equiv B(u)$ in the problem (2) and proved that it is asymptotically autonomous.

The paper is organized as follows. In Section 2 we prove existence of solution for inclusion (1) following Papageorgiou and Papalini [13]. In Section 3 we provide estimates on the solutions. In Section 4 we establish the existence of a pullback attractor. In Section 5 we prove the asymptotic upper semicontinuity of the elements of the pullback attractor, i.e., we prove that the inclusion (1) is, in fact, asymptotically autonomous when $F$ does not depend explicitily on $t$.

## 2. Existence of solution

In this section we present the operator and its properties and we establish existence of solution for the multivalued Cauchy problem (1).

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain, $H := L^2(\Omega)$ and $Y := W^{1,p(\cdot)}(\Omega)$ with $p^- > 2$. Then $Y \subset H \subset Y^*$ with continuous and dense embeddings. Moreover, the inclusion $Y \subset H$ is compact (see Proposition 2.1 (ii) and Proposition 2.5 (ii) in [7]). We refer the reader to [6, 7] and references therein to see properties of the Lebesgue and Sobolev spaces with variable exponents.

In particular, with

$$L^{p(\cdot)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$$

and $L^\infty_+(\Omega) := \{q \in L^\infty(\Omega) : \text{ ess inf } q \geq 1\}$, define

$$\rho(u) := \int_\Omega |u(x)|^{p(x)} dx, \qquad \|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho\Big(\frac{u}{\lambda}\Big) \leq 1 \right\},$$

for $u \in L^{p(\cdot)}(\Omega)$ and $p \in L^\infty_+(\Omega)$.

Consider the operator $A(t)$ defined in $Y$ such that for each $u \in Y$ associate the following element of $Y^*$, $A(t)u : Y \to \mathbb{R}$ given by

$$A(t)u(v) := \int_\Omega D(t,x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) dx + \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx.$$

The authors in [9] proved that:

- For each $t \in [\tau, T]$ the operator $A(t) : Y \to Y^*$, with domain $Y = W^{1,p(\cdot)}(\Omega)$, is maximal monotone and $A(t)(Y) = Y^*$.
- The realization operator of $A(t)$ at $H = L^2(\Omega)$, i.e., $A_H(t)u = -\text{div}(D(t)|\nabla u(t)|^{p(x)-2}\nabla u(t)) + |u(t)|^{p(x)-2}u(t)$, is maximal monotone in $H$ for each $t \in [\tau, T]$.
- The operator $A_H(t)$ is the subdifferential $\partial \varphi^t_{p(\cdot)}$ of the convex, proper and lower semicontinuous map $\varphi^t_{p(\cdot)} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi^t_{p(\cdot)}(u) = \begin{cases} \left[ \displaystyle\int_\Omega \frac{D(t,x)}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx \right]; \text{ if } u \in Y \\ +\infty, \text{ otherwise.} \end{cases} \tag{3}$$

As our operator is of subdifferential type, we can obtain existence of global solution for problem (1) using results in the paper of Papageorgiou and Papalini [13].

We recall the definition of solution and a result of existence from [13]. Consider a multivalued Cauchy problem of the form

$$\begin{cases} \dfrac{du}{dt}(t) + \partial \varphi^t(u(t)) + F(t, u(t)) \ni 0, & \text{a.e. on } [0,T], \\ u(0) = \xi, \quad \xi \in H. \end{cases} \tag{4}$$

in a real Hilbert space $H$, where, for all $t \in [0,T]$, $\partial\varphi^t$ is the subdifferential of a lower semicontinuous proper convex function $\varphi^t$ from $H$ into $(-\infty,\infty]$.

**Definition 1.** *A solution of (4) is a continuous function $u : [0,T] \to H$ such that $u(\cdot)$ is absolutely continuous on any closed subinterval of $(0,T)$ and with the property*

*1) $u(t) \in D(\partial\varphi^t)$, a.e. on $[0,T]$;*

*2) There exists $f \in L^2([0,T];H)$ such that $f(t) \in F(t,u(t))$ and*

$$\frac{du}{dt}(t) + \partial\varphi^t(u(t)) + f(t) \ni 0, \quad a.e.\ on\ [0,T];$$

*3) $u(0) = \xi$.*

Under the following assumption Papageorgiou and Papalini [13] established the existence of solution to problem (4) in Theorem 1 below.

**Assumption A.** Let $T > 0$ be fixed.
$(A.1)$ For each $t \in [0,T]$, $\varphi^t : H \to (-\infty,\infty]$ is proper, convex and lower semicontinuous.
$(A.2)$ For any positive integer $r$ there exist a constant $K_r > 0$, an absolutely continuous function $g_r : [0,T] \to \mathbb{R}$ with $g_r' \in L^\beta(0,T)$ and a function of bounded variation $h_r : [0,T] \to \mathbb{R}$ such that if $t \in [0,T]$, $w \in D(\varphi^t)$ with $|w| \le r$ and $s \in [t,T]$, then there exists an element $\tilde{w} \in D(\varphi^s)$ satisfying

$$\begin{aligned} |\tilde{w} - w| &\le |g_r(s) - g_r(t)|(\varphi^t(w)) + K_r)^\alpha, \\ \varphi^s(\tilde{w}) &\le \varphi^t(w) + |h_r(s) - h_r(t)|(\varphi^t(w) + K_r), \end{aligned}$$

where $\alpha$ is some fixed constant with $0 \le \alpha \le 1$ and

$$\beta := \begin{cases} 2 & \text{if } \ 0 \le \alpha \le \frac{1}{2}, \\ \dfrac{1}{1-\alpha} & \text{if } \ \frac{1}{2} \le \alpha \le 1. \end{cases}$$

**Theorem 1.** *[Theorem 6 in [13]] Suppose that Assumption $(A)$ and Assumption $(F)$ are satisfied. Then for each $\xi \in cl(D(\varphi^0))$, the multivalued problem (4) has a solution $u$ on $[0,T]$ with $u(0) = \xi$.*

Let us now return to our particular problem. We already saw that $(A.1)$ is satisfied. To check condition $(A.2)$ consider a positive integer $r$ and define

$K_r := r$, $g_r(t) := t + r$ and $h_r(t) := r$. For $t \in [0, T]$, $w \in D(\varphi^t_{p(\cdot)}) = Y$ with $\|w\|_Y \leq r$ and $s \in [t, T]$, consider the element $\tilde{w} := w \in Y = D(\varphi^s_{p(\cdot)})$. Taking $\alpha := 1/2$, condition $(A.2)$ is then satisfied with $\beta = 2$. In particular, we need Assumption (D2) that $D(t, x) \geq D(s, x)$ for each $x \in \Omega$ and $t \leq s$ in $[0, T]$ here. Thus, we obtain the existence of a global solution for the following problem.

**Theorem 2.** *If Assumption* $(D)$ *and Assumption* $(F)$ *hold, then the multi-valued problem (1) has a solution for every* $u_0 \in H$.

## 3. Estimates on the solutions

In this section we provide estimates on the solutions in the spaces $H$ and $Y$.

**Theorem 3.** *Let* $u$ *be a solution of problem (1). Then there exist a constant* $T_1$ *and a function* $B_1 : \mathbb{R} \to \mathbb{R}$ *which does not depend on the initial data, such that*

$$\|u(t)\|_H \leq B_1(t), \quad \forall\, t \geq T_1 + \tau.$$

**Proof:** As $u$ is a solution of (1) there exists $f \in L^2([\tau, T]; H)$ such that $f(t) \in F(t, u(t))$ and

$$\frac{du}{dt}(t) + A(t)u(t) + f(t) = 0, \tag{5}$$

a.e. on $[\tau, T]$. Multiplying the equation (5) by $u(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_H + \langle A(t)u(t), u(t)\rangle + \langle f(t), u(t)\rangle = 0.$$

We know that if $p(x) > q(x)$ then $L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega)$ with $\|u\|_{L^{q(\cdot)}(\Omega)} \leq 2(|\Omega| + 1)\|u\|_{L^{p(\cdot)}(\Omega)}$ for all $u \in L^{p(x)}(\Omega)$ (see [6]). Thus

$$\|u(t)\|_H \leq 2(|\Omega| + 1)\|u(t)\|_{L^{p(\cdot)}(\Omega)} \leq 2(|\Omega| + 1)\|u(t)\|_Y.$$

If $\|u(t)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ and $\|\nabla u(t)\|_{L^{p(\cdot)}(\Omega)} \geq 1$, then by Lemma 2.3 in [9]

$$\langle A(t)u(t), u(t)\rangle \geq \frac{\min\{1, \beta\}}{2^{(p^- - 1)}}\|u(t)\|^{p^-}_Y,$$

and then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 \le -\frac{\min\{1,\beta\}}{2^{(p^- -1)}}\|u(t)\|_Y^{p^-} - \langle f(t), u(t)\rangle.$$

Using the Cauchy-Schwarz inequality and (F3), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 &\le -\frac{\min\{1,\beta\}}{2^{(p^- -1)}}\|u(t)\|_Y^{p^-} + \|f(t)\|_H\|u(t)\|_H \\ &\le -\frac{\min\{1,\beta\}}{2^{(p^- -1)}}\|u(t)\|_Y^{p^-} + (a(t)+c(t)\|u(t)\|_H)\,\|u(t)\|_H \quad (6)\\ &\le -\frac{\min\{1,\beta\}}{2^{(p^- -1)}}\|u(t)\|_Y^{p^-} + C_1(t)\|u(t)\|_Y^2 + C_2(t)\|u(t)\|_Y\end{aligned}$$

where $C_1(t) := [2(|\Omega|+1)]^2 c(t)$ and $C_2(t) := 2(|\Omega|+1)a(t)$.

If $\theta := \frac{1}{2}p^-$, $\theta' := \frac{\theta}{\theta-1}$ and $\epsilon > 0$, it follows from Young's inequality

$$\begin{aligned}C_1(t)\|u(t)\|_Y^2 + C_2(t)\|u(t)\|_Y &= \frac{C_1(t)\epsilon}{\epsilon}\|u(t)\|_Y^2 + \frac{C_2(t)\epsilon}{\epsilon}\|u(t)\|_Y \\ &\le \frac{1}{\theta'}\left(\frac{C_1(t)}{\epsilon}\right)^{\theta'} + \frac{1}{\theta}\epsilon^{\theta}\|u(t)\|_Y^{p^-} \quad (7)\\ &\quad + \frac{1}{(p^-)'}\left(\frac{C_2(t)}{\epsilon}\right)^{(p^-)'} + \frac{1}{p^-}\epsilon^{p^-}\|u(t)\|_Y^{p^-}.\end{aligned}$$

Choose $\epsilon_0 > 0$ such that

$$\gamma := \frac{\min\{1,\beta\}}{2^{(p^- -1)}} - \frac{1}{\theta}\epsilon_0^{\theta} - \frac{1}{p^-}\epsilon_0^{p^-} > 0,$$

we have from (6) and (7),

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \gamma\|u(t)\|_Y^{p^-} \le \frac{1}{\theta'}\left(\frac{C_1(t)}{\epsilon_0}\right)^{\theta'} + \frac{1}{(p^-)'}\left(\frac{C_2(t)}{\epsilon_0}\right)^{(p^-)'}.$$

Let $\delta(t) := \frac{2}{\theta'}\left(\frac{C_1(t)}{\epsilon_0}\right)^{\theta'} + \frac{2}{(p^-)'}\left(\frac{C_2(t)}{\epsilon_0}\right)^{(p^-)'}$, $\tilde{\gamma} := \frac{2\gamma}{[4(|\Omega|+1)^2]^{p^-}}$ and $y(t) := \|u(t)\|_H^2$. Then

$$y'(t) + \tilde{\gamma}y(t)^{p^-/2} \le \delta(t), \quad \forall\, t \ge \tau.$$

From a slight generalization of Lemma 5.1 in [16], we obtain

$$y(t) \leq \left(\frac{\delta(t)}{\tilde{\gamma}}\right)^{2/p^-} + \left(\tilde{\gamma}\left(\frac{p^- - 2}{2}\right)(t-\tau)\right)^{-\frac{2}{p^- - 2}}.$$

Let $\tau_1 > 0$ be such that $\left[\tilde{\gamma}\left(\frac{p^- -2}{2}\right)\tau_1\right]^{-\frac{2}{p^- -2}} \leq 1$. Then

$$\|u(t)\|_H \leq \left(\frac{\delta(t)}{\tilde{\gamma}}\right)^{1/p^-} + 1 =: K_1(t),$$

for all $t \geq \tau_1 + \tau$. Observe that $K_1(t)$ depends on $a(t)$ and $c(t)$ of Assumption (F3).

Similarly for each of the cases: $\|u(t)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ and $\|\nabla u(t)\|_{L^{p(\cdot)}(\Omega)} \leq 1$; $\|u(t)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and $\|\nabla u(t)\|_{L^{p(\cdot)}(\Omega)} \geq 1$; $\|u(t)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and $\|\nabla u(t)\|_{L^{p(\cdot)}(\Omega)} \leq 1$, we obtain $(K_2(t),\tau_2)$, $(K_3(t),\tau_3)$ and $(K_4(t),\tau_4)$ such that

$$\|u(t)\|_H \leq K_i(t), \forall\, t \geq \tau_i + \tau,$$

for $i = 2$, 3, 4, respectively. Taking $B_1(t) := \max\{K_1(t), K_2(t), K_3(t), K_4(t)\}$ and $T_1 := \max\{\tau_1,\tau_2,\tau_3,\tau_4\}$ we obtain

$$\|u(t)\|_H \leq B_1(t), \forall\, t \geq T_1 + \tau,$$

where $B_1(t)$ depends on $a(t)$ and $c(t)$ of Assumption (F3). This completes the proof of the theorem. $\square$

**Remark 1.** *Note that the function $B_1$ depends on the functions $a$ and $c$ of condition (F3) in the sense that if $a$ and $c$ are nondecreasing functions then $B_1(t)$ in Theorem 3 is a nondecreasing function.*

**Theorem 4.** *Let $u$ be a solution of problem (1). Then there exist a constant $T_2$ and a function $B_2 : \mathbb{R} \to \mathbb{R}$, which does not depend on the initial data, such that*

$$\|u(t)\|_Y \leq B_2(t), \quad \forall\, t \geq T_2 + \tau.$$

**Proof:** If $u$ is a solution of (1) then, there exists $f \in L^2([\tau, T]; H)$ such that $f(t) \in F(t, u(t))$ and

$$\frac{du}{dt}(t) + A(t)u(t) + f(t) = 0,$$

a.e. on $[\tau, T]$. Furthermore, by Theorem 3

$$\|u(t)\|_H \leq B_1(t), \quad \forall\, t \geq T_1 + \tau.$$

Using the notation $\varphi^t_{p(\cdot)}(\cdot) := \varphi_{p(\cdot)}(t, \cdot)$ and $y(t) := \varphi^t_{p(\cdot)}(u(t))$ we have

$$\frac{d}{dt}y(t) = \frac{\partial}{\partial t}\varphi_{p(\cdot)}(t, u(t)) + \langle \partial\varphi^t_{p(\cdot)}(u(t)), \frac{d}{dt}u(t)\rangle.$$

Differentiating under the integral sign in (3) and using assumption $(D2)$, we obtain $\frac{\partial}{\partial t}\varphi_{p(\cdot)}(t, u(t)) \leq 0$. Therefore

$$\frac{d}{dt}\varphi^t_{p(\cdot)}(u(t)) \leq \left\langle \partial\varphi^t_{p(\cdot)}(u(t)), \frac{du}{dt}(t)\right\rangle.$$

Then

$$\begin{aligned} \frac{d}{dt}\varphi^t_{p(\cdot)}(u(t)) &\leq \left\langle -f(t) - \frac{du}{dt}(t), \frac{du}{dt}(t)\right\rangle \\ &= -\left\| f(t) + \frac{du}{dt}(t)\right\|_H^2 + \left\langle f(t) + \frac{du}{dt}(t), f(t)\right\rangle \\ &\leq -\frac{1}{2}\left\| f(t) + \frac{du}{dt}(t)\right\|_H^2 + \frac{1}{2}\|f(t)\|_H^2. \end{aligned}$$

Thus

$$\frac{d}{dt}\varphi^t_{p(\cdot)}(u(t)) + \frac{1}{2}\left\| f(t) + \frac{du}{dt}(t)\right\|_H^2 \leq \frac{1}{2}\|f(t)\|_H^2,$$

and we obtain

$$\begin{aligned} \frac{d}{dt}\varphi^t_{p(\cdot)}(u(t)) &\leq \frac{1}{2}\|f(t)\|_H^2 \leq \frac{1}{2}\left(a(t) + c(t)\|u(t)\|_H\right)^2 \\ &\leq \frac{1}{2}\left(a(t) + c(t)B_1(t)\right)^2 = \frac{1}{2}M_1(t)^2 \end{aligned}$$

for all $t \geq T_1 + \tau$, where $M_1(t) := a(t) + c(t)B_1(t)$.

From the definition of subdifferential, we have

$$\varphi^t_{p(\cdot)}(u(t)) \leq \left\langle \partial\varphi^t_{p(\cdot)}(u(t)), u(t)\right\rangle.$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 + \varphi_{p(\cdot)}^t(u(t)) &\leq \left\langle \frac{du}{dt}(t), u(t) \right\rangle + \left\langle \partial\varphi_{p(\cdot)}^t(u(t)), u(t) \right\rangle \\
&= \langle -f(t), u(t) \rangle \leq \|f(t)\|_H \|u(t)\|_H \qquad (8) \\
&\leq M_1(t)B_1(t), \qquad \forall\, t \geq T_1 + \tau.
\end{aligned}$$

Fixing $r > 0$ and integrating both sides of (8) over $(t, t+r)$ for $t \geq T_1 + \tau$,

$$\begin{aligned}
\int_t^{t+r} \varphi_{p(\cdot)}^s(u(s))ds &\leq \frac{1}{2}\|u(t)\|_H^2 + \int_t^{t+r} M_1(s)B_1(s)ds \\
&\leq \frac{1}{2}B_1(t)^2 + \int_t^{t+r} M_1(s)B_1(s)ds =: a_3(t).
\end{aligned}$$

Let $y(s) := \varphi_{p(\cdot)}^s(u(s))$, $g := 0$ and $h(s) := \frac{1}{2}M_1(s)^2$. Then

$$\int_t^{t+r} g(s)ds = 0 =: a_1(t), \qquad \int_t^{t+r} h(s)ds =: a_2(t), \qquad \int_t^{t+r} y(s)ds \leq a_3(t),$$

so, by a slight generalization of the Uniform Gronwall Lemma [16], we obtain

$$y(t+r) \leq \left( \frac{a_3(t)}{r} + a_2(t) \right) e^0 =: \tilde{r}_1(t), \qquad \forall\, t \geq T_1 + \tau.$$

Therefore,

$$\int_\Omega \frac{D(\ell, x)}{p(x)} \left|\nabla u(\ell, x)\right|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \left|u(\ell, x)\right|^{p(x)} dx \;\leq\; \tilde{r}_1(\ell),$$

for all $\ell \geq T_1 + \tau + r$. Then

$$\frac{\min\{1, \beta\}}{p^+} \left[\rho\left(\nabla u(\ell)\right) + \rho\left(u(\ell)\right)\right] \;\leq\; \tilde{r}_1(\ell)$$

for all $\ell \geq T_1 + \tau + r$, and hence,

$$\rho\left(\nabla u(\ell)\right) + \rho\left(u(\ell)\right) \;\leq\; \frac{p^+}{\min\{1, \beta\}} \tilde{r}_1(\ell) \qquad (9)$$

for all $\ell \geq T_1 + \tau + r$.

If $\ell \geq T_1 + \tau + r$ and $\|u(\ell)\|_Y \geq 1$ there are four cases to analyze:

**Case 1:** If $\|\nabla u(\ell)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ and $\|u(\ell)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ we know that

$$\|\nabla u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho(\nabla u(\ell)) \leq \|\nabla u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)},$$

and

$$\|u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho(u(\ell)) \leq \|u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)}.$$

Using (9) gives

$$\|u(t)\|_Y \leq R_1(t), \qquad t \geq T_2 + \tau,$$

where $R_1(t) := 2\left[\frac{p^+}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{1/p^-}$ and $T_2 := T_1 + r$.

**Case 2:** If $\|\nabla u(\ell)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ and $\|u(\ell)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ we know that

$$\|\nabla u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho(\nabla u(\ell)) \leq \|\nabla u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)},$$

and

$$\|u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho(u(\ell)) \leq \|u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)}.$$

Using (9) we obtain

$$\|u(t)\|_Y \leq R_2(t), \qquad t \geq T_2 + \tau,$$

where $R_2(t) := \left[\frac{p^+}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{1/p^-} + \left[\frac{p^+}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{1/p^+}$.

**Case 3:** If $\|\nabla u(\ell)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and $\|u(\ell)\|_{L^{p(\cdot)}(\Omega)} \geq 1$ we know that

$$\|\nabla u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho(\nabla u(\ell)) \leq \|\nabla u(\ell)\|^{p_s^-}_{L^{p(\cdot)}(\Omega)},$$

and

$$\|u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho(u(\ell)) \leq \|u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)}.$$

By (9) it follows that

$$\|u(t)\|_Y \leq R_3(t), \qquad t \geq T_2 + \tau,$$

where $R_3(t) := R_2(t)$.

**Case 4:** If $\|\nabla u(\ell)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ and $\|u(\ell)\|_{L^{p(\cdot)}(\Omega)} \leq 1$ then we know that

$$\|\nabla u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho(\nabla u(\ell)) \leq \|\nabla u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)},$$

and

$$\|u(\ell)\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho(u(\ell)) \leq \|u(\ell)\|^{p^-}_{L^{p(\cdot)}(\Omega)}.$$

Using (9) then gives

$$\|u(t)\|_Y \leq R_4(t), \qquad t \geq T_2 + \tau,$$

where $R_4(t) := 2\left[\frac{p^+}{\min\{1,\beta\}}\tilde{r_1}(t)\right]^{1/p^+}$.

In summary, defining

$$B_2(t) := \max\left\{1, 2\left[\left(\frac{p^+}{\min\{1,\beta\}}\tilde{r_1}(t)\right)^{1/p^-} + \left(\frac{p^+}{\min\{1,\beta\}}\tilde{r_1}(t)\right)^{1/p^+}\right]\right\}$$

we have $\|u(t)\|_Y \leq B_2(t)$ for all $t \geq T_2 + \tau$. □

**Remark 2.** *Note that the function $B_2$ depends on the functions $a$ and $c$ of condition (F3) in the sense that if $a$ and $c$ are nondecreasing functions then $B_2(t)$ in Theorem 4 is a nondecreasing function.*

## 4. Existence of a pullback attractor

We start this subsection with some definitions, see e.g., [2, 3, 12].

**Definition 2.** *Let $X$ be a complete metric space, $P(X)$ the set of all nonempty subsets of $X$ and $\mathbb{R}_d := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$. The map $U : \mathbb{R}_d \times X \to P(X)$ is called a multivalued evolution process on $X$ if*
*(1) $U(t,t,\cdot) = \mathbf{1}$ is the identity map;*
*(2) $U(t,s,x) \subset U(t,\tau,U(\tau,s,x))$, for all $x \in X$, $s \leq \tau \leq t$, where*

$$U(t,\tau,U(\tau,s,x)) = \bigcup_{y \in U(\tau,s,x)} U(t,\tau,y).$$

*The multivalued evolution process $U$ is called strict if*

$$U(t,s,x) = U(t,\tau,U(\tau,s,x)), \text{ for all } x \in X,\ s \leq \tau \leq t.$$

**Definition 3.** *Let $U$ be a multivalued evolution process on $X$ and $t \in \mathbb{R}$. The set $D(t) \subset X$ attracts (pullback) the nonempty bounded subset $B$ of $X$ at time $t$ if*

$$\lim_{\tau \to -\infty} dist(U(t,\tau)B, D(t)) = 0. \tag{10}$$

*The set $D(t)$ is said to be (pullback) attracting at time $t$ if (10) is satisfied for any nonempty bounded subset $B \subset X$.*

For a nonempty and bounded subset $B \subset X$ and $t \in \mathbb{R}$, put $\gamma^s(t,B) = \bigcup_{\tau \le s} U(t,\tau,B)$ and $\omega(t,B) = \bigcap \overline{\gamma^s(t,B)}$. The set $\omega(t,B)$ is called the pullback $\omega$-limit set of $B$ at time $t$ with respect to the multivalued evolution process $U$.

**Theorem 5.** *[Theorem 6 in [3]] Suppose that for $t \in \mathbb{R}$ and $B$ a nonempty and bounded subset of $X$ there exists a nonempty compact subset $D(t,B)$ of $X$ such that*

$$\lim_{s \to -\infty} dist(U(t,s)B, D(t,B)) = 0.$$

*Then $\omega(t,B)$ is nonempty, compact and the minimal closed set attracting $B$ at time $t$.*

**Definition 4.** *A family of sets $\{A(t) : t \in \mathbb{R}\}$ of $X$ is called a pullback attractor for the multivalued evolution process $U$ if*
*(1) $A(t)$ is pullback attracting at time $t$ for all $t \in \mathbb{R}$;*
*(2) it is semi-invariant (or negatively invariant), that is,*

$$A(t) \subset U(t,s,A(s)), \quad \text{for any } (t,s) \in \mathbb{R}_d;$$

*(3) it is minimal, that is, for any closed attracting set $Y$ at time $t$, we have $A(t) \subset Y$.*

**Theorem 6.** *[Theorem 18 in [3]] Let us suppose that for all $(t,s) \in \mathbb{R}_d$ the map $x \mapsto U(t,s,x) \in P(X)$ is closed. If, moreover, for any $t \in \mathbb{R}$ there exists a nonempty compact set $D(t)$ which is attracting, then the set $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$, with*

$$A(t) = \overline{\bigcup_{B \in \mathcal{B}(X)} \omega(t,B)}$$

*where $\mathcal{B}(X) = \{B \in P(X) : B \text{ is bounded}\}$, is the pullback attractor of $U$. Moreover, the sets $A(t)$ are compact.*

The multivalued evolution process associated with problem (1) is the map $U : \mathbb{R}_d \times H \to P(H)$ define by

$$U(t, \tau, \xi) = \{z : \text{there exists } u(\cdot) \in \mathcal{D}_\tau(\xi) \text{ such that } u(t) = z\}$$

where $\mathcal{D}_\tau(\xi)$ denotes the set of all solutions of problem (1) corresponding to the initial condition $u(\tau) = \xi$.

**Theorem 7.** *Let $t \in \mathbb{R}$ and let $B$ be a nonempty and bounded subset of $H$. Then the $\omega$-limit set $\omega(t, B)$ corresponding to the multivalued evolution process associated with problem (1) is nonempty, compact and the minimal closed set attracting $B$ at time $t$.*

**Proof:** Note that Theorem 4 shows that the family $K(t) = \overline{B_Y(0, B_2(t))}^H$ of compact sets of $H$ pullback attracts bounded sets of $H$ at time $t$. Hence, by Theorem 5, $\omega(t, B)$ is nonempty, compact and the minimal closed set attracting $B$ at time $t$. □

**Lemma 1.** *Let $\xi \in H$ fixed. If $g_n \to g$ weakly in $L^2([\tau, T]; H)$, then the solution $u_n$ of the problem*

$$\begin{cases} \frac{\partial u_n}{\partial t}(t) + A_H(t)u_n(t) + g_n(t) = 0 \quad \textit{a.e. on } [\tau, T], \\ u_n(\tau) = \xi. \end{cases}$$

*converges in $C([\tau, T]; H)$ to the solution $u$ of the problem*

$$\begin{cases} \frac{\partial u}{\partial t}(t) + A_H(t)u(t) + g(t) = 0 \quad \textit{a.e. on } [\tau, T], \\ u(\tau) = \xi. \end{cases}$$

**Proof:** If $\xi \in Y = W^{1,p(\cdot)}(\Omega) = D(\varphi^\tau_{p(\cdot)})$ the result follows from [8, 13].

If $\xi \in H \setminus Y$, given $n \in \mathbb{N}$ there exists $\xi_n \in Y$ with $\|\xi_n - \xi\| < \frac{1}{n}$. Let $v_{n,j}(\cdot)$ and $v_j(\cdot)$ be respectively the unique solution to

$$\begin{cases} \frac{\partial v_{n,j}}{\partial t}(t) + A_H(t)v_{n,j}(t) + g_n(t) = 0 \quad \text{a.e. on } [\tau, T], \\ v_{n,j}(\tau) = \xi_j, \end{cases}$$

and

$$\begin{cases} \frac{\partial v_j}{\partial t}(t) + A_H(t)v_j(t) + g(t) = 0 \quad \text{a.e. on } [\tau, T], \\ v_j(\tau) = \xi_j. \end{cases}$$

Since $\xi_j \in Y$ we have from [8, 13] that $v_{n,j} \to v_j$ in $C([\tau,T];H)$ as $n \to \infty$. Moreover,

$$\|v_j(t) - u(t)\|_H \leq \|\xi_j - \xi\|_H < \frac{1}{j}, \quad \forall\, t \in [\tau, T],$$

and $v_j \to u$ in $C([\tau,T];H)$ as $j \to \infty$. Hence, the subsequence $v_{n,n} \to u$ in $C([\tau,T];H)$ as $n \to \infty$.

Since

$$\begin{aligned} \|u_n(t) - u(t)\|_H &\leq \|u_n(t) - v_{n,n}(t)\|_H + \|v_{n,n}(t) - u(t)\|_H \\ &\leq \|\xi_n - \xi\|_H + \|v_{n,n}(t) - u(t)\|_H, \end{aligned}$$

we conclude that $u_n \to u$ in $C([\tau,T];H)$ as $n \to \infty$. □

**Theorem 8.** *The multivalued evolution process associated with problem (1) has a pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$. Moreover, the sets $A(t)$ are compact.*

**Proof:** By Theorem 4 we have that the family $K(t) = \overline{B_Y(0, B_2(t))}^H$ of compact sets of $H$ is attracting. Therefore, in order to conclude the existence of the pullback attractor using Theorem 6 we need to prove that for all $(T,\tau) \in \mathbb{R}_d$ the map $H \ni \xi \mapsto U(T,\tau,\xi) \in P(H)$ is closed. Then, for fixed $(T,\tau) \in \mathbb{R}_d$ let $y_n \in U(T,\tau,\xi_n)$ be such that

$$\xi_n \to \xi \quad \text{in } H,$$

$$y_n \to y \quad \text{in } H.$$

We have to prove that $y \in U(T,\tau,\xi)$.

There exists a sequence $u_n(\cdot) \in \mathcal{D}_\tau(\xi_n)$ such that $u_n(T) = y_n$. As $u_n(\cdot) \in \mathcal{D}_\tau(\xi_n)$ also there exists a sequence $f_n \in L^2([\tau,T];H)$ such that $f_n(t) \in F(t, u_n(t))$ a.e. on $[\tau,T]$ and

$$\begin{cases} \frac{\partial u_n}{\partial t}(t) + A_H(t)u_n(t) + f_n(t) = 0 \quad \text{a.e. on } [\tau,T], \\ u_n(\tau) = \xi_n. \end{cases} \tag{11}$$

In view of (F3), we have

$$\|f_n(t)\|_H \leq a(t) + c(t)\|u_n(t)\|_H \quad \text{a.e. on } [\tau,T]. \tag{12}$$

Let $z(\cdot)$ be the unique solution to

$$\begin{cases} \frac{\partial z}{\partial t}(t) + A_H(t)z(t) = 0 \quad \text{a.e. on } [\tau, T], \\ z(\tau) = \xi. \end{cases} \tag{13}$$

As a particular case of Theorem 3.1 in [9] we have that $\|z(t)\|_H \le r_0$ where the constant $r_0$ does not depend on $t$, since that we are in the case $B \equiv 0$.

Subtracting the first equation on (11) from the first equation on (13) and multiplying by $u_n(t) - z(t)$ we obtain

$$\frac{1}{2}\|u_n(t) - z(t)\|_H^2 \le -\langle f_n(t), u_n(t) - z(t)\rangle \le \|f_n(t)\|_H \|u_n(t) - z(t)\|_H$$

a.e. on $[\tau, T]$. Integrating the above inequality from $\tau$ to $t \le T$ we obtain

$$\frac{1}{2}\|u_n(t) - z(t)\|_H^2 \le \frac{1}{2}\|\xi_n - \xi\|_H^2 + \int_\tau^t \|f_n(s)\|_H \|u_n(s) - z(s)\|_H ds.$$

It follows by the Gronwall inequality that

$$\|u_n(t) - z(t)\|_H \le \|\xi_n - \xi\|_H + \int_\tau^t \|f_n(s)\|_H ds, \quad \forall\, t \in [\tau, T],$$

and then by (12),

$$\begin{aligned} \|u_n(t)\|_H &\le \|z(t)\|_H + \|\xi_n - \xi\|_H + \int_\tau^t (a(s) + c(s)\|u_n(s)\|_H)\, ds \\ &\le r_0 + r_1 + \int_\tau^T a(s)ds + \int_\tau^t c(s)\|u_n(s)\|_H ds \end{aligned}$$

where $r_1$ is a constant such that $\|\xi_n - \xi\|_H \le r_1$ for all $n \in \mathbb{N}$. So

$$\|u_n(t)\|_H \le K(\tau, T, a) + \int_\tau^t c(s)\|u_n(s)\|_H ds, \quad \forall\, t \in [\tau, T],$$

where $K(\tau, T, a) = r_0 + r_1 + \int_\tau^T a(s)ds$. Hence, by the Gronwall-Bellman inequality we obtain

$$\|u_n(t)\|_H \le K(\tau, T, a)e^{\int_\tau^t c(s)ds} = r(t), \quad \forall\, t \in [\tau, T].$$

Therefore again using (12), we obtain

$$\|f_n(t)\|_H \le a(t) + c(t)r(t) = m(t) \quad \text{a.e. on } [\tau, T].$$

Hence, $\{f_n\}$ is a bounded sequence on the reflexive Banach space $L^2([\tau,T];H)$, and we obtain that there exists a subsequence such that

$$f_n \to f \quad \text{weakly in } L^2([\tau,T];H).$$

Let $v_n(\cdot)$ be the unique solution to

$$\begin{cases} \frac{\partial v_n}{\partial t}(t) + A_H(t)v_n(t) + f_n(t) = 0 \quad \text{a.e. on } [\tau,T], \\ v_n(\tau) = \xi. \end{cases} \tag{14}$$

By Lemma 1 we have that $v_n$ converges in $C([\tau,T];H)$ to the solution $v$ of

$$\begin{cases} \frac{\partial v}{\partial t}(t) + A_H(t)v(t) + f(t) = 0 \quad \text{a.e. on } [\tau,T], \\ v(\tau) = \xi. \end{cases}$$

Now, using (11) and (14) we have

$$\|u_n(t) - v_n(t)\|_H \leq \|\xi_n - \xi\|_H, \quad \forall\, t \in [\tau,T].$$

So we conclude that $u_n \to v$ in $C([\tau,T];H)$ and $y = v(T)$.

Now, to conclude the proof, it follows by Theorem 3.3 in [5] that $f(t) \in F(t,v(t))$ a.e. on $[\tau,T]$. □

## 5. Asymptotic upper semicontinuity

In this section we assume that $F$ does not depend explicitly in $t$, i.e., we consider $F(t,u) \equiv F(u)$ and $F : H \to P_f(H)$ is a multifunction such that there exists a constant $K > 0$ such that $h(F(x),F(y)) \leq K\|x-y\|$ for all $x,y \in H$.

We will prove the asymptotic upper semicontinuity of the elements of the pullback attractor, i.e., we prove that in fact the inclusion (1) is asymptotically autonomous.

### *5.1. Theoretical results*

In this subsection motivated by problem (1), we study the asymptotic behaviour of an abstract non-autonomous multivalued problem in a Hilbert space $H$ of the form

$$\frac{\partial u}{\partial t}(t) + A(t)u(t) + F(u(t)) \ni 0, \quad u(\tau) = \psi_\tau, \tag{15}$$

compared with that of an autonomous multivalued problem of the form

$$\frac{\partial v}{\partial t}(t) + A_\infty v(t) + F(v(t)) \ni 0, \quad v(0) = \psi_0, \tag{16}$$

where $A(t)$, $A_\infty$ are univalued operators in $H$ and $F : H \to P(H)$ is a multivalued map.

The autonomous problem (16) is thus the asymptotic autonomous version of the non-autonomous problem (15). In particular, we establish the convergence in the Hausdorff semi-distance of the component subsets of the pullback attractor of the non-autonomous problem (15) to the global autonomous attractor of the autonomous problem (16).

Some definitions on multivalued semigroups are recalled here, see for example [4, 11, 14] for more details.

**Definition 5.** *Let $X$ be a complete metric space. The map $G : \mathbb{R}^+ \times X \to P(X)$ is called a multivalued semigroup (or m-semiflow) if*
*(1) $G(0, \cdot) = \mathbf{1}$ is the identity map;*
*(2) $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, for all $x \in X$ and $t_1, t_2 \in \mathbb{R}^+$.*

**Definition 6.** *Let $G$ be a multivalued semigroup on $X$. The set $A \subset X$ attracts the subset $B$ of $X$ if $\lim_{t\to\infty} dist(G(t, B), A) = 0$. The set $M$ is said to be a global $B$-attractor for $G$ if $M$ attracts any nonempty bounded subset $B \subset X$ and it is negatively invariant, i.e., $M \subset G(t, M)$, $\forall t \geq 0$.*

Suppose that the multivalued evolution process $\{U(t, \tau) : t \geq \tau\}$ in $H$ associated with problem (15) has a pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$, with $\mathcal{A}(t)$ compact for each $t \in \mathbb{R}$, and that the multivalued semigroup $G : \mathbb{R}^+ \times H \to P(H)$ associated with problem (16) has a compact global autonomous $B$-attractor $\mathcal{A}_\infty$ in the Hilbert space $H$. The following result will be used later to establish the convergence in the Hausdorff semi-distance of the component subsets $\mathcal{A}(t)$ of the pullback attractor $\mathfrak{A}$ to $\mathcal{A}_\infty$ as $t \to \infty$.

**Theorem 9.** *Suppose that $\mathcal{C} := \overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}$ is a compact subset of $H$. In addition, suppose that for each solution $u$ of problem (15) there exists a solution $v$ of problem (16) such that $u(t + \tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $t \geq 0$ whenever $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$. Then*

$$\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0.$$

**Proof:** Suppose that this is not true. Then there would exists an $\epsilon_0 > 0$ and a real sequence $\{\tau_n\}_{n\in\mathbb{N}}$ with $\tau_n > n$ for $n \in \mathbb{N}$ such that $\mathrm{dist}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0$ for all $n \in \mathbb{N}$. Since the sets $\mathcal{A}(\tau_n)$ are compact, there exist $a_n \in \mathcal{A}(\tau_n)$ such that

$$\mathrm{dist}(a_n, \mathcal{A}_\infty) = \mathrm{dist}(\mathcal{A}(\tau_n), \mathcal{A}_\infty) \geq 3\epsilon_0, \tag{17}$$

for each $n \in \mathbb{N}$. By attraction for the multivalued semigroup we have $\mathrm{dist}(G(\ell, \mathcal{C}), \mathcal{A}_\infty) \leq \epsilon_0$ for $\ell > 0$ large enough. Moreover, by the semi-invariance of the pullback attractor there exist $b_n \in \mathcal{A}\left(\frac{1}{2}\tau_n\right) \subset \mathcal{C}$ for $n \in \mathbb{N}$ such that $a_n \in U\left(\tau_n, \frac{1}{2}\tau_n\right) b_n$ for each $n \in \mathbb{N}$. Since $\mathcal{C}$ is compact, there is a convergent subsequence $b_{n'} \to b \in \mathcal{C}$. Since $a_{n'} \in U\left(\tau_{n'}, \frac{1}{2}\tau_{n'}\right) b_{n'}$ there exists a solution $u_{n'}$ of

$$\frac{\partial u_{n'}}{\partial t}(t) + A(t)u_{n'}(t) + F(u_{n'}(t)) \ni 0, \quad u_{n'}\left(\frac{1}{2}\tau_{n'}\right) = b_{n'},$$

such that $a_{n'} = u_{n'}(\tau_{n'})$. From the hypotheses, there exists a solution $v_{n'}$ of

$$\frac{\partial v_{n'}}{\partial t}(t) + A_\infty v_{n'}(t) + F(v_{n'}(t)) \ni 0, \quad v_{n'}(0) = b,$$

such that

$$\left\| u_{n'}(\tau_{n'}) - v_{n'}\left(\frac{1}{2}\tau_{n'}\right) \right\|_H < \epsilon_0$$

for $n'$ large enough (since $\frac{1}{2}\tau_{n'} > \frac{1}{2}n'$ is large enough). Hence,

$$\begin{aligned}
\mathrm{dist}\left(a_{n'}, \mathcal{A}_\infty\right) &= \mathrm{dist}\left(u_{n'}(\tau_{n'}), \mathcal{A}_\infty\right) \\
&\leq \left\| u_{n'}(\tau_{n'}) - v_{n'}\left(\frac{1}{2}\tau_{n'}\right) \right\|_H + \mathrm{dist}\left(v_{n'}\left(\frac{1}{2}\tau_{n'}\right), \mathcal{A}_\infty\right) \\
&\leq \left\| u_{n'}(\tau_{n'}) - v_{n'}\left(\frac{1}{2}\tau_{n'}\right) \right\|_H + \mathrm{dist}\left(G\left(\frac{1}{2}\tau_{n'}, \mathcal{C}\right), \mathcal{A}_\infty\right) \\
&\leq 2\epsilon_0,
\end{aligned}$$

which contradicts (17). $\square$

*5.2. Application to inclusion* (1)

The result in Subsection 5.1 is applied here to the nonlinear inclusion with spatially variable exponents (1) in the Hilbert space $H := L^2(\Omega)$.

We assume in this subsection that $F$ has compact values, i.e., $F(u)$ is a compact set for each $u \in H$ and the coefficient $D$ satisfies Assumption D and the additional Assumption D3 that follows:

**Assumption D3.** $D(t+\tau,\cdot) \to D^*(\cdot)$ in $L^\infty(\Omega)$ as $\tau \to +\infty$ uniformly in $t \geq 0$.

Assumptions (D1)-(D2) imply that the pointwise limit $D^*(x)$ as $t \to \infty$ exists and satisfies $0 < \beta \leq D^*(x) \leq M$ for almost all $x \in \Omega$. Then the problem (1) with $D^*(x)$ is autonomous and has a global autonomous $B$-attractor as a particular case of the results in Section 4.

It will be shown that the dynamics of the original non-autonomous problem is asymptotically autonomous and its pullback attractor converges uppersemi continuously to the autonomous global $B$-attractor $\mathcal{A}_\infty$ of the problem

$$\begin{cases} \frac{\partial v}{\partial t}(t) - \operatorname{div}\left(D^*|\nabla v(t)|^{p(x)-2}\nabla v(t)\right) + |v(t)|^{p(x)-2}v(t) + F(v(t)) \ni 0, \\ v(0) = \psi_0. \end{cases} \tag{18}$$

In particular, we consider the operators

$$\begin{aligned} A(t)u &:= -\operatorname{div}\left(D(t)|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u, \\ A_\infty v &:= -\operatorname{div}\left(D^*|\nabla v|^{p(x)-2}\nabla v\right) + |v|^{p(x)-2}v. \end{aligned}$$

Applying Theorem 3 for the particular case $F(t,u) \equiv F(u)$, there exist positive constants $T_1$, $B_1$ such that $\|u(t)\|_H \leq B_1$ for all $t \geq T_1+\tau$. Moreover, applying Theorem 4 for the particular case $F(t,u) \equiv F(u)$ and space $Y = W^{1,p(x)}(\Omega)$, there exist positive constants $T_2$, $B_2$ such that

$$\|u(t)\|_Y \leq B_2, \quad \forall\, t \geq T_2 + \tau. \tag{19}$$

Since, also $\|v(t)\|_Y \leq B_2$ for all $t \geq T_2 + \tau$ and $Y \subset H$ with compact embedding, it follows that

**Corollary 1.** $\overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}$ *is a compact subset of* $H$.

Using estimate (19), the proof of the next result follows the same lines as the proof of Theorem 4.2 of [10], and therefore is omitted here.

**Lemma 2.** *If $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in $Y$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$, then for each $\tau \in \mathbb{R}$ there exists a function $g_\tau : [0, +\infty) \to [0, +\infty)$ given by $g_\tau(t) = K\|D(t+\tau, \cdot) - D^*(\cdot)\|_{L^\infty(\Omega)}$, where $K$ is a positive constant, such that*

$$\langle A(t+\tau)u(t+\tau) - A_\infty v(t), u(t+\tau) - v(t)\rangle \geq -g_\tau(t), \text{ for all } t \in \mathbb{R}^+,\ \tau \in \mathbb{R},$$

*for any solution $u$ of* (15) *and any uniformly bounded function $v$ with $v(t) \in D(A_\infty)$ for all $t \geq 0$.*

Observe that by Assumption D3 the function $g_\tau : [0, +\infty) \to [0, +\infty)$ given in Lemma 2 satisfies $g_\tau(t) \to 0$ as $\tau \to +\infty$ uniformly in $t \geq 0$.

In the next result we check the hypothesis of asymptotic continuity of the non-autonomous flow in the Theorem 9 for problems like (15).

**Theorem 10.** *If $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$ as $\tau \to +\infty$, then for each solution $u$ of* (15) *there exists a solution $v$ of* (18) *such that $u(t+\tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $t \geq 0$.*

**Proof:** Let $u$ be a solution of (15) then there exists $f \in L^2([\tau, T]; H)$ such that $f(t) \in F(u(t))$ a.e. and

$$\begin{cases} \frac{\partial u}{\partial t}(t) + A(t)u(t) + f(t) = 0, & \text{a.e in } [\tau, T], \\ u(\tau) = \psi_\tau. \end{cases} \tag{20}$$

Using the semi-invariance of the pullback attractor and the estimate (19) it follows that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in $Y$. From the semi-invariance of the pullback attractor, $\mathcal{A}(\tau) \subset U(\tau, s)\mathcal{A}(s),\ \forall\ (\tau, s) \in \mathbb{R}_d$. So, there exists a solution $w$ of (15) with $\psi_\tau = w(\tau)$ and $w(s) \in \mathcal{A}(s)$. Consider the concatenate solution

$$\theta_s(\ell) := \begin{cases} u(\ell), & \ell \geq \tau, \\ w(\ell), & s \leq \ell \leq \tau. \end{cases}$$

Using the pullback attracting property, we have that for each given $\epsilon > 0$ thre exists $s_\epsilon \in \mathbb{R}$ such that

$$\text{dist}\left(U(t+\tau, s_\epsilon)\overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}, \mathcal{A}(t+\tau)\right) < \epsilon.$$

In particular,

$$u(t+\tau) = \theta_{s_\epsilon}(t+\tau) \in O_\epsilon\left(\mathcal{A}(t+\tau)\right) \subset O_\epsilon\left(\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)\right), \ \forall\, \epsilon > 0.$$

Then,

$$u(t+\tau) \in \bigcap_{\epsilon>0} O_\epsilon\left(\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)\right) = \overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}.$$

Considering $z_\tau(t) := f(t+\tau)$, we have

$$z_\tau(t) \in F(u(t+\tau)) \subset \mathcal{K} := F\left(\overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}\right). \tag{21}$$

Using Corollary 1 and that $F$ is Lipschitz continuous with compact values we have from Proposition 3 p. 42 in [1] that $\mathcal{K}$ is a compact set in $H$. So, for each $t \geq 0$ there exist $z(t) \in \mathcal{K}$ and a subnet of $\{z_\tau(t)\}_{\tau\in\mathbb{R}}$, which we do not relabel, such that $z_\tau(t) \to z(t)$ as $\tau \to +\infty$. Let $v$ be the unique solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t}(t) + A_\infty v(t) + z(t) = 0, \\ v(0) = \psi_0. \end{cases} \tag{22}$$

With similar computations as in Theorem 3 one can show that $v$ is uniformly bounded. Subtracting the equation in (20) from the equation in (22) gives

$$\frac{d}{dt}(u(t+\tau) - v(t)) + A(t+\tau)u(t+\tau) - A_\infty v(t) + f(t+\tau) - z(t) = 0$$

for a.e. $t \in [0,T]$. Multiplying by $u(t+\tau) - v(t)$ and using Lemma 2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t+\tau) - v(t)\|_H^2 \leq g_\tau(t) + \|z_\tau(t) - z(t)\|_H \|u(t+\tau) - v(t)\|_H.$$

Integrating this last inequality from 0 to $t$, gives

$$\begin{aligned} \frac{1}{2}\|u(t+\tau) - v(t)\|_H^2 \ &\leq \ \frac{1}{2}\|\psi_\tau - \psi_0\|_H^2 + T \text{ ess sup}_{t\in[0,+\infty)} g_\tau(t) \\ &+ \ \int_0^t \|z_\tau(s) - z(s)\|_H \|u(s+\tau) - v(s)\|_H ds. \end{aligned}$$

Hence, by the Gronwall inequality

$$\begin{aligned} \|u(t+\tau) - v(t)\|_H \ &\leq \ \left(\|\psi_\tau - \psi_0\|_H^2 + 2T \text{ ess sup}_{t\in[0,+\infty)} g_\tau(t)\right)^{1/2} \\ &+ \ \int_0^T \|z_\tau(s) - z(s)\|_H ds. \end{aligned}$$

Using (21) and the Dominated Convergence Theorem, we have

$$\int_0^T \|z_\tau(s) - z(s)\|_H ds \to 0$$

as $\tau \to +\infty$.

Since $\psi_\tau \to \psi_0$ in $H$ and ess $\sup_{t\in[0,+\infty)} g_\tau(t) \to 0$ as $\tau \to +\infty$, we obtain $u(t+\tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $[0, T]$.

By Theorem 4, $u(t+\tau) \in K$ for all $t \geq T_2$ where $K$ is a compact set in $H$ which is independent of $t$ in this case of $F(t,u) \equiv F(u)$.

Consider $\epsilon > 0$ given. Since $K$ is a compact set we have $K \subset \bigcup_i^n B_{\epsilon/4}(x_i)$, $x_1, \cdots x_n \in K$. Then there exists $\tau_1 = \tau_1(T_2)$ such that $u(T_2+\tau), v(T_2) \in B_{\epsilon/4}(x_{i_0})$ for $\tau > \tau_1$ for some $1 \leq i_0 \leq n$. For $t > T_2$ and $\tau > \tau_1$ we have $t = T_2 + s$ with $s + \tau > \tau_1$ for some $s > 0$. Then

$$u(t+\tau) = u(T_2 + s + \tau) \in B_{\epsilon/4}(x_{i_0}).$$

Once $u(t+\tau) \to v(t)$ in $H$ as $\tau \to +\infty$ we have $v(t) \in \overline{B_{\epsilon/4}(x_{i_0})}$.

We have from the previous part that there exists $\tau_2$ such that $\tau > \tau_2$ implies $\sup_{t\in[0,T_2]} \|u(t+\tau) - v(t)\|_H < \frac{\epsilon}{2}$. Taking $\tau_0 := \max\{\tau_1, \tau_2\}$ we have that $\tau > \tau_0$ implies

$$\begin{aligned}
\sup_{t\in[0,\infty)} \|u(t+\tau) - v(t)\|_H &\leq \sup_{t\in[0,T_2]} \|u(t+\tau) - v(t)\|_H \\
&+ \sup_{t\in[T_2,\infty)} \|u(t+\tau) - v(t)\|_H \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, $u(t+\tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $[0, \infty)$.

From Theorem 3.3 in [5], $z \in SelF(v)$ and the result follows. □

The next result gives the desired asymptotic upper semi-continuous convergence.

**Theorem 11.** $\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$.

**Proof:** Suppose that $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in $H$. From Theorem 10, for each solution $u$ of (15) there exists a solution $v$ of (18) such that $u(t+\tau) \to v(t)$ in $H$ as $\tau \to +\infty$ uniformly in $t \geq 0$. Corollary 1 and Theorem 9 then yield $\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$. □

**Acknowledgements**

This work has been partially supported by Science without Borders-CAPES-PVE-Process 88881.030388/2013-01. The first author was partially supported by the Spanish Ministerio de Economía y Competitividad project MTM2011-22411, the Consejería de Innovación, Ciencia y Empresa (Junta de Andalucía) under the Ayuda 2009/FQM314, the Proyecto de Excelencia: P12-FQM-1492. The second author was supported with CNPq scholarship - process 202645/2014-2 (Brazil). The third author was supported with CAPES scholarship - process N. 99999.006893/2014-07 (Brazil).

The authors thank the anonymous referees for their useful suggestions.

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