



Haar meager sets, their hulls, and relationship to compact sets



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ABSTRACT

Let G be an abelian Polish group. We show that there is a strongly Haar meager set in G without any F_σ Haar meager hull (and that this still remains true if we replace F_σ by any other class of the Borel hierarchy). We also prove that there is a coanalytic naively strongly Haar meager set without any Haar meager hull. Further, we investigate the relationship of the collection of all compact sets to the collection of all Haar meager sets in non-locally compact Polish groups.

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1. Introduction

The notion of Haar meager sets (in abelian Polish groups) was introduced by Darji [4] (and straightforwardly generalized to the case of arbitrary Polish groups in [5]). Haar meager sets form a topological counterpart to the so called Haar null sets defined by Christensen in [3]. Let us recall the definitions (and some of its variants).

Definition 1. Let G be a Polish group. A set $A \subseteq G$ is said to be

- (i) *Haar null* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$;
- (ii) *generalized Haar null* if there are a universally measurable set $B \subseteq G$ such that $A \subseteq B$, and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

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Definition 2. Let G be a Polish group. A set $A \subseteq G$ is said to be

- (i) *Haar meager* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, a compact metric space K and a continuous function $f: K \rightarrow G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$;
- (ii) *strongly Haar meager* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, and a compact set $K \subseteq G$ such that $gBh \cap K$ is meager in K for every $g, h \in G$;
- (iii) *naively Haar meager* if there is a compact metric space K and a continuous function $f: K \rightarrow G$ such that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$;
- (iv) *naively strongly Haar meager* if there is a compact set $K \subseteq G$ such that $gAh \cap K$ is meager in K for every $g, h \in G$.

Darji proved that in any abelian Polish group G , Haar meager sets form a σ -ideal contained in the σ -ideal of all meager sets, and that these two σ -ideals coincide if and only if G is locally compact. These results clearly correlate to the analogous well known results concerning Haar null sets proved by Christensen in [3]. Other similarities of the σ -ideals of Haar meager sets and of Haar null sets were investigated in [7] and [5]. We should note that it is not known whether every (naively) Haar meager set is (naively) strongly Haar meager.

In this paper we study two different topics. First, we investigate whether it is possible to replace the Borel hull B from (i) in Definition 2 by a hull from some other class of sets, e.g. by an F_σ hull. Next, we investigate the relationship of the collection of all compact sets to the collection of all Haar meager sets in non-locally compact Polish groups.

Let us look at the content of this paper a little closer. In Chapter 2 we introduce the notation and recall some facts needed later. Chapter 3 is devoted to the descriptive complexity of hulls of Haar meager sets. This chapter is very closely inspired by a paper of Elekes and Vidnyánszky [6] (both by the results and by the proof methods). Elekes and Vidnyánszky answered a question posed by Mycielski [10, Problem (P1)] by proving that in any non-locally compact abelian Polish group, there is a Haar null set which cannot be covered by a G_δ Haar null set [6, Theorem 1.3]. They also offered a more general statement where G_δ may be replaced by any other class of the Borel hierarchy [6, Theorem 1.4]. Finally, they answered a question asked by Fremlin at his webpage by showing that there are generalized Haar null sets which are not Haar null [6, Theorem 1.8]. Comparing Definitions 1 and 2, it is reasonable to believe that analogous results could be proved also for Haar meager sets where ‘Haar null’ corresponds to ‘Haar meager’ and the multiplicative class G_δ corresponds to the additive class F_σ . The purpose of Chapter 3 is showing that this is indeed possible by making only subtle changes in the proofs from [6]. The main result of Chapter 3 is Theorem 10 whose special cases are Theorems 12 and 13. Note that instead of the family of all universally measurable sets, the topological counterpart to the generalized Haar null sets should consider the family of all sets $B \subseteq G$ such that for every compact metric space K and every continuous function $f: K \rightarrow G$, the preimage $f^{-1}(B) \subseteq K$ has the Baire property. However, we do not need to formulate this definition (and its strong variant) since Theorem 13 uses a coanalytic naively strongly Haar meager set which is a stronger notion.

The results from Chapter 4 are inspired by the question posed in [5, Question 6] which asks whether compact subsets of a non-locally compact Polish group are Haar meager. We provide a sufficient condition for F_σ subsets of a Polish group to be strongly Haar meager. Then we show that if G is either the symmetric group S_∞ or any non-locally compact Polish group with a translation invariant metric then all compact subsets of G satisfy this sufficient condition, and thus they are strongly Haar meager. This improves the result by Jabłońska who proved that every compact subset of a non-locally compact abelian Polish group is Haar meager [7].

2. Notation

For a Polish group G let $\mathcal{K}(G)$ denote the family of all nonempty compact subsets of G with the Vietoris topology. If Z is a Polish space then $\mathcal{C}(Z, G)$ denotes the family of all continuous functions from Z to G with the topology of uniform convergence.

For every $1 \leq \xi < \omega_1$, let Σ_ξ^0 be the ξ th additive Borel class and Π_ξ^0 be the ξ th multiplicative Borel class. Let Δ_1^1 denote the class of all Borel sets, Σ_1^1 the class of all analytic sets and Π_1^1 the class of all coanalytic sets. Whenever Γ is one of these classes then we define its dual class $\check{\Gamma}$ by $\check{\Gamma} = \{\sim A : A \in \Gamma\}$.

Recall that if Γ is a class of sets in Polish spaces and X is a Polish space then a set $U \subseteq 2^\omega \times X$ is called *universal for $\Gamma(X)$* if $U \in \Gamma(2^\omega \times X)$ and $\{U_y : y \in 2^\omega\} = \Gamma(X)$. By [8, Theorems 22.3 and 26.1], if Γ is one of the classes Σ_ξ^0 , Π_ξ^0 , Σ_1^1 or Π_1^1 , and if X is a Polish space then there exists a universal set for $\Gamma(X)$.

Recall that if X, Y are Polish spaces and $\Phi : X \rightarrow 2^{2^Y}$ is a function (whose values are collections of subsets of Y) then Φ is said to be *Borel on Borel* if for every Polish space Z and every Borel set $A \subseteq Z \times X \times Y$, the set $\{(z, x) \in Z \times X : A_{z,x} \in \Phi(x)\}$ is Borel.

Whenever $s \in \omega^{<\omega}$ is a finite sequence of elements of ω , we write $|s|$ for the length of s .

3. The descriptive complexity of hulls of Haar meager sets

First of all, we prove the following simple characterization of Haar meager sets which states that the witnessing compact K from (i) in Definition 2 can be always chosen to be the Cantor set, denoted by C . Note that this was essentially proved in [4] (even though not explicitly written down).

Proposition 3. *Let G be a Polish group and let A be a subset of G . Then the following assertions are equivalent:*

- (a) A is Haar meager;
- (b) there are a Borel set $B \subseteq G$ such that $A \subseteq B$, and $f \in \mathcal{C}(C, G)$ such that $f^{-1}(gBh)$ is meager in C for every $g, h \in G$.

Proof. The implication (b) \Rightarrow (a) is trivial. So suppose that A is Haar meager. Let $B \subseteq G$ be a Borel set such that $A \subseteq B$, let K be a compact metric space, and let $f : K \rightarrow G$ be a continuous function such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. It is well known that every nonempty compact metric space is a continuous image of the Cantor set C , so there is a continuous surjection $\phi : C \rightarrow K$. By [4, Lemma 2.10], the surjection ϕ can be chosen in such a way that the preimages of meager sets (in K) are also meager (in C). Then the composition $\phi \circ f : C \rightarrow G$ also witnesses that A is Haar meager. \square

Note that in the further text, we use the Cantor set also in other contexts (for example in Lemma 5 where we work with a subset of $2^\omega \times 2^\omega \times G$). To avoid any misunderstandings, we use the symbol C only when we consider the Cantor set as a (possible) witnessing compact for some set being Haar meager while we use the designation 2^ω in all other contexts.

The following two lemmas will be used in the proof of Proposition 7. Lemma 4 is a modification of the theorem by Montgomery and Novikov (see [8, Theorem 16.1]) and is needed for the case $\Gamma = \Delta_1^1$ and $\Lambda = \Sigma_\xi^0$ (for some $1 \leq \xi < \omega_1$) of Proposition 7. Lemma 5 is a modification of the theorem by Novikov (see [8, Theorem 29.22]) and is needed for the complementary case $\Gamma = \Pi_1^1$ and $\Lambda = \Delta_1^1$. These two lemmas form the only new ingredients to the proof methods from [6]. Indeed, Proposition 7 is a straightforward analogy of [6, Theorem 3.1], Proposition 9 is an easy consequence of [6, Proposition 3.5] and Theorem 10 just puts all the ingredients together (in the same way as [6, Theorem 4.1]).

Lemma 4. Let G be a Polish group and let Z be a Polish space. Then for every nonempty open set $V \subseteq C$ and for every Borel set $A \subseteq Z \times \mathcal{C}(C, G) \times G$, the set $\{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap V \text{ is meager in } V\}$ is also Borel.

In particular, the mapping $f \in \mathcal{C}(C, G) \mapsto \{B \subseteq G : f^{-1}(B) \text{ is meager in } C\} \in 2^{2^G}$ is Borel on Borel.

Proof. Let \mathcal{A} be the collection of all Borel sets $A \subseteq Z \times \mathcal{C}(C, G) \times G$ for which the set $\{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap V \text{ is meager in } V\}$ is Borel for every nonempty open set $V \subseteq C$. We need to show that \mathcal{A} contains all Borel sets.

First of all, let $A = A_Z \times A_{\mathcal{C}(C, G)} \times A_G$ where the sets $A_Z \subseteq Z$, $A_{\mathcal{C}(C, G)} \subseteq \mathcal{C}(C, G)$ and $A_G \subseteq G$ are open. We want to verify that $A \in \mathcal{A}$. So let us fix a nonempty open set $V \subseteq C$. For every $f \in \mathcal{C}(C, G)$, the preimage $f^{-1}(A_G)$ is an open subset of C . Therefore for every $f \in \mathcal{C}(C, G)$, the set $f^{-1}(A_G) \cap V$ is meager in V if and only if $f^{-1}(A_G) \cap V = \emptyset$. So we have

$$\begin{aligned} & \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap V \text{ is meager in } V\} \\ &= ((Z \times \mathcal{C}(C, G)) \setminus (A_Z \times A_{\mathcal{C}(C, G)})) \cup \{(z, f) \in A_Z \times A_{\mathcal{C}(C, G)} : f^{-1}(A_G) \cap V = \emptyset\}. \end{aligned}$$

To see that this set is Borel, it is clearly enough to verify that the set $\{f \in \mathcal{C}(C, G) : f^{-1}(A_G) \cap V = \emptyset\}$ is Borel, and this is easy (in fact, this set is even closed).

Next, we show that \mathcal{A} is closed under countable unions and under taking complements which will finish the proof.

Let A^n , $n \in \omega$, be sets from \mathcal{A} and let $V \subseteq C$ be a nonempty open set. Then we have

$$\begin{aligned} & \left\{ (z, f) \in Z \times \mathcal{C}(C, G) : f^{-1} \left(\left(\bigcup_{n \in \omega} A^n \right)_{z,f} \right) \cap V \text{ is meager in } V \right\} \\ &= \left\{ (z, f) \in Z \times \mathcal{C}(C, G) : \bigcup_{n \in \omega} f^{-1}(A^n_{z,f}) \cap V \text{ is meager in } V \right\} \\ &= \bigcap_{n \in \omega} \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A^n_{z,f}) \cap V \text{ is meager in } V\}. \end{aligned}$$

This set is the countable intersection of Borel sets, and so it is Borel.

Finally, suppose that $A \in \mathcal{A}$. Let $V \subseteq C$ be a nonempty open set and let \mathcal{B} be a countable open basis of C . Recall that for every Borel set $M \subseteq C$, the set $M \cap V$ is comeager in V if and only if $M \cap B$ is non-meager in B whenever $B \in \mathcal{B}$ is such that $\emptyset \neq B \subseteq V$ (see e.g. [8, Proposition 8.27]). Therefore we have

$$\begin{aligned} & \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}((\sim A)_{z,f}) \cap V \text{ is meager in } V\} \\ &= \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap V \text{ is comeager in } V\} \\ &= \bigcap_{\substack{B \in \mathcal{B} \\ \emptyset \neq B \subseteq V}} \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap B \text{ is non-meager in } B\} \\ &= \bigcap_{\substack{B \in \mathcal{B} \\ \emptyset \neq B \subseteq V}} \sim \{(z, f) \in Z \times \mathcal{C}(C, G) : f^{-1}(A_{z,f}) \cap B \text{ is meager in } B\}. \end{aligned}$$

This set is the countable intersection of complements of Borel sets, and so it is Borel. \square

In the proof of Lemma 5 we will use the unfolded $**$ -game from [8, 21.C] which we recall here. Let Y be a Polish space with a fixed compatible complete metric, and let \mathcal{V} be a countable basis of nonempty open sets in Y . Then for every $E \subseteq Y \times \omega^\omega$, let $G_u^{**}(E)$ be the unfolded $**$ -game defined as follows:

$$\begin{array}{ccccccc}
 \text{I} & & U_0 & & U_1 & & \dots \\
 & & & & & & \\
 \text{II} & & (z(0), V_0) & & (z(1), V_1) & &
 \end{array}$$

Players I and II play $U_i, V_i \in \mathcal{V}$ such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$ and $\text{diam}(U_i), \text{diam}(V_i) < 2^{-i}$. Player II also plays $z(i) \in \omega$. Let $y \in Y$ be the unique point such that $\{y\} = \bigcap_{i \in \omega} \overline{U_i} = \bigcap_{i \in \omega} \overline{V_i}$, and let $z = (z(i))_{i \in \omega} \in \omega^\omega$. Player II wins the game if and only if $(y, z) \in E$ (the pair (y, z) is called the outcome of the run). Recall that if E is closed then it holds by [8, Theorem 21.5 and the following comments] that Player II has a winning strategy in the game $G_u^{**}(E)$ if and only if $\text{proj}_Y E$ is comeager in Y .

Lemma 5. *Let G be a Polish group, let Z be a Polish space, and let $V \subseteq 2^\omega \times 2^\omega \times G$ be in Σ_1^1 . Then the sets*

$$\tilde{V} := \{(f, x) \in \mathcal{C}(Z, G) \times 2^\omega : f^{-1}(V_{x,x}) \text{ is non-meager in } Z\}$$

and

$$\tilde{\tilde{V}} := \{(f, x) \in \mathcal{C}(Z, G) \times 2^\omega : f^{-1}(V_{x,x}) \text{ is comeager in } Z\}$$

are also in Σ_1^1 .

Proof. Let \mathcal{B} be a countable open basis of Z . Whenever $f \in \mathcal{C}(Z, G)$ and $x \in 2^\omega$ then by [8, Proposition 8.27], the set $f^{-1}(V_{x,x})$ is non-meager in Z if and only if there is a nonempty set $B \in \mathcal{B}$ such that $f^{-1}(V_{x,x}) \cap B$ is comeager in B . So it suffices to show that for every nonempty $B \in \mathcal{B}$, the set

$$\tilde{V}_0 := \{(f, x) \in \mathcal{C}(Z, G) \times 2^\omega : f^{-1}(V_{x,x}) \cap B \text{ is comeager in } B\}$$

is in Σ_1^1 (as we may suppose that $Z \in \mathcal{B}$). So let us fix a nonempty $B \in \mathcal{B}$ together with a compatible complete metric on B . By [8, Exercise 14.3] there is a closed set $F \subseteq 2^\omega \times 2^\omega \times G \times \omega^\omega$ with $V = \text{proj}_{2^\omega \times 2^\omega \times G}(F)$. For every $f \in \mathcal{C}(Z, G)$ and every $x \in 2^\omega$, let

$$F(f, x) := \{(y, z) \in B \times \omega^\omega : (x, x, f(y), z) \in F\},$$

which is clearly a closed subset of $B \times \omega^\omega$. Then for every $f \in \mathcal{C}(Z, G)$ and every $x \in 2^\omega$, Player II has a winning strategy in the game $G_u^{**}(F(f, x))$ if and only if $\text{proj}_B F(f, x)$ is comeager in B . But $\text{proj}_B F(f, x) = f^{-1}(V_{x,x}) \cap B$, and so we have

$$\tilde{V}_0 = \{(f, x) \in \mathcal{C}(Z, G) \times 2^\omega : \text{Player II has a winning strategy in } G_u^{**}(F(f, x))\}.$$

Let T be the tree of all legal positions in the games G_u^{**} (clearly, the tree T does not depend on the choice of the subset of $B \times \omega^\omega$ the game is played with). We can view the tree T , as well as every strategy $\sigma \subseteq T$ for Player II, as an element of the space Tr of all trees on \mathbb{N} (which is a Polish space by [8, Exercise 4.32]). Then the set \tilde{V}_0 is the projection of the set

$$\begin{aligned}
 \tilde{V}_1 &:= \{(f, x, \sigma) \in \mathcal{C}(Z, G) \times 2^\omega \times \text{Tr} : \\
 &\quad \sigma \subseteq T \text{ is a winning strategy for Player II in the game } G_u^{**}(F(f, x))\}
 \end{aligned}$$

to the first two coordinates, and so it is enough to verify that \tilde{V}_1 is analytic.

For every legal position $p \in T$ in the games G_u^{**} we define a (closed) subset $Q(p)$ of $B \times \omega^\omega$ as the set of all possible outcomes of those runs of the games G_u^{**} which start according to the position p . This means that if $p = (U_0, (z(0), V_0), U_1, \dots, (z(n), V_n))$ is of even length then we put

$$Q(p) = \overline{V_n} \times \{z \in \omega^\omega : (z(0), \dots, z(n)) \text{ is an initial segment of } z\}.$$

And if $p = (U_0, (z(0), V_0), U_1, \dots, (z(n), V_n), U_{n+1})$ is of odd length then we put

$$Q(p) = \overline{U_{n+1}} \times \{z \in \omega^\omega : (z(0), \dots, z(n)) \text{ is an initial segment of } z\}.$$

The proof of the following claim is already included in the proof of [8, Theorem 29.22] but for completeness sake we briefly repeat it here.

Claim 6. *Let $\sigma \subseteq T$ be a strategy for Player II in the games G_u^{**} . Let $f \in \mathcal{C}(Z, G)$ and $x \in 2^\omega$. Then σ is a winning strategy for Player II in the game $G_u^{**}(F(f, x))$ if and only if for every $p \in \sigma$ we have $Q(p) \cap F(f, x) \neq \emptyset$.*

Proof. Let us fix a compatible complete metric on ω^ω , and consider the product metric on $B \times \omega^\omega$ (recall that we have fixed a compatible complete metric on B , so the product metric is also complete). For every infinite branch $b \in [T]$, the sequence $(Q(b|n))_{n \in \omega}$ is a decreasing (with respect to inclusion) sequence of closed subsets of $B \times \omega^\omega$ whose diameters tend to 0. So for every $b \in [T]$, the intersection $\bigcap_{n \in \omega} Q(b|n)$ is a singleton whose only element $q(b)$ is the outcome of the run in the game $G_u^{**}(F(f, x))$ corresponding to b . Now the strategy σ is winning for Player II in the game $G_u^{**}(F(f, x))$ if and only if $q(b) \in F(f, x)$ for every $b \in [\sigma]$, and this clearly holds if and only if $Q(b|n) \cap F(f, x) \neq \emptyset$ for every $b \in [\sigma]$ and $n \in \omega$. This finishes the proof of the claim. \square

By the previous claim we have

$$\begin{aligned} \tilde{V}_1 &= \{(f, x, \sigma) \in \mathcal{C}(Z, G) \times 2^\omega \times \text{Tr} : \\ &\quad \sigma \subseteq T \text{ is a strategy for Player II and } \forall_{p \in \sigma} Q(p) \cap F(f, x) \neq \emptyset\} \\ &= \{(f, x, \sigma) \in \mathcal{C}(Z, G) \times 2^\omega \times \text{Tr} : \\ &\quad \sigma \subseteq T \text{ is a strategy for Player II and } \forall_{p \in \sigma} \exists_{(y, z) \in Q(p)} (x, x, f(y), z) \in F\}, \end{aligned}$$

which is clearly an analytic set. \square

Proposition 7. *Let either $\Gamma = \Delta_1^1$ and $\Lambda = \Sigma_\xi^0$ for some $1 \leq \xi < \omega_1$, or let $\Gamma = \Pi_1^1$ and $\Lambda = \Delta_1^1$. Let G be a Polish group. Then there exists a partial function $\phi: \mathcal{C}(C, G) \times 2^\omega \rightarrow G$ such that $\text{graph}(\phi) \in \Gamma$, and such that for every $f \in \mathcal{C}(C, G)$ it holds*

- (i) $\forall x \in 2^\omega : (f, x) \in \text{dom}(\phi) \Rightarrow \phi(f, x) \in f(C),$
- (ii) $\forall S \in \Lambda(2^\omega \times G) : \text{graph}(\phi_f) \subseteq S \Rightarrow \text{there is } x \in 2^\omega \text{ such that } f^{-1}(S_x) \text{ is non-meager in } C.$

Proof. Let $U \subseteq 2^\omega \times 2^\omega \times G$ be a set which is

- universal for $\Pi_\xi^0(2^\omega \times G)$ if $[\Gamma = \Delta_1^1 \text{ and } \Lambda = \Sigma_\xi^0 \text{ for some } 1 \leq \xi < \omega_1];$
- universal for $\Pi_1^1(2^\omega \times G)$ if $[\Gamma = \Pi_1^1 \text{ and } \Lambda = \Delta_1^1].$

We put

$$\tilde{U} := \{(f, x, g) \in \mathcal{C}(C, G) \times 2^\omega \times G : f^{-1}(U_{x,x}) \text{ is non-meager in } C, g \in U_{x,x} \cap f(C)\}.$$

Claim 8. *The set \tilde{U} is from the class Γ .*

Proof. It is easy to see that the set $\{(f, g) \in \mathcal{C}(C, G) \times G : g \in f(C)\}$ is closed. The set U is in Γ by its definition, and so the set $\{(x, g) \in 2^\omega \times G : g \in U_{x,x}\}$ is also in Γ . Putting these two facts together we get that the set

$$\{(f, x, g) \in \mathcal{C}(C, G) \times 2^\omega \times G : g \in U_{x,x} \cap f(C)\}$$

is in Γ . So it remains to show that the set

$$\{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(U_{x,x}) \text{ is non-meager in } C\}$$

is in Γ as well.

In the case of $\Gamma = \Delta_1^1$ it suffices to use (the first part of) [Lemma 4](#) on the Borel set

$$A := \{(x, f, g) \in 2^\omega \times \mathcal{C}(C, G) \times G : (x, x, g) \in U\},$$

as for every $x \in 2^\omega$ and every $f \in \mathcal{C}(C, G)$ it holds $f^{-1}(A_{x,f}) = f^{-1}(U_{x,x})$.

In the case of $\Gamma = \Pi_1^1$ let \mathcal{B} be a countable basis of nonempty open sets in C . Then (using [\[8, Proposition 8.27\]](#) in the first equality) we have

$$\begin{aligned} & \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(U_{x,x}) \text{ is non-meager in } C\} \\ &= \bigcup_{B \in \mathcal{B}} \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(U_{x,x}) \cap B \text{ is comeager in } B\} \\ &= \sim \bigcap_{B \in \mathcal{B}} \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(\sim U_{x,x}) \cap B \text{ is non-meager in } B\}. \end{aligned}$$

Therefore it is enough to show that for every $B \in \mathcal{B}$, the set

$$\begin{aligned} & \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(\sim U_{x,x}) \cap B \text{ is non-meager in } B\} \\ &= \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : (f \upharpoonright_B)^{-1}((\sim U)_{x,x}) \text{ is non-meager in } B\} \end{aligned}$$

is in Σ_1^1 . But this easily follows from [Lemma 5](#) used on the Σ_1^1 set $V := \sim U$. \square

Whenever $f \in \mathcal{C}(C, G)$ and $x \in 2^\omega$ are such that $f^{-1}(U_{x,x})$ is non-meager in C then $\tilde{U}_{f,x} = U_{x,x} \cap f(C)$. On the other hand, if $f \in \mathcal{C}(C, G)$ and $x \in 2^\omega$ are such that $f^{-1}(U_{x,x})$ is meager in C then $\tilde{U}_{f,x} = \emptyset$. It follows that for every $(f, x) \in \text{proj}_{\mathcal{C}(C, G) \times 2^\omega} \tilde{U}$ we have $f^{-1}(\tilde{U}_{f,x})$ is non-meager in C . This enables us to define ϕ as the uniformization of the set $\tilde{U} \subseteq (\mathcal{C}(C, G) \times 2^\omega) \times G$ such that $\text{graph}(\phi) \in \Gamma$. Indeed, in the case of $\Gamma = \Delta_1^1$ the existence of such a uniformization follows by (the second part of) [Lemma 4](#) and by [\[8, Theorem 18.6\]](#). And in the case of $\Gamma = \Pi_1^1$ the existence follows by the uniformization property of the class Π_1^1 (see [\[8, Theorem 36.14\]](#)).

We have $\text{dom}(\phi) = \{(f, x) \in \mathcal{C}(C, G) \times 2^\omega : f^{-1}(U_{x,x}) \text{ is non-meager in } C\}$, and $\phi(f, x) \in U_{x,x} \cap f(C)$ for every $(f, x) \in \text{dom}(\phi)$. In particular, the partial function ϕ satisfies (i) for every $f \in \mathcal{C}(C, G)$.

Next we fix $f \in \mathcal{C}(C, G)$, and we verify (ii). Assume towards a contradiction that there is $S \in \Lambda(2^\omega \times G)$ such that $\text{graph}(\phi_f) \subseteq S$ and for every $x \in 2^\omega$ the set $f^{-1}(S_x)$ is meager in C . By the universality of U , there is $x \in 2^\omega$ such that $U_x = (2^\omega \times G) \setminus S$. Then $f^{-1}(U_{x,x}) = f^{-1}(G \setminus S_x) = C \setminus f^{-1}(S_x)$ is non-meager in C (even comeager in C), and so $(f, x) \in \text{dom}(\phi)$ and $\phi(f, x) \in U_{x,x} \cap f(C) \subseteq U_{x,x} = G \setminus S_x$. On the other hand, the inclusion $\text{graph}(\phi_f) \subseteq S$ implies that $\phi(f, x) \in S_x$ which is a contradiction. \square

Proposition 9. *Let G be a non-locally compact abelian Polish group and let $K \in \mathcal{K}(G)$ be fixed. Then there is a Borel map $u : \mathcal{C}(C, G) \times 2^\omega \times 2^\omega \rightarrow G$ such that*

(A) if $(f, x, y) \neq (f', x', y')$ are elements of $\mathcal{C}(C, G) \times 2^\omega \times 2^\omega$ then

$$(f(C) - K + u(f, x, y)) \cap (f'(C) - K + u(f', x', y')) = \emptyset;$$

(B) for every $f \in \mathcal{C}(C, G)$ and $y \in 2^\omega$ the map $u(f, \cdot, y)$ is continuous.

Proof. By [6, Proposition 3.5] there is a Borel map $t: \mathcal{K}(G) \times 2^\omega \times 2^\omega \rightarrow G$ such that

(\tilde{A}) if $(L, x, y) \neq (L', x', y')$ are elements of $\mathcal{K}(G) \times 2^\omega \times 2^\omega$ then

$$(L - K + t(L, x, y)) \cap (L' - K + t(L', x', y')) = \emptyset;$$

(\tilde{B}) for every $L \in \mathcal{K}(G)$ and $y \in 2^\omega$ the map $t(L, \cdot, y)$ is continuous.

Let us fix a Borel bijection $\theta: \mathcal{C}(C, G) \times 2^\omega \rightarrow 2^\omega$. For $(f, x, y) \in \mathcal{C}(C, G) \times 2^\omega \times 2^\omega$ we define $u(f, x, y) := t(f(C), x, \theta(f, y))$. This map clearly works. \square

The following theorem is the main result of this chapter.

Theorem 10. *Let either $\Gamma = \Delta_1^1$ and $\Lambda = \Sigma_\xi^0$ for some $1 \leq \xi < \omega_1$, or let $\Gamma = \Pi_1^1$ and $\Lambda = \Delta_1^1$. Let G be a non-locally compact abelian Polish group. Then there is a strongly Haar meager (naively strongly Haar meager in the case of $\Gamma = \Pi_1^1$) set $E \in \Gamma(G)$ without any Haar meager hull in $\Lambda(G)$.*

Proof. We fix a Borel bijection $c: \mathcal{C}(C, G) \rightarrow 2^\omega$ and a perfect compact set $K \in \mathcal{K}(G)$ such that $0 \in K$. Let $\phi: \mathcal{C}(C, G) \times 2^\omega \rightarrow G$ be the partial function from Proposition 7, and let $u: \mathcal{C}(C, G) \times 2^\omega \times 2^\omega \rightarrow G$ be the map from Proposition 9. Let $\Psi: \mathcal{C}(C, G) \times 2^\omega \times G \rightarrow G$ be defined by $\Psi(f, x, g) = g + u(f, x, c(f))$. We put $E := \Psi(\text{graph}(\phi))$.

We first prove that E is in the class Γ . The map Ψ is clearly Borel. Let $D := \{(f, x, g) \in \mathcal{C}(C, G) \times 2^\omega \times G : g \in f(C)\}$, then D is a Borel set (even closed). Let (f, x, g) and (f', x', g') be two distinct elements of D . If $(f, x) \neq (f', x')$ then by condition (A) from Proposition 9 we have $\Psi(f, x, g) \neq \Psi(f', x', g')$ (as $g \in f(C)$, $g' \in f'(C)$ and $0 \in K$). And if $(f, x) = (f', x')$ and $g \neq g'$ then it clearly holds $\Psi(f, x, g) \neq \Psi(f', x', g')$ as well. It follows that Ψ is injective on the set D , and so it is a Borel isomorphism of the Borel sets D and $\Psi(D)$. By condition (i) from Proposition 7 we have $\text{graph}(\phi) \subseteq D$. Moreover $\text{graph}(\phi)$ is in the class Γ (again by Proposition 7), and so it follows that $E = \Psi(\text{graph}(\phi))$ is also in the class Γ .

Next we show that E is strongly Haar meager (naively strongly Haar meager in the case of $\Gamma = \Pi_1^1$), witnessed by the compact K . We have

$$\begin{aligned} E &= \{\Psi(f, x, \phi(f, x)) : (f, x) \in \text{dom}(\phi)\} \\ &= \{\phi(f, x) + u(f, x, c(f)) : (f, x) \in \text{dom}(\phi)\} \\ &\subseteq \bigcup_{(f, x) \in \text{dom}(\phi)} (f(C) + u(f, x, c(f))) \end{aligned}$$

where the sets from the last union are pairwise disjoint by condition (A) from Proposition 9. It follows that for every $(f, x) \in \text{dom}(\phi)$, the intersection of E and $f(C) + u(f, x, c(f))$ is a singleton $\{\phi(f, x) + u(f, x, c(f))\}$. Let us fix $g \in G$. Then $K - g$ intersects at most one of the sets $f(C) + u(f, x, c(f))$ (otherwise $-g$ would be in the intersection of two distinct sets of the form $f(C) - K + u(f, x, c(f))$ which would contradict Proposition 9). Therefore $K - g$ intersects E in at most one point, and so $E + g$ intersects K in at most one point. Thus $E + g$ is meager in K , as K is perfect.

Finally, we show that there is no Haar meager hull of E in $\Lambda(G)$. Suppose for a contradiction that H is such a hull and let $f: C \rightarrow G$ be the witnessing continuous function. By condition (B) from [Proposition 9](#) we easily have that the map $\Psi_f = \Psi(f, \cdot, \cdot)$ is continuous. Thus $S := \Psi_f^{-1}(H) \in \Lambda(2^\omega \times G)$. For every $(x, g) \in \text{graph}(\phi_f)$, we have $\Psi_f(x, g) = \phi(f, x) + u(f, x, c(f)) \in E \subseteq H$, and so $\text{graph}(\phi_f) \subseteq S$. By condition (ii) from [Proposition 7](#) there is $x \in 2^\omega$ such that $f^{-1}(S_x)$ is non-meager in C . But we also have $S_x + u(f, x, c(f)) = \Psi_{f,x}(S_x) \subseteq \Psi_f(S) \subseteq H$, and so $f^{-1}(H - u(f, x, c(f))) \supseteq f^{-1}(S_x)$ is non-meager in C . This is a contradiction with the fact that f is a witnessing function for H being Haar meager. \square

As it was already noted in [\[6\]](#) in case of Haar null sets, [Theorem 10](#) has the following easy consequence concerning the additivity of the σ -ideal of all (strongly) Haar meager sets.

Corollary 11. *Let G be a non-locally compact abelian Polish group. Then the additivity of the σ -ideal of all (strongly) Haar meager subsets of G is ω_1 .*

Proof. For every $1 \leq \xi < \omega_1$, let $E_\xi \subseteq G$ be a (strongly) Haar meager set without any Haar meager hull in Σ_ξ^0 . Then $\bigcup_{1 \leq \xi < \omega_1} E_\xi$ has no Borel Haar meager hull, and so it is not (strongly) Haar meager. \square

The next two theorems are just special cases of [Theorem 10](#) but we write them down explicitly as they were the main motivation for this chapter. [Theorem 12](#) answers in negative a question posed in [\[5, Question 9\]](#). [Theorem 13](#) is in a sharp contrast with [\[5, Proposition 8\]](#) which states that every Σ_1^1 naively Haar meager set is Haar meager.

Theorem 12. *Let G be a non-locally compact abelian Polish group. Then there is a strongly Haar meager set without any F_σ Haar meager hull.*

Theorem 13. *Let G be a non-locally compact abelian Polish group. Then there is a coanalytic naively strongly Haar meager set without any Haar meager hull.*

4. Relationship between Haar meager sets and compact sets

In the following definitions we introduce two conditions, both of which are sufficient for F_σ sets to be strongly Haar meager. Note that a related result is also proved in [\[1, Proposition 5.9\]](#) where it is shown that a closed set is Haar meager if and only if it is not prethick.

Definition 14. Let G be a Polish group. We say that a set $A \subseteq G$ satisfies the *finite translation property* \mathcal{FTP} if for every open set $\emptyset \neq U \subseteq G$ there exists a finite set $M \subseteq U$ such that for every $g, h \in G$ we have $gMh \not\subseteq A$.

Definition 15. Let G be a Polish group. We say that a set $A \subseteq G$ satisfies the *finite open translation property* \mathcal{FOTP} if for every open set $\emptyset \neq U \subseteq G$ there exists a finite collection \mathcal{M} of open nonempty subsets of U such that for every $g, h \in G$ there exists $V \in \mathcal{M}$ with $gVh \cap A = \emptyset$.

Remark 16. Let G be a Polish group, and let $A \subseteq G$. Then clearly

- A satisfies \mathcal{FOTP} if and only if \overline{A} satisfies \mathcal{FTP} ,
- if A satisfies \mathcal{FOTP} then A satisfies \mathcal{FTP} .

Theorem 17. *Let G be a Polish group. Suppose that $A \subseteq G$ is an F_σ set which satisfies \mathcal{FTP} . Then A is strongly Haar meager.*

Proof. We fix a compatible complete metric on G such that $\text{diam}(G) \leq 1$. We will inductively construct a nonempty finitely branching pruned tree $T \subseteq \omega^{<\omega}$, together with nonempty open sets $U_t \subseteq G$ and points $x_t \in G$ for $t \in T$ such that

- (i) $\text{diam}(U_t) \leq 2^{-|t|}$, $t \in T$,
- (ii) if $s, t \in T$ are such that s is an initial segment of t then $\overline{U_t} \subseteq U_s$,
- (iii) if $s, t \in T$ are such that $|s| = |t|$ and $s \neq t$, then $\overline{U_s} \cap \overline{U_t} = \emptyset$,
- (iv) $x_t \in U_t$, $t \in T$,
- (v) for every $t \in T$ we have $t^\frown 0 \in T$ and $x_{t^\frown 0} = x_t$,
- (vi) for every $t \in T$ and $g, h \in G$ we have $\{x_{t^\frown i} \in G : i \in \omega \text{ such that } t^\frown i \in T\} \not\subseteq gAh$.

We start the construction by setting $U_\emptyset = G$, and by choosing x_\emptyset to be an arbitrary element of G . Now, suppose that for some $n \in \omega$, we have already constructed all elements of the tree T whose length is at most n , as well as the sets U_t and points x_t for $t \in T$, $|t| \leq n$. Let $t \in T$ with $|t| = n$ be fixed. Since A satisfies \mathcal{FTP} there exists a finite set $M \subseteq U_t$ such that for all $g, h \in G$ we have $gMh \not\subseteq A$. Let $x_{t^\frown 0}, \dots, x_{t^\frown k}$ be pairwise distinct elements of G such that $x_{t^\frown 0} = x_t$ and $\{x_{t^\frown 0}, \dots, x_{t^\frown k}\} = M \cup \{x_t\}$. We define $t^\frown i \in T$ if and only if $i \leq k$. We find $U_{t^\frown i}$ for $i \leq k$ such that the conditions (i)–(iv) are satisfied. This construction clearly works.

Let $[T] = \{t \in \omega^\omega : t \restriction_k \in T \text{ for every } k \in \omega\}$ be the set of all infinite branches of the tree T . We put

$$L = \bigcup_{t \in [T]} \bigcap_{k \in \omega} U_{t \restriction_k}.$$

Then $L \subseteq G$ is a compact set by (i)–(iii). We show that L is the witnessing compact for the fact that A is strongly Haar meager. Since A is F_σ , it suffices to show that the relative (in L) interior of each translation of A is empty. By (ii), (iv) and (v) we have that $\{x_t : t \in T\} \subseteq L$. By this fact together with (ii), (iv) and (vi) we obtain that for every $t \in T$ and every $g, h \in G$ we have $U_t \cap L \not\subseteq gAh$. This finishes the proof as the sets $U_t \cap L$, $t \in T$, clearly form a basis of relatively open sets in L . \square

Corollary 18. *Let G be a Polish group. Suppose that $A \subseteq G$ satisfies $FOTP$. Then A is strongly Haar meager.*

Proof. Since A satisfies $FOTP$, its closure \overline{A} also satisfies $FOTP$. Thus \overline{A} satisfies \mathcal{FTP} . By Theorem 17 we have that \overline{A} is strongly Haar meager, so A is strongly Haar meager as well. \square

Theorems 19 and 21 provide examples of Polish groups in which all compact sets satisfy \mathcal{FTP} (even $FOTP$ in the latter case).

Theorem 19. *Every compact subset of S_∞ satisfies \mathcal{FTP} .*

Proof. Let $K \subseteq S_\infty$ be a compact set. Let $s \in \omega^\omega$ be a nondecreasing sequence such that for all $n \in \omega$ we have

$$\#\{x(n) \in \omega : x \in K\} \leq s(n).$$

Let $\emptyset \neq U \subseteq S_\infty$ be open. Let us fix $z \in U$ and find $k \in \omega$ such that

$$\{y \in S_\infty : z \restriction_k \text{ is an initial segment of } y\} \subseteq U.$$

Clearly, we can find $M = \{x_0, \dots, x_{s(k)}\} \subseteq S_\infty$ such that $z \restriction_k$ is an initial segment of x_i for every $i \leq s(k)$, and such that $x_i(n) \neq x_j(n)$ for every $n \geq k$ and $i \neq j$. Then $M \subseteq U$. Let us fix $g, h \in S_\infty$, and let us fix some $i \geq k$ with $h^{-1}(i) \leq k$. Then we have

$$\begin{aligned} \#\{y(h^{-1}(i)): y \in gMh\} &= \#\{gyh(h^{-1}(i)): y \in M\} \\ &= \#\{gy(i): y \in M\} \\ &= \#\{y(i): y \in M\} \\ &= s(k) + 1 \\ &> s(h^{-1}(i)) \\ &\geq \#\{y(h^{-1}(i)): y \in K\}. \end{aligned}$$

Thus $gMh \not\subseteq K$. \square

Corollary 20. *All compact subsets of S_∞ are strongly Haar meager.*

Theorem 21. *Let G be a non-locally compact Polish group which admits a translation invariant metric. Then every compact subset of G satisfies \mathcal{FOTP} .*

Proof. Let us fix a translation invariant metric on G . By [2, Corollary 1.2.2] this metric is complete. Let $K \subseteq G$ be a compact set, and let $\emptyset \neq U \subseteq G$ be an arbitrary open set. Since G is not locally compact, the set U is not totally bounded. Thus, there exists $\varepsilon > 0$ and an infinite set $N = \{x_i: i \in \omega\} \subseteq U$ such that for every $i \in \omega$ we have $\text{dist}(\{x_i\}, N \setminus \{x_i\}) \geq 4\varepsilon$. By the compactness of K there is $n \in \omega$ and a finite set $\{y_0, \dots, y_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{i=0}^n B(y_i, \varepsilon).$$

We set

$$\mathcal{M} = \{B(x_i, \varepsilon) \cap U: i = 0, \dots, n+1\}.$$

Using the fact that the fixed metric is translation invariant, it is easy to see that for every $g, h \in G$ and every $i \leq n$ we have

$$\#\{B \in \mathcal{M}; gBh \cap B(y_i, \varepsilon)\} \leq 1.$$

Thus, for every $g, h \in G$ there exists $B \in \mathcal{M}$ such that

$$gBh \cap K \subseteq gBh \cap \bigcup_{i=0}^n B(y_i, \varepsilon) = \emptyset.$$

This shows that K satisfies \mathcal{FOTP} . \square

The following corollary improves the result by Jabłońska who proved that every compact subset of a non-locally compact abelian Polish group is Haar meager [7, Corollary 1].

Corollary 22. *Let G be a non-locally compact Polish group which admits a translation invariant metric. Then all compact subsets of G are strongly Haar meager.*

The following theorem is a weaker reformulation of the result proved by Matoušková in [9, Theorem 2.3].

Theorem 23. *Let X be a separable reflexive Banach space. Then every bounded, convex and nowhere dense subset of X satisfies \mathcal{FTP} .*

Corollary 24. *Let X be a separable reflexive Banach space. Then every convex and nowhere dense subset of X is Haar meager.*

Proof. Each convex and nowhere dense subset of X is the countable union of bounded, convex and nowhere dense sets. Thus the assertion follows from the fact that Haar meager sets form a σ -ideal, together with Theorems 17 and 23. \square

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