J. Math. Anal. Appl. 448 (2017) 1006–1041

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On nonlocal Choquard equations with Hardy–Littlewood–Sobolev critical exponents ☆

Fashun Gao, Minbo Yang *

Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, PR China

ARTICLE INFO

Article history:
Received 3 August 2016
Available online 11 November 2016
Submitted by V. Radulescu

Keywords:
Choquard equation
Hardy–Littlewood–Sobolev critical exponent
Concave and convex nonlinearities
Brezis–Nirenberg problem

ABSTRACT

We consider the following nonlinear Choquard equation with Dirichlet boundary condition

$$-\Delta u=\left(\int_{\Omega}\frac{|u|^{2_{\mu}^{*}}}{|x-y|^{\mu}}dy\right)|u|^{2_{\mu}^{*}-2}u+\lambda f(u)\quad \text{in }\Omega,$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $\lambda>0$, $N\geq 3$, $0<\mu<N$ and 2_{μ}^{*} is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. Under suitable assumptions on different types of nonlinearities $f(u)$, we are able to prove some existence and multiplicity results for the equation by variational methods.

1. Introduction and main results

In a pioneering paper [14], Brezis and Nirenberg studied the problems of the type

$$\begin{cases}-\Delta u=|u|^{2^{*}-2}u+\lambda u^{q} & \text{in }\ \Omega,\\ u=0 & \text{on }\ \partial\Omega,\end{cases}\tag{1.1}$$

where 2^* is the critical exponent for the embedding of $H_0^1(\Omega)$ to $L^p(\Omega)$. By developing some skillful techniques in estimating the Minimax level, the authors were able to prove the existence of nontrivial solutions for the equation under linear or subcritical superlinear perturbation. To complement the results in [14], the sublinear case was studied by Ambrosetti, Brezis and Cerami in [6]. There the authors investigated the combined effects of the critical term and the sublinear term on the existence of solutions and established

☆ Partially supported by NSFC (11571317, 11101374, 11271331) and ZJNSF (LY15A010010).
* Corresponding author.
E-mail address: mbyang@zjnu.edu.cn (M. Yang).

http://dx.doi.org/10.1016/j.jmaa.2016.11.015
0022-247X/

the existence and multiplicity results depending on the parameter λ. Ever since then, the elliptic equations with critical exponents have been a hot subject. For example, [7,11,18,20,21,29,31,48] studied the critical problems on bounded or unbounded domains, [26,32] investigated the quasilinear critical growth problems driven by p-Laplacian, [19] studied the regularity of stable solutions of elliptic problems on Riemannian manifold, [17,24,27,32,33,44,45] considered the Hardy–Sobolev type inequalities with remaining terms and the Hardy–Sobolev singular critical exponent problems, while [8,9,46] investigated the critical problems driven by the fractional power of the Laplacian $(-\Delta)^s$ $(0 < s < 1)$.

In the present paper we are going to study the existence and multiplicity results for the following *critical nonlocal equation*:

$$\begin{cases} -\Delta u = \Big(\displaystyle\int_\Omega \frac{|u|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|u|^{2^*_\mu-2}u + \lambda f(u) \quad \text{in } \Omega, \\ u \in H^1_0(\Omega), \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $\lambda > 0$, $N \geq 3$, $0 < \mu < N$, $2^*_\mu = (2N-\mu)/(N-2)$ and $f(u)$ is a nonlinearity satisfying certain assumptions. This type of nonlocal elliptic equation is closely related to the nonlinear Choquard equation

$$-\Delta u + V(x)u = \Big(\frac{1}{|x|^\mu} * |u|^p\Big)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3, \tag{1.3}$$

where $\frac{2N-\mu}{N} \leq p \leq \frac{2N-\mu}{N-2}$. For $p = 2$ and $\mu = 1$, the problem goes back to the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [42] and the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard, as a certain approximation to Hartree–Fock theory of one-component plasma [34]. In some particular cases, this equation is also known as the Schrödinger–Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse of a quantum mechanical wave function [43]. For $p = \frac{2N-1}{N-2}$ and $\mu = 1$, by using the Green function, it is obvious that (1.3) can be regarded as a generalized version of Schrödinger–Newton

$$\begin{cases} -\Delta u + V(x)u = u^{\frac{N+1}{N-2}}\phi \quad \text{in } \mathbb{R}^N, \\ -\Delta \phi = u^{\frac{2N-1}{N-2}} \quad \text{in } \mathbb{R}^N. \end{cases}$$

And thus equation (1.2) can be viewed as a generalized Schrödinger–Newton restricted on bounded domain with Dirichlet boundary condition.

The starting point of the variational approach to the problem (1.2) is the following well-known Hardy–Littlewood–Sobolev inequality, see [35], which leads to a new type of critical problem with nonlocal nonlinearities driven by Riesz potential.

Proposition 1.1 *(Hardy–Littlewood–Sobolev inequality). Let $t, r > 1$ and $0 < \mu < N$ with $1/t+\mu/N+1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}dxdy \leq C(t, N, \mu, r)|f|_t|h|_r, \tag{1.4}$$

where $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$. If $t = r = 2N/(2N-\mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

In this case there is equality in (1.4) if and only if $f \equiv (const.)h$ *and*

$$h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$.

The Hardy–Littlewood–Sobolev inequality plays an important role in studying nonlocal problems and we'd like to mention that other nonlocal version inequalities are considered in some recent literature, for example, the authors in [25] studied the Hardy–Littlewood inequalities in fractional weighted Sobolev spaces.

Notice that, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dxdy$$

is well defined if

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-2}.$$

Here, $\frac{2N-\mu}{N}$ is called the lower critical exponent and $2_\mu^* = \frac{2N-\mu}{N-2}$ is the upper critical exponent due to the Hardy–Littlewood–Sobolev inequality. In this sense we can call the problem (1.2) a critical nonlocal elliptic equation. In a recent paper [41] by Moroz and Van Schaftingen, the authors considered the nonlinear Choquard equation (1.3) in $\mathbb{R}^N$ with lower critical exponent $\frac{2N-\mu}{N}$ if the potential $1-V$ should not decay to zero at infinity faster then the inverse square of $|x|$. However, nothing is known about the upper critical exponent case and how the behavior of the potential will affect the existence results.

The existence and qualitative properties of solutions of Choquard type equations (1.3) have been widely studied in the last decades. In [34], Lieb proved the existence and uniqueness, up to translations, of the ground state. Later, in [36], Lions showed the existence of a sequence of radially symmetric solutions. In [22, 37,38] the authors showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. Nowadays the existence and asymptotic behavior of the solutions of the Choquard equations under different assumptions of the potentials and the nonlinearities attract a lot of attention and interest. We refer the readers to [1,16] for the strongly indefinite case with sign-changing periodic potential V, where the existence of ground state solutions and infinitely many geometrically distinct weak solutions were obtained by critical point theorem; [39] for the existence of ground states under the assumptions of Berestycki–Lions type; [3] for the existence of multibump shaped solution for the equation with deepening potential well; [23,30] for the existence of sign-changing solutions; [4,5,40,47] for the existence and concentration behavior of the semiclassical solutions.

To the best knowledge of the authors, there are not so many papers considering the Choquard equation with critical growth, for problem set on $\mathbb{R}^2$ the second author and his collaborators in [2] considered the case of critical growth in the sense of Trudinger–Moser inequality and studied the existence and concentration of the ground states. For the problem set on a bounded domain of $\mathbb{R}^N$, $N \geq 3$, the authors of the present paper considered in [28] the critical Choquard equation (1.2) with upper critical exponent $2_\mu^* = \frac{2N-\mu}{N-2}$ under a linear perturbation term and established the existence results corresponding to the well-known results in [14].

The aim of the present paper is to continue to study the existence and multiplicity of the critical Choquard equation (1.2) with upper critical exponent $2_\mu^* = \frac{2N-\mu}{N-2}$ on a bounded domain of $\mathbb{R}^N$, $N \geq 3$. We are interested in the problem that how the subcritical superlinear or the sublinear perturbation term will affect the existence and multiplicity of the critical Choquard equation (1.2). In [28], $S_{H,L}$ denotes the best constant defined by

$$S_{H,L} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy)^{\frac{N-2}{2N-\mu}}}. \tag{1.5}$$

Proposition 1.2. *(See [28].) The constant $S_{H,L}$ defined in (1.5) is achieved if and only if*

$$u = C\Big(\frac{b}{b^2+|x-a|^2}\Big)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0,\infty)$ are parameters. What is more,

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S is the best Sobolev constant.

Let $U(x) := \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ be a minimizer for S, see [48] for example, then

$$\tilde{U}(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N,\mu)^{\frac{2-N}{2(N-\mu+2)}} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}} \tag{1.6}$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\Delta u = \Big(\int\limits_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|u|^{2^*_\mu-2}u \quad \text{in} \quad \mathbb{R}^N$$

with

$$\int\limits_{\mathbb{R}^N} |\nabla \tilde{U}|^2 dx = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|\tilde{U}(x)|^{2^*_\mu}|\tilde{U}(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Moreover, for every open subset Ω of $\mathbb{R}^N$,

$$S_{H,L}(\Omega) := \inf_{u\in D_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int\limits_{\Omega} |\nabla u|^2 dx}{\Big(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} dxdy\Big)^{\frac{N-2}{2N-\mu}}} = S_{H,L}, \tag{1.7}$$

and $S_{H,L}(\Omega)$ is never achieved except $\Omega = \mathbb{R}^N$, see [28].

Through this paper we will assume that Ω is a smooth bounded domain of $\mathbb{R}^N$, the main results of this paper are the following Theorems.

For the critical Choquard equation (1.2) with a subcritical local term $f = u^q$, $1 < q < 2^* - 1$,

$$\begin{cases} -\Delta u = \Big(\int\limits_{\Omega} \frac{|u|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|u|^{2^*_\mu-2}u + \lambda u^q & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.8}$$

we have the following existence result.

Theorem 1.3. *Assume that* $1 < q < 2^* - 1$, $N \geq 3$ *and* $0 < \mu < N$. *Then, problem* (1.8) *has at least one nontrivial solution provided that either*

(1). $N > \max\{\min\{\frac{2(q+3)}{q+1}, 2 + \frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ *and* $\lambda > 0$, *or*
(2). $N \leq \max\{\min\{\frac{2(q+3)}{q+1}, 2 + \frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ *and* λ *is sufficiently large.*

We are also interested in the critical Choquard equation (1.2) with a subcritical nonlocal term, that is to consider the following equation,

$$\begin{cases} -\Delta u = \Big(\int\limits_{\Omega} \frac{|u|^{2^*_\mu}}{|x-y|^{\mu}} dy\Big)|u|^{2^*_\mu - 2}u + \lambda\Big(\int\limits_{\Omega} \frac{|u|^{q}}{|x-y|^{\mu}} dy\Big)|u|^{q-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.9}$$

For this case, we establish the following existence result.

Theorem 1.4. *Assume that* $1 < q < 2^*_\mu$, $N \geq 3$ *and* $0 < \mu < N$. *Then, problem* (1.9) *has at least one nontrivial solution provided that either*

(1). $N > \frac{2(q+1)-\mu}{q-1}$ *and* $\lambda > 0$, *or*
(2). $N \leq \frac{2(q+1)-\mu}{q-1}$ *and* λ *is sufficiently large.*

Analogously, we have the existence result for the Choquard equation with Sobolev critical exponent and subcritical nonlocal perturbation.

Theorem 1.5. *Assume that* $1 < q < 2^*_\mu$, $N \geq 3$ *and* $0 < \mu < N$. *Then, problem*

$$\begin{cases} -\Delta u = \lambda\Big(\int\limits_{\Omega} \frac{|u|^{q}}{|x-y|^{\mu}} dy\Big)|u|^{q-2}u + |u|^{2^*-2}u \quad \textit{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{1.10}$$

has at least one nontrivial solution provided that either

(1). $N > \frac{2(q+1)-\mu}{q-1}$ *and* $\lambda > 0$, *or*
(2). $N \leq \frac{2(q+1)-\mu}{q-1}$ *and* λ *is sufficiently large.*

As observed by Ambrosetti, Brezis and Cerami in [6], the combination of a critical term and sublinear term influence the existence and multiplicity of the equation (1.1) greatly. Here we can also establish a similar result for the critical Choquard equation under the perturbation of both sublinear and suplinear subcritical terms. Consider the following equation

$$\begin{cases} -\Delta u = \Big(\int\limits_{\Omega} \frac{|u|^{2^*_\mu}}{|x-y|^{\mu}} dy\Big)|u|^{2^*_\mu - 2}u + u^p + \lambda u^q \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.11}$$

we can draw the following conclusions.

Theorem 1.6. *Assume that* $0 < q < 1$, $1 < p < 2^* - 1$, $N \geq 3$ *and* $0 < \mu < N$. *Then, there exists* $0 < \Lambda < \infty$ *such that*

(1). *problem* (1.11) *has no positive solution for* $\lambda > \Lambda$;
(2). *problem* (1.11) *has a minimal positive solution* u_λ *for any* $0 < \lambda < \Lambda$ *and the family of minimal solutions is increasing with respect to* λ;
(3). *problem* (1.11) *has at least two positive solutions if* $0 < \lambda < \Lambda$.

From the statements above, we had completely studied how the different perturbation will affect the existence results. In fact, Theorem 1.3 states the existence under a superlinear perturbation while Theorem 1.4 and Theorem 1.5 focuses on the nonlocal perturbation both for the critical problem in the sense of the Hardy–Littlewood–Sobolev inequality and the classical Sobolev inequality. Finally Theorem 1.6 says that how the appearance of sublinear perturbation will lead to the existence results.

Next we are going to investigate the existence of infinitely many solutions for the critical Choquard equation (1.8) under the effect of only a sublinear local perturbation.

Theorem 1.7. *Assume that* $0 < q < 1$, $N \geq 3$ *and* $0 < \mu < N$. *Then, there exists* $\lambda^* > 0$ *such that, for every* $0 < \lambda < \lambda^*$, *problem* (1.8) *has a sequence of solutions* $\{u_n\} \subset H_0^1(\Omega)$ *such that* $J_\lambda(u_n) \to 0$, $n \to \infty$.

We would like to point out that, for the convex and concave problem with subcritical growth, it is standard to apply the Fountain Theorem to obtain an unbounded sequence of solutions. We state the multiplicity result here for the completeness of this paper.

Theorem 1.8. *Assume that* $0 < q < 1$, $N \geq 3$, $0 < \mu < N$ *and* $1 \leq p < 2_\mu^*$. *Then, for every* $\lambda > 0$, *the problem*

$$\begin{cases} -\Delta u = \Big(\displaystyle\int_\Omega \frac{|u|^p}{|x-y|^\mu}dy\Big)|u|^{p-2}u + \lambda u^q & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.12}$$

has an unbounded sequence of solutions $\{u_n\} \subset H_0^1(\Omega)$ *such that* $J_\lambda(u_n) \to \infty$, $n \to \infty$.

The main results will be proved by variational methods. For this, we introduce the energy functional associated to equation (1.2) by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - \lambda\int_\Omega F(u)dx,$$

where $F(u) = \int_0^u f(t)dt$. Then, under the assumptions in the theorems above, the Hardy–Littlewood–Sobolev inequality implies that J_λ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ with

$$\langle J_\lambda'(u), \varphi\rangle = \int_\Omega \nabla u \nabla\varphi dx - \int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)\varphi(y)}{|x-y|^\mu}dxdy - \lambda\int_\Omega f(u)\varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

And so u is a weak solution of (1.2) if and only if u is a critical point of functional J_λ.

We need to point out the main features of the present problem are two-fold: the first is the loss of compactness due to the appearance of the Hardy–Littlewood–Sobolev upper critical exponent which makes it difficult to verify the (PS) condition. The second is the nonlocal nature of the critical Choquard equation where the convolution type nonlinearities, no longer locally defined, are totally determined by the behavior on the domain ω. This feature not only makes it difficult to verify the geometric conditions of the critical point theorems, but also causes a lot of trouble in applying the well-known Brezis–Nirenberg type arguments and showing the regularity of the solutions.

For the proof of the main results, we will show that the energy functional J_λ satisfies a compactness property and has suitable geometrical features of some critical point theorems. Frequently, we will apply the Mountain Pass Theorem to study the critical Choquard equation with subcritical superlinear perturbation and obtain the existence of at least one solution for (1.2) for suitable values of λ, the space dimension N and the order of the Riesz potential μ. As in [14], the key step is to use the extreme function of the best constant in Proposition 1.2 to estimate the Mountain Pass value. By showing that the Minimax value is below the level where the (PS) condition holds, one can easily obtain the existence results. The proof of Theorem 1.5 is similar to that of Theorem 1.3 and the proof of Theorem 1.4 will only be sketched. For the critical Choquard equation with only sublinear perturbation, we will use the Dual Fountain Theorem established in [11] to prove that the problem (1.2) has a sequence of solutions $\{u_n\} \subset H_0^1(\Omega)$ such that $J_\lambda(u_n) \to 0$, $n \to \infty$ and we also write some lines for the subcritical nonlocal case for the completeness of the paper. For the critical Choquard equation with both sublinear and superlinear perturbation, we will follow the idea in [6] to prove the existence of at least two positive solutions for an admissible small range of λ. Firstly, we can obtain a positive solution that is a local minimum for the functional J_λ by sub- and supersolution techniques. Secondly, in order to find a second solution of (1.11), we suppose that this local minimum is the only critical point of the functional J_λ and then we prove a local Palais–Smale ((PS) for short) condition for $c \in \mathbb{R}$ below a critical level related with $S_{H,L}$ defined in (1.5). Finally, we apply the Mountain Pass Theorem and its refined version in [31] to get the conclusion. In order to overcome the difficulties, we need to adjust the arguments in [6,14] to suit the new environment.

Throughout the paper, we will use the following notations:

- We denote positive constants by $C, C_1, C_2, C_3, \cdots$ and $\lambda > 0$ is a real parameter.
- We denote the standard norm on $H_0^1(\Omega)$ by $\|u\| := \left(\int\limits_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ and write $|\cdot|_q$ for the $L^q(\Omega)$-norm for $q \in [1, \infty]$.
- We always assume Ω is a smooth bounded domain of $\mathbb{R}^N$ with $0 \in \Omega$.
- Let E be a real Hilbert space and $I : E \to \mathbb{R}$ a functional of class C^1. We say that $(u_n) \subset E$ is a Palais–Smale ((PS) for short) sequence at c for I if (u_n) satisfies

$$I(u_n) \to c \text{ and } I'(u_n) \to 0, \text{ as } n \to \infty.$$

Moreover, I satisfies the (PS) condition at c, if any (PS) sequence at c possesses a convergent subsequence.

This paper is organized as follows: In Section 2, we investigate the Hardy–Littlewood–Sobolev critical Choquard equation perturbed by a subcritical superlinear local term. In Section 3, we consider the critical Choquard equation perturbed by a subcritical superlinear nonlocal term. In Section 4, we investigate the combined effects of the superlinear and sublinear nonlinearities on the existence and multiplicity of the critical Choquard equation. Finally, we study the existence of infinitely many solutions of the critical Choquard equation under the effect of a sublinear term.

2. Perturbation with a superlinear local term

In this section we will study the existence of solutions for the critical Choquard equation with superlinear local perturbation, i.e.

$$\begin{cases} -\Delta u = \Big(\int\limits_{\Omega} \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u|^{2_\mu^*-2}u + \lambda u^q \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where $1 < q < 2^* - 1$.

By introducing the energy functional by

$$J_\lambda(u) = \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{q+1}\int\limits_{\Omega} u^{q+1}dx - \frac{1}{2\cdot 2_\mu^*}\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy,$$

we check that the functional J_λ satisfies the Mountain Pass Geometry, that is

Lemma 2.1. *If $1 < q < 2^* - 1$ and $\lambda > 0$, then, the functional J_λ satisfies the following properties:*

(1). *There exist $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.*
(2). *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.*

Proof. (1). By the Sobolev embedding and the Hardy–Littlewood–Sobolev inequality, for all $u \in H_0^1(\Omega)\setminus \{0\}$ we have

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}C_1\|u\|^{q+1} - \frac{1}{2\cdot 2_\mu^*}C_0\|u\|^{2(\frac{2N-\mu}{N-2})}.$$

Since $2 < q+1 < 2\cdot 2_\mu^*$, we can choose some ρ small enough such that $J_\lambda(u) \geq \alpha > 0$ for all u satisfying $\|u\| = \rho$.

(2). For some $u_1 \in H_0^1(\Omega)\setminus \{0\}$, we have

$$J_\lambda(tu_1) = \frac{t^2}{2}\int\limits_{\Omega} |\nabla u_1|^2 dx - \frac{\lambda t^{q+1}}{q+1}\int\limits_{\Omega} u_1^{q+1}dx - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u_1(x)|^{2_\mu^*}|u_1(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy < 0$$

for $t > 0$ large enough. Hence, we can take an $e := t_1u_1$ for some $t_1 > 0$ and the conclusion (2) follows. □

Lemma 2.2. *Let $1 < q < 2^* - 1$, $\lambda > 0$. If $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, then $\{u_n\}$ is bounded. Let $u_0 \in H_0^1(\Omega)$ be the weak limit of $\{u_n\}$, then u_0 is a weak solution of problem (1.8).*

Proof. It is easy to see that there exists $C_1 > 0$ such that

$$|J_\lambda(u_n)| \leq C_1, \quad |\langle J_\lambda'(u_n), \frac{u_n}{\|u_n\|}\rangle| \leq C_1.$$

Then we have

$$\begin{aligned} J_\lambda(u_n) &- \frac{1}{q+1}\langle J_\lambda'(u_n), u_n\rangle \\ &= (\frac{1}{2} - \frac{1}{q+1})\int\limits_{\Omega} |\nabla u_n|^2 dx + (\frac{1}{q+1} - \frac{1}{2\cdot 2_\mu^*})\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\ &\leq C_1(1 + \|u_n\|). \end{aligned}$$

Since $2 < q+1 < 2\cdot 2_\mu^*$, we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Up to a subsequence, there exists $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \rightharpoonup u_0$ in $L^{2^*}(\Omega)$ as $n \to +\infty$. Then

$$|u_n|^{2_\mu^*} \rightharpoonup |u_0|^{2_\mu^*} \quad \text{in} \quad L^{\frac{2N}{2N-\mu}}(\Omega)$$

as $n \to +\infty$. By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$, hence

$$|x|^{-\mu} * |u_n|^{2_\mu^*} \rightharpoonup |x|^{-\mu} * |u_0|^{2_\mu^*} \quad \text{in} \quad L^{\frac{2N}{\mu}}(\Omega)$$

as $n \to +\infty$. Combining this with the fact that

$$|u_n|^{2_\mu^*-2}u_n \rightharpoonup |u_0|^{2_\mu^*-2}u_0 \quad \text{in} \quad L^{\frac{2N}{N-\mu+2}}(\Omega),$$

as $n \to +\infty$, we have

$$(|x|^{-\mu} * |u_n|^{2_\mu^*})|u_n|^{2_\mu^*-2}u_n \rightharpoonup (|x|^{-\mu} * |u_0|^{2_\mu^*})|u_0|^{2_\mu^*-2}u_0 \quad \text{in} \quad L^{\frac{2N}{N+2}}(\Omega)$$

as $n \to +\infty$. Since, for any $\varphi \in H_0^1(\Omega)$,

$$0 \leftarrow \langle J_\lambda'(u_n), \varphi\rangle = \int_\Omega \nabla u_n \nabla \varphi dx - \lambda \int_\Omega u_n^q \varphi dx - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*-2}u_n(y)\varphi(y)}{|x-y|^\mu} dxdy.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\int_\Omega \nabla u_0 \nabla \varphi dx - \lambda \int_\Omega u_0^q \varphi dx - \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-2}u_0(y)\varphi(y)}{|x-y|^\mu} dxdy = 0$$

for any $\varphi \in H_0^1(\Omega)$, which means u_0 is a weak solution of the problem (1.8).

Finally, taking $\varphi = u_0 \in H_0^1(\Omega)$ as a test function in equation (1.8), we have

$$\int_\Omega |\nabla u_0|^2 dx = \lambda \int_\Omega u_0^{q+1} dx + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy,$$

and so, for $1 < q < 2^* - 1$,

$$J_\lambda(u_0) = (\frac{\lambda}{2} - \frac{\lambda}{q+1}) \int_\Omega u_0^{q+1} dx + \frac{N+2-\mu}{4N-2\mu} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \geq 0. \quad \square$$

The following Brezis–Lieb type lemma for the nonlocal term is proved in [1] (the subcritical case) and [28] (the critical case).

Lemma 2.3. *Let $N \geq 3$, $0 < \mu < N$ and $(2N-\mu)/2N \leq p \leq 2_\mu^*$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\Omega)$ such that $u_n \to u$ almost everywhere in Ω as $n \to \infty$, then the following hold,*

$$\int_\Omega (|x|^{-\mu} * |u_n|^p)|u_n|^p dx - \int_\Omega (|x|^{-\mu} * |u_n - u|^p)|u_n - u|^p dx \to \int_\Omega (|x|^{-\mu} * |u|^p)|u|^p dx$$

as $n \to \infty$.

In the next Lemma we prove a convergence criterion for the $(PS)_c$ sequences which will play an important role in applying the critical point theorems.

Lemma 2.4. *Let $1 < q < 2^* - 1$ and $\lambda > 0$. If $\{u_n\}$ is a $(PS)_c$ sequence of J_λ with*

$$c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \tag{2.1}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Let u_0 be the weak limit of $\{u_n\}$ obtained in Lemma 2.2 and define $v_n := u_n - u_0$, then we know $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n \to 0$ a.e. in Ω. Moreover, by the Brézis–Lieb Lemma in [13] and Lemma 2.3, we know

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla v_n|^2 dx + \int_\Omega |\nabla u_0|^2 dx + o_n(1),$$

$$\int_\Omega |u_n|^{q+1} dx = \int_\Omega |v_n|^{q+1} dx + \int_\Omega |u_0|^{q+1} dx + o_n(1)$$

and

$$\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy = \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1).$$

Consequently, we have

$$\begin{aligned} c \leftarrow J_\lambda(u_n) &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{q+1}\int_\Omega u_n^{q+1} dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \\ &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{\lambda}{q+1}\int_\Omega v_n^{q+1} dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{\lambda}{q+1}\int_\Omega u_0^{q+1} dx \\ &\quad - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1) \qquad (2.2) \\ &= J_\lambda(u_0) + \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{\lambda}{q+1}\int_\Omega v_n^{q+1} dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1) \\ &\geq \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1), \end{aligned}$$

since $J_\lambda(u_0) \geq 0$ and $\int_\Omega v_n^{q+1} dx \to 0$, as $n \to +\infty$. Similarly, since $\langle J_\lambda'(u_0), u_0\rangle = 0$, we have

$$
\begin{aligned}
o_n(1) &= \langle J_\lambda'(u_n), u_n\rangle \\
&= \int_\Omega |\nabla u_n|^2 dx - \lambda \int_\Omega u_n^{q+1} dx - \int_\Omega\int_\Omega \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \\
&= \int_\Omega |\nabla v_n|^2 dx - \lambda \int_\Omega v_n^{q+1} dx + \int_\Omega |\nabla u_0|^2 dx - \lambda \int_\Omega u_0^{q+1} dx \\
&\quad - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy - \int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1) \\
&= \langle J_\lambda'(u_0), u_0\rangle + \int_\Omega |\nabla v_n|^2 dx - \lambda \int_\Omega v_n^{q+1} dx - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1) \\
&= \int_\Omega |\nabla v_n|^2 dx - \int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy + o_n(1).
\end{aligned} \tag{2.3}
$$

From (2.3), we know there exists a nonnegative constant b such that

$$\int_\Omega |\nabla v_n|^2 dx \to b$$

and

$$\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \to b$$

as $n \to +\infty$. Thus from (2.2), we obtain

$$c \geq \frac{N+2-\mu}{4N-2\mu} b. \tag{2.4}$$

By the definition of the best constant $S_{H,L}$ in (1.5), we have

$$S_{H,L}\Big(\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy\Big)^{\frac{N-2}{2N-\mu}} \leq \int_\Omega |\nabla v_n|^2 dx,$$

which yields $b \geq S_{H,L} b^{\frac{N-2}{2N-\mu}}$. Thus we have either $b = 0$ or $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$, then we obtain from (2.4) that

$$\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leq \frac{N+2-\mu}{4N-2\mu} b \leq c,$$

which contradicts with the fact that $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Thus $b = 0$, and

$$\|u_n - u_0\| \to 0$$

as $n \to +\infty$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *There exists u_ε such that*

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} \tag{2.5}$$

provided that either

(1). $N > \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ *and* $\lambda > 0$, *or*
(2). $N \leq \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ *and* λ *is sufficiently large.*

Proof. From Theorem 1.42 in [48], we know $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for S, the best Sobolev constant. By Proposition 1.2, we know that $U(x)$ is also a minimizer for $S_{H,L}$. Assume that $B_\delta \subset \Omega \subset B_{2\delta}$ and let $\psi \in C_0^\infty(\Omega)$ be such that

$$\begin{cases} \psi(x) = \begin{cases} 1 & \text{if } x \in B_\delta, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases} \\ 0 \leq \psi(x) \leq 1 & \forall x \in \mathbb{R}^N, \\ |D\psi(x)| \leq C = const & \forall x \in \mathbb{R}^N. \end{cases}$$

We define, for $\varepsilon > 0$,

$$\begin{aligned} U_\varepsilon(x) &:= \varepsilon^{\frac{2-N}{2}} U(\frac{x}{\varepsilon}), \\ u_\varepsilon(x) &:= \psi(x) U_\varepsilon(x). \end{aligned} \tag{2.6}$$

From [28], we know that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = C(N,\mu))^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2}) \tag{2.7}$$

and

$$\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \geq C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}}). \tag{2.8}$$

Case 1. $N > \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$.

First, by the proof of Lemma 4.1 in [26], since $q > 1$ and $N > \frac{2(q+1)}{q}$ we know $N < (N-2)(q+1)$, thus we have

$$\int_\Omega u_\varepsilon^{q+1} dx = O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}}) + O(\varepsilon^{\frac{(N-2)(q+1)}{2}}) = O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}}), \tag{2.9}$$

for $\varepsilon > 0$ sufficiently small.

Using the estimates in (2.7), (2.8) and (2.9), we have

$$\begin{aligned}J_\lambda(tu_\varepsilon) &= \frac{t^2}{2}\int\limits_\Omega |\nabla u_\varepsilon|^2 dx - \frac{\lambda t^{q+1}}{q+1}\int\limits_\Omega |u_\varepsilon|^{q+1}dx - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\ &\le \frac{t^2}{2}\big(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})\big) - \lambda t^{q+1}O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}})\\ &\quad - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})\big)\\ &:= g(t).\end{aligned}$$

It is clear that $g(t)\to-\infty$ as $t\to+\infty$. It follows that there exists $t_\varepsilon>0$ such that $\sup_{t>0} g(t)$ is attained at t_ε. Differentiating $g(t)$ and equaling to zero, we obtain that

$$t_\varepsilon\big(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})\big) - \lambda t^q O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}}) - t_\varepsilon^{2\cdot 2_\mu^*-1}\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})\big) = 0$$

and so

$$t_\varepsilon < \Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})}\Big)^{\frac{1}{2\cdot 2_\mu^*-2}} := S_{H,L}(\varepsilon)$$

and there exists $t_0>0$ independent of ε such that for $\varepsilon>0$ small enough

$$t_\varepsilon > t_0.$$

Notice that the function

$$t\mapsto \frac{t^2}{2}\big(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})\big) - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})\big)$$

is increasing on $[0, S_{H,L}(\varepsilon)]$, we have

$$\begin{aligned}\max_{t\ge 0} J_\lambda(tu_\varepsilon) &\le \frac{N+2-\mu}{4N-2\mu}\Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})\big)^{\frac{N-2}{2N-\mu}}}\Big)^{\frac{2N-\mu}{N+2-\mu}} - O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}})\\ &\le \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + O(\varepsilon^{\min\{N-2,N-\frac{\mu}{2}\}}) - O(\varepsilon^{N-\frac{(N-2)(q+1)}{2}})\\ &< \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},\end{aligned}$$

thanks to $t_0 < t_\varepsilon < S_{H,L}(\varepsilon)$, (2.9) and $N > \min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}$.

Case 2. $N \le \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$.

For any fixed ε in (2.6), from

$$J_\lambda(tu_\varepsilon)\to-\infty$$

as $t\to+\infty$, we have that $\max_{t\ge0} J_\lambda(tu_\varepsilon)$ is attained at some $t_\lambda>0$ and t_λ satisfies

$$t_\lambda \int\limits_\Omega |\nabla u_\varepsilon|^2 dx = \lambda t_\lambda^q \int\limits_\Omega |u_\varepsilon|^{q+1} dx + t_\lambda^{2\cdot 2_\mu^*-1} \int\limits_\Omega \int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy,$$

that is

$$\int\limits_\Omega |\nabla u_\varepsilon|^2 dx = \lambda t_\lambda^{q-1} \int\limits_\Omega |u_\varepsilon|^{q+1} dx + t_\lambda^{2\cdot 2_\mu^*-2} \int\limits_\Omega \int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy,$$

thanks to $\frac{\partial J_\lambda(tu_\varepsilon)}{\partial t}|_{t=t_\lambda} = 0$. Thus $t_\lambda \to 0$ as $\lambda \to +\infty$. Then,

$$\max_{t\geq 0} J_\lambda(tu_\varepsilon) = \frac{t_\lambda^2}{2} \int\limits_\Omega |\nabla u_\varepsilon|^2 dx - \frac{\lambda t_\lambda^{q+1}}{q+1} \int\limits_\Omega |u_\varepsilon|^{q+1} dx - \frac{t_\lambda^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*} \int\limits_\Omega \int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \to 0$$

as $\lambda \to +\infty$, which easily yields the desired conclusion for the case $N \leq \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$. $\square$

Proof of Theorem 1.3. By Lemma 2.1 and the Mountain Pass Theorem without (PS) condition (cf. [48]), there exists a (PS) sequence $\{u_n\}$ such that $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ in $H_0^1(\Omega)^{-1}$ at the minimax level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}.$$

From Lemma 2.5 and the definition of c, we know $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, provided that either

(1). $N > \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ and $\lambda > 0$, or
(2). $N \leq \max\{\min\{\frac{2(q+3)}{q+1}, 2+\frac{\mu}{q+1}\}, \frac{2(q+1)}{q}\}$ and λ is sufficiently large.

Applying Lemma 2.4, we know $\{u_n\}$ contains a convergent subsequence. And so, we have J_λ has a critical value $c \in (0, \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}})$ and thus the problem (1.8) has a nontrivial solution. $\square$

3. Perturbation with a superlinear nonlocal term

In this section we will study the existence of solutions in the case of a nonlocal perturbation,

$$\begin{cases} -\Delta u = \Big(\int\limits_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu} dy\Big)|u|^{2_\mu^*-2}u + \lambda\Big(\int\limits_\Omega \frac{|u|^q}{|x-y|^\mu} dy\Big)|u|^{q-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

Since the problem is set in a bounded domain, the Sobolev embedding and the Hardy–Littlewood–Sobolev inequality imply that the integral

$$\int\limits_\Omega \int\limits_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu} dxdy$$

is well defined if

$$\frac{2N-\mu}{2N} \leq q \leq \frac{2N-\mu}{N-2}.$$

Thus, associated to the equation (1.9), we can introduce the energy functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - \frac{\lambda}{2q}\int_\Omega\int_\Omega \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dxdy,$$

which belongs to $C^1(H_0^1(\Omega),\mathbb{R})$, and so u is a weak solution of (1.9) if and only if u is a critical point of functional J_λ.

Similar to the proof of Lemma 2.1 and Lemma 2.2, we have the following conclusions.

Lemma 3.1. *If $1 < q < 2_\mu^*$ and $\lambda > 0$, then, the functional J_λ satisfies the following properties:*

(1). *There exist $\alpha, \rho > 0$ such that $J_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.*
(2). *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $J_\lambda(e) < 0$.*

Lemma 3.2. *If $1 < q < 2_\mu^*$ and $\lambda > 0$ and $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, then $\{u_n\}$ is bounded. Let $u_0 \in H_0^1(\Omega)$ be the weak limit of $\{u_n\}$, then u_0 is a weak solution of problem (1.9).*

Since $1 < q < 2_\mu^*$ it is easy to see that

$$J_\lambda(u_0) = (\frac{\lambda}{2} - \frac{\lambda}{2q})\int_\Omega\int_\Omega \frac{|u_0(x)|^q|u_0(y)|^q}{|x-y|^\mu}dxdy + \frac{N+2-\mu}{4N-2\mu}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \geq 0.$$

Lemma 3.3. *If $1 < q < 2_\mu^*$, $\lambda > 0$ and $\{u_n\}$ is a $(PS)_c$ sequence of J_λ with*

$$c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \tag{3.1}$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Let u_0 be the weak limit of $\{u_n\}$ obtained in Lemma 3.2 and define $v_n := u_n - u_0$, then we know $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n \to 0$ a.e. in Ω. Moreover, by Lemma 2.3, we know

$$\int_\Omega\int_\Omega \frac{|u_n(x)|^q|u_n(y)|^q}{|x-y|^\mu}dxdy = \int_\Omega\int_\Omega \frac{|v_n(x)|^q|v_n(y)|^q}{|x-y|^\mu}dxdy + \int_\Omega\int_\Omega \frac{|u_0(x)|^q|u_0(y)|^q}{|x-y|^\mu}dxdy + o_n(1).$$

Similar to the proof of Lemma 2.4, we have

$$c \leftarrow J_\lambda(u_n) \geq \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1),$$

since $J_\lambda(u_0) \geq 0$ and

$$\int_\Omega\int_\Omega \frac{|v_n(x)|^q|v_n(y)|^q}{|x-y|^\mu}dxdy \leq C(N,\mu)|v_n^q|^2_{\frac{2N}{2N-\mu}} \to 0,$$

as $n \to +\infty$. Repeating the same arguments in the proof of Lemma 2.4, we have

$$\|u_n - u_0\| \to 0$$

as $n \to +\infty$. This ends the proof of Lemma 3.3. □

Lemma 3.4. *Let $1 < q < 2_\mu^*$ and u_ε as defined in (2.6). If one of the following conditions is holds:*

(1) $N > \frac{2(q+1)-\mu}{q-1}$ *and* $\lambda > 0$,
(2) $N \le \frac{2(q+1)-\mu}{q-1}$ *and λ is sufficiently large,*

then there exists ε such that

$$\sup_{t \ge 0} J_\lambda(tu_\varepsilon) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \tag{3.2}$$

Proof. Case 1. $N > \frac{2(q+1)-\mu}{q-1}$.

To estimate the convolution part, we know

$$\begin{aligned}
\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^q |u_\varepsilon(y)|^q}{|x-y|^\mu} dxdy &\ge \int_{B_\delta} \int_{B_\delta} \frac{|u_\varepsilon(x)|^q |u_\varepsilon(y)|^q}{|x-y|^\mu} dxdy \\
&= \int_{B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy \\
&= \int_\Omega \int_\Omega \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy - 2 \int_{\Omega \setminus B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy \\
&\quad - \int_{\Omega \setminus B_\delta} \int_{\Omega \setminus B_\delta} \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy \\
&:= \mathbb{A} - 2\mathbb{B} - \mathbb{C},
\end{aligned} \tag{3.3}$$

where

$$\mathbb{A} = \int_\Omega \int_\Omega \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy, \quad \mathbb{B} = \int_{\Omega \setminus B_\delta} \int_{B_\delta} \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy$$

and

$$\mathbb{C} = \int_{\Omega \setminus B_\delta} \int_{\Omega \setminus B_\delta} \frac{|U_\varepsilon(x)|^q |U_\varepsilon(y)|^q}{|x-y|^\mu} dxdy.$$

We are going to estimate $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. By direct computation, we know, for $\varepsilon < 1$,

$$\begin{aligned}
\mathbb{A} &= \varepsilon^{-(N-2)q} [N(N-2)]^{\frac{(N-2)q}{2}} \int_\Omega \int_\Omega \frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)q}{2}} |x-y|^\mu (1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}} dxdy \\
&\ge \varepsilon^{-(N-2)q} [N(N-2)]^{\frac{(N-2)q}{2}} \int_{B_\delta} \int_{B_\delta} \frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)q}{2}} |x-y|^\mu (1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}} dxdy
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^{-(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\varepsilon^{2N-\mu}\int\limits_{B_{\frac{\delta}{\varepsilon}}}\int\limits_{B_{\frac{\delta}{\varepsilon}}}\frac{1}{(1+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(1+|y|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&\geq O(\varepsilon^{2N-\mu-(N-2)q})\int\limits_{B_\delta}\int\limits_{B_\delta}\frac{1}{(1+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(1+|y|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&= O(\varepsilon^{2N-\mu-(N-2)q}),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\mathbb{B} &= \varepsilon^{-(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\int\limits_{\Omega\backslash B_\delta}\int\limits_{B_\delta}\frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&= \varepsilon^{(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\int\limits_{\Omega\backslash B_\delta}\int\limits_{B_\delta}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&\leq O(\varepsilon^{(N-2)q})\Big(\int\limits_{\Omega\backslash B_\delta}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)qN}{2N-\mu}}}dx\Big)^{\frac{2N-\mu}{2N}}\Big(\int\limits_{B_\delta}\frac{1}{(\varepsilon^2+|y|^2)^{\frac{(N-2)qN}{2N-\mu}}}dy\Big)^{\frac{2N-\mu}{2N}} \\
&= O(\varepsilon^{\frac{2N-\mu}{2}})\Big(\int_0^{\frac{\delta}{\varepsilon}}\frac{z^{N-1}}{(1+z^2)^{\frac{(N-2)qN}{2N-\mu}}}dz\Big)^{\frac{2N-\mu}{2N}} \\
&\leq O(\varepsilon^{\frac{2N-\mu}{2}})\Big(\int_0^{+\infty}\frac{z^{N-1}}{(1+z^2)^{\frac{(N-2)qN}{2N-\mu}}}dz\Big)^{\frac{2N-\mu}{2N}}.
\end{aligned}
\tag{3.5}
$$

Since $N > \frac{2(q+1)-\mu}{q-1} > 2 + \frac{4-\mu}{2(q-1)}$ if $\mu < 4$ and $N > 2 + \frac{4-\mu}{2(q-1)}$ if $\mu \geq 4$, we know $\frac{2(N-2)qN}{2N-\mu} > N$, therefore

$$
\mathbb{B} \leq O(\varepsilon^{\frac{2N-\mu}{2}}),
\tag{3.6}
$$

$$
\begin{aligned}
\mathbb{C} &= \varepsilon^{-(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\int\limits_{\Omega\backslash B_\delta}\int\limits_{\Omega\backslash B_\delta}\frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(1+|\frac{y}{\varepsilon}|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&= \varepsilon^{(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\int\limits_{\Omega\backslash B_\delta}\int\limits_{\Omega\backslash B_\delta}\frac{1}{(\varepsilon^2+|x|^2)^{\frac{(N-2)q}{2}}|x-y|^{\mu}(\varepsilon^2+|y|^2)^{\frac{(N-2)q}{2}}}dxdy \\
&\leq \varepsilon^{(N-2)q}[N(N-2)]^{\frac{(N-2)q}{2}}\int\limits_{\Omega\backslash B_\delta}\int\limits_{\Omega\backslash B_\delta}\frac{1}{|x|^{(N-2)q}|x-y|^{\mu}|y|^{(N-2)q}}dxdy \\
&= O(\varepsilon^{(N-2)q}).
\end{aligned}
\tag{3.7}
$$

It follows from (3.3)–(3.7) that

$$
\begin{aligned}
\int\limits_{\Omega}\int\limits_{\Omega}\frac{|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{\mu}}dxdy &\geq O(\varepsilon^{2N-\mu-(N-2)q}) - O(\varepsilon^{\frac{2N-\mu}{2}}) - O(\varepsilon^{(N-2)q}) \\
&= O(\varepsilon^{2N-\mu-(N-2)q}) - O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}}).
\end{aligned}
\tag{3.8}
$$

By (2.7), (2.8) and (3.8), we have

$$
J_\lambda(tu_\varepsilon) = \frac{t^2}{2}\int\limits_{\Omega}|\nabla u_\varepsilon|^2dx - \frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int\limits_{\Omega}\int\limits_{\Omega}\frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy - \frac{\lambda t^{2q}}{2q}\int\limits_{\Omega}\int\limits_{\Omega}\frac{|u_\varepsilon(x)|^q|u_\varepsilon(y)|^q}{|x-y|^{\mu}}dxdy
$$

$$
\begin{aligned}
&\leq \frac{t^2}{2}\big(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})\big)-\frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})\big)\\
&\quad -\frac{t^{2q}}{2q}\big(O(\varepsilon^{2N-\mu-(N-2)q})-O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\big)\\
&:= g(t).
\end{aligned}
$$

It is clear that $g(t)\to-\infty$ as $t\to+\infty$. It follows that there exists $t_\varepsilon>0$ such that $\sup_{t>0}g(t)$ is attained at t_ε. Differentiating $g(t)$ and equaling to zero, we obtain that

$$
\begin{aligned}
t_\varepsilon\big(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})\big)-t_\varepsilon^{2\cdot 2_\mu^*-1}\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})\big)\\
-t^{2q-1}\big(O(\varepsilon^{2N-\mu-(N-2)q})-O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\big)=0,
\end{aligned}
$$

since $N>\frac{2(q+1)-\mu}{q-1}$ and $N\geq 3$ we know

$$
2N-\mu-(N-2)q<\min\{\frac{2N-\mu}{2},(N-2)q\}, \tag{3.9}
$$

which means

$$
\big(O(\varepsilon^{2N-\mu-(N-2)q})-O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\big)\geq 0
$$

if ε is small enough. And so

$$
t_\varepsilon<\Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})}{C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})}\Big)^{\frac{1}{2\cdot 2_\mu^*-2}}:=S_{H,L}(\varepsilon)
$$

and there exists $t_0>0$ such that for $\varepsilon>0$ small enough

$$
t_\varepsilon>t_0.
$$

Notice that the function

$$
t\mapsto\frac{t^2}{2}(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2}))-\frac{t^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))
$$

is increasing on $[0,S_{H,L}(\varepsilon)]$, thanks to $t_0<t_\varepsilon<S_{H,L}(\varepsilon)$ and (3.8), we have

$$
\begin{aligned}
\max_{t\geq 0}J_\lambda(tu_\varepsilon)&\leq\frac{N+2-\mu}{4N-2\mu}\Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})}{\big(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})\big)^{\frac{N-2}{2N-\mu}}}\Big)^{\frac{2N-\mu}{N+2-\mu}}\\
&\quad-O(\varepsilon^{2N-\mu-(N-2)q})+O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\\
&\leq\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}+O(\varepsilon^{\min\{\frac{2N-\mu}{2},N-2\}})-O(\varepsilon^{2N-\mu-(N-2)q})+O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}}).
\end{aligned}
$$

The assumptions $N>\frac{2(q+1)-\mu}{q-1}$ and $1<q<2_\mu^*$ together with (3.9) imply that

$$
\max_{t\geq 0}J_\lambda(tu_\varepsilon)<\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},
$$

if ε is small enough.

Case 2. $N \le \frac{2(q+1)-\mu}{q-1}$.

For any fixed ε in (2.6), assume that $\max_{t\ge 0} J_\lambda(tu_\varepsilon)$ is attained at some $t_\lambda > 0$, repeat the arguments in the proof of Lemma 2.5, we know $t_\lambda \to 0$ as $\lambda \to +\infty$ and $\max_{t\ge 0} J_\lambda(tu_\varepsilon) \to 0$, as $\lambda \to +\infty$, which leads to the conclusion for the case $N \le \frac{2(q+1)-\mu}{q-1}$. □

Proof of Theorem 1.4. By Lemma 3.1, Lemma 3.4 and the Mountain Pass Theorem without (PS) condition (cf. [48]), there exists a $(PS)_c$ sequence $\{u_n\}$ of J_λ with $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$ if one of the following conditions is holds:

(1). $N > \frac{2(q+1)-\mu}{q-1}$ and $\lambda > 0$,
(2). $N \le \frac{2(q+1)-\mu}{q-1}$ and λ is sufficiently large.

Applying Lemma 3.3, we know $\{u_n\}$ contains a convergent subsequence. And so, we have J_λ has a critical value $c \in (0, \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}})$ and problem (1.9) has a nontrivial solution. □

Proof of Theorem 1.5. The proof of Theorem 1.5 is similar to that of Theorem 1.4 and the main difference is that the $(PS)_c$ condition holds below the critical level $\frac{1}{N} S^{\frac{N}{2}}$. From Lemma 1.46 of [48], we know that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) \tag{3.10}$$

and

$$\int_\Omega |u_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^{N}). \tag{3.11}$$

For the Case 1 of Lemma 3.4, we have

$$\begin{aligned}\max_{t\ge 0} J_\lambda(tu_\varepsilon) &\le \frac{1}{N}\Big(\frac{S^{\frac{N}{2}} + O(\varepsilon^{N-2})}{\big(S^{\frac{N}{2}} + O(\varepsilon^{N})\big)^{\frac{2}{2^*}}}\Big)^{\frac{N}{2}} - O(\varepsilon^{2N-\mu-(N-2)q}) + O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\\ &< \frac{1}{N} S^{\frac{N}{2}} + O(\varepsilon^{N-2}) - O(\varepsilon^{2N-\mu-(N-2)q}) + O(\varepsilon^{\min\{\frac{2N-\mu}{2},(N-2)q\}})\\ &< \frac{1}{N} S^{\frac{N}{2}},\end{aligned}$$

thanks to $N > \frac{2(q+1)-\mu}{q-1}$. The rest of the proof is omitted here. □

4. An Ambrosetti–Brezis–Cerami type concave and convex result

In this section we discuss the problem (1.11) with both suplinear and sublinear local perturbation, i.e.

$$\begin{cases} -\Delta u = \Big(\displaystyle\int_\Omega \frac{|u|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|u|^{2^*_\mu-2}u + u^p + \lambda u^q & \text{in } \Omega,\\ u \in H_0^1(\Omega), \end{cases}$$

where $0 < q < 1$ and $1 < p < 2^* - 1$. Then we may define

$$J_\lambda(u) = \frac{1}{2}\int\limits_\Omega |\nabla u|^2 dx - \frac{\lambda}{q+1}\int\limits_\Omega u^{q+1}dx - \frac{1}{p+1}\int\limits_\Omega u^{p+1}dx - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy.$$

The proof of the main results in Theorem 1.6 will be separated into several Lemmas. We begin with a standard comparison method as well as some ideas given in [6, Lemma 3.1]. Let Λ be defined by

$$\Lambda = \sup\{\lambda > 0 : \text{problem (1.11) has a solution}\}.$$

Lemma 4.1. $0 < \Lambda < \infty$.

Proof. To prove that $\Lambda > 0$ we use the sub- and supersolution technique to construct a solution for any small λ by using some ideas from [6,29]. Let (λ_1, e_1) be the first eigenvalue and a corresponding positive eigenfunction of the Laplacian in Ω. We can obtain a subsolution by taking a small multiple of e_1. We denote it as εe_1. Let function v denote the solution of

$$\begin{cases} -\Delta v = 1 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $0 < q < 1 < p < 2\cdot 2_\mu^* - 1$, we can find $\lambda_0 > 0$ such that for $0 < \lambda \le \lambda_0$ there exists $M(\lambda) > 0$ satisfying

$$M \ge \lambda M^q |v|_\infty^q + M^p|v|_\infty^p + c_0 M^{2\cdot 2_\mu^* - 1}|v|_\infty^{2\cdot 2_\mu^*-1},$$

where $c_0 = \int_{\Omega+\Omega}\frac{1}{|y|^\mu}dy$, $\Omega + \Omega := \{x + y \in \mathbb{R}^N : x, y \in \Omega\}$. As a consequence, the function Mv verifies

$$M = -\Delta(Mv) \ge \lambda(Mv)^q + (Mv)^p + M^{2\cdot 2_\mu^*-1}\int\limits_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy|v(x)|^{2_\mu^*-2}v(x)$$

and so Mv is a supersolution of (1.11). It follows that (1.11) has a positive solution $\varepsilon e_1 \le u \le Mv$ for $0 < \lambda \le \lambda_0$ and so $\Lambda > 0$.

Using e_1 as a test function in (1.11), we have that

$$\begin{aligned} \lambda_1\int\limits_\Omega ue_1 dx &= \int\limits_\Omega\int\limits_\Omega \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)e_1(y)}{|x-y|^\mu}dxdy + \lambda\int\limits_\Omega u^q e_1 dx + \int\limits_\Omega u^p e_1 dx \\ &\ge \lambda\int\limits_\Omega u^q e_1 dx + \int\limits_\Omega u^p e_1 dx. \end{aligned} \tag{4.1}$$

Since there exist positive constants c_1, c_2 such that $\lambda t^q + t^p > c_1\lambda^{c_2}t$, for any $t > 0$, we obtain from (4.1) that $c_1\lambda^{c_2} < \lambda_1$ which implies $\Lambda < \infty$. □

Lemma 4.2. *For all $0 < \lambda < \Lambda$, the equation (1.11) has a minimal solution.*

Proof. Let v_λ be the unique positive solution of

$$\begin{cases} -\Delta v = \lambda v^q & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We already know that there exists a solution $u > 0$ of (1.11) for every $0 < \lambda < \Lambda$. Since $-\Delta u \geq \lambda u^q$ we can use Lemma 3.3 of [6] with $w = u$ to deduce that any solution u of (1.11) must satisfy $u \geq v_\lambda$. Clearly, v_λ is a subsolution of (1.11). The monotone iteration

$$-\Delta u_{n+1} = \lambda u_n^q + u_n^p + \Big(\int\limits_\Omega \frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_n(x)|^{2_\mu^*-2}u_n(x), \quad u_0 = v_\lambda,$$

satisfies $u_n \nearrow u_\lambda$, with u_λ solution of (1.11). It is easy to check that u_λ is a minimal solution of (1.11). Indeed, if u is any solution of (1.11), then $u \geq v_\lambda$ and u is a supersolution of (1.11). Thus $u_n \leq u$, $\forall n$, by induction, and $u \geq u_\lambda$.

Moreover, this minimal solution is increasing with respect to λ. In fact, if $u_{\lambda'}$ is a minimal solution of (1.11) with $\lambda = \lambda'$, then we have

$$-\Delta u_{\lambda'} \leq \lambda'' u_{\lambda'}^q + u_{\lambda'}^p + \Big(\int\limits_\Omega \frac{|u_{\lambda'}(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_{\lambda'}(x)|^{2_\mu^*-2}u_{\lambda'}(x)$$

for $0 < \lambda' < \lambda'' < \Lambda$, that is $u_{\lambda'}$ is a subsolution of (1.11) with $\lambda = \lambda''$. So $u_{\lambda'} \leq u_{\lambda''}$ and $u_{\lambda'} \not\equiv u_{\lambda''}$ for $0 < \lambda' < \lambda'' < \Lambda$. $\square$

The solutions for equation (1.11) are classical in fact. First let us recall an important inequality for nonlocal nonlinearities by Moroz and Van Schaftingen [39] which complement the results by Brezis and Kato in [12].

Lemma 4.3. *(See [39].) Let $N \geq 2$, $\mu \in (0, N)$ and $\theta \in (0, N)$. If $H, K \in L^{\frac{2N}{N-\mu+2}}(\mathbb{R}^N) + L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$, $(1 - \frac{\mu}{N}) < \theta < (1 + \frac{\mu}{N})$, then for any $\varepsilon > 0$, there exists $C_{\varepsilon,\theta} \in \mathbb{R}$ such that for every $u \in H^1(\mathbb{R}^N)$,*

$$\int\limits_{\mathbb{R}^N} \Big(|x|^{-\mu} * (H|u|^\theta)\Big)K|u|^{2-\theta}dx \leq \varepsilon^2 \int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + C_{\varepsilon,\theta} \int\limits_{\mathbb{R}^N} |u|^2 dx.$$

We have the following regularity Lemma and L^∞ estimates for the solutions.

Lemma 4.4. *Let u be a solution to the problem*

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{4.2}$$

and assume that $|g(x,u)| \leq C(1+|u|^p) + \Big(\int_\Omega \frac{|u|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u|^{2_\mu^*-2}u$, $1 < p < 2^*-1$ and $C > 0$, then $u \in L^\infty(\Omega)$ and $u \in C^2(\overline{\Omega})$.

Proof. Let us define the truncation: $\Omega \to \mathbb{R}$, for $\tau > 0$ large,

$$u_\tau(x) = \begin{cases} -\tau & \text{if } u \leq -\tau, \\ u(x) & \text{if } -\tau < u < \tau, \\ \tau & \text{if } u \geq \tau. \end{cases}$$

Since $|u_\tau|^{s-2}u_\tau \in H_0^1(\Omega)$ for $s \geq 2$, we take $|u_\tau|^{s-2}u_\tau$ as a test function in (4.2), we obtain

$$\begin{aligned}
\frac{4(s-1)}{s^2}&\int\limits_\Omega |\nabla (u_\tau)^{\frac{s}{2}}|^2dx \\
&= (s-1)\int\limits_\Omega |u_\tau|^{s-2}|\nabla u_\tau|^2dx \\
&\le \int\limits_\Omega\int\limits_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*-1}|u_\tau(x)|^{s-1}dx + C\int\limits_\Omega |u_\tau|^{s-2}u_\tau u^p dx + C\int\limits_\Omega |u_\tau|^{s-1}u_\tau dx \\
&\le \int\limits_\Omega\int\limits_\Omega \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*-1}|u_\tau(x)|^{s-1}dx + 2C\int\limits_\Omega (1+u^2)|u_\tau|^{s-2}\frac{1+u^p}{1+u}dx.
\end{aligned}$$

We denote $a(u) := \frac{1+u^p}{1+u}$ and have $0 \le a(u) \le C_1(1+u^{p-1}) \in L^{\frac{N}{2}}(\Omega)$. If $2 \le s < \frac{2N}{N-\mu}$, using Lemma 4.3 with $\theta = \frac{2}{s}$, there exists $C_2 > 0$ such that

$$\int\limits_\Omega\int\limits_\Omega \frac{|u_\tau(y)|^{2_\mu^*}}{|x-y|^\mu}dy|u_\tau(x)|^{2_\mu^*-1}|u_\tau(x)|^{s-1}dx \le \frac{2(s-1)}{s^2}\int\limits_\Omega |\nabla (u_\tau)^{\frac{s}{2}}|^2dx + C_2\int\limits_\Omega ||u_\tau|^{\frac{s}{2}}|^2dx.$$

Since $|u_\tau| \le |u|$, we have

$$\begin{aligned}
\frac{2(s-1)}{s^2}\int\limits_\Omega |\nabla (u_\tau)^{\frac{s}{2}}|^2dx &\le C_2\int\limits_\Omega |u|^s dx + \int\limits_{A_\tau}\int\limits_\Omega \frac{|u(y)|^{2_\mu^*-1}|u(y)|^{s-1}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*}dx \\
&\quad + 2C\int\limits_\Omega a(u)(1+u^2)|u_\tau|^{s-2}dx,
\end{aligned}$$

where $A_\tau = \{x \in \Omega : |u| > \tau\}$.

Since $2 \le s < \frac{2N}{N-\mu}$, applying the Hardy–Littlewood–Sobolev inequality again,

$$\int\limits_{A_\tau}\int\limits_\Omega \frac{|u(y)|^{2_\mu^*-1}|u(y)|^{s-1}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*}dx \le C_3\Big(\int\limits_\Omega ||u|^{2_\mu^*-1}|u|^{s-1}|^r dx\Big)^{\frac{1}{r}}\Big(\int\limits_{A_\tau} ||u|^{2_\mu^*}|^l dx\Big)^{\frac{1}{l}},$$

with $\frac{1}{r} = 1 + \frac{N-\mu}{2N} - \frac{1}{s}$ and $\frac{1}{l} = \frac{N-\mu}{2N} + \frac{1}{s}$. By Hölder's inequality, if $u \in L^s(\Omega)$, then $|u|^{2_\mu^*} \in L^l(\Omega)$ and $|u|^{2_\mu^*-1}|u|^{s-1} \in L^r(\Omega)$, whence by Lebesgue's dominated convergence theorem

$$\lim_{\tau\to\infty}\int\limits_{A_\tau}\int\limits_\Omega \frac{|u(y)|^{2_\mu^*-1}|u(y)|^{s-1}}{|x-y|^\mu}dy|u(x)|^{2_\mu^*}dx = 0.$$

On the other hand,

$$\begin{aligned}
\int\limits_\Omega a(u)u^2|u_\tau|^{s-2}dx &\le \tau_0\int\limits_{a<\tau_0} u^2|u_\tau|^{s-2}dx + \int\limits_{a\ge\tau_0} a(u)u^2|u_\tau|^{s-2}dx \\
&\le C_4\tau_0 + (\int\limits_{a\ge\tau_0} a(u)^{\frac{N}{2}}dx)^{\frac{2}{N}}(\int\limits_{a\ge\tau_0} (u|u_\tau|^{\frac{s-2}{2}})^{2^*}dx)^{\frac{2}{2^*}}
\end{aligned}$$

and

$$\int_\Omega a(u)|u_\tau|^{s-2}dx \le C_5\tau_0 + (\int_{a\ge\tau_0} a(u)^{\frac{N}{2}}dx)^{\frac{2}{N}}(\int_{a\ge\tau_0} (|u_\tau|^{\frac{s-2}{2}})^{2^*}dx)^{\frac{2}{2^*}},$$

where, since $|u_\tau|^{\frac{s}{2}}$, C_4 and C_5 can be taken independent of τ. Hence, by $a(u) \in L^{\frac{N}{2}}$ it follows that

$$(\int_{a\ge\tau_0} a(u)^{\frac{N}{2}}dx)^{\frac{2}{N}} \to 0$$

as $\tau_0 \to \infty$. Therefore, choosing τ_0 large enough such that

$$C(\int_{a\ge\tau_0} a(u)^{\frac{N}{2}}dx)^{\frac{2}{N}} < \frac{1}{2},$$

by Sobolev embedding theory, we obtain that there exists a constant $K(\tau_0)$, independent of τ, for which it holds

$$\Big(\int_\Omega |u_\tau|^{\frac{sN}{N-2}}dx\Big)^{1-\frac{2}{N}} \le C_2\int_\Omega |u|^s dx + K(\tau_0).$$

Letting $\tau \to \infty$ we conclude that $u \in L^{\frac{sN}{N-2}}(\Omega)$. By iterating over s a finite number of times we cover the range $s \in [2, \frac{2N}{N-\mu})$. So we can get weak solution $u \in L^s(\Omega)$ of (1.11) for every $s \in [2, \frac{2N^2}{(N-\mu)(N-2)})$. Thus, $|u|^{2^*_\mu} \in L^s(\Omega)$ for every $s \in [\frac{2(N-2)}{2N-\mu}, \frac{2N^2}{(N-\mu)(2N-\mu)})$. Since $\frac{2(N-2)}{2N-\mu} < \frac{N}{N-\mu} < \frac{2N^2}{(N-\mu)(2N-\mu)}$, we have

$$\int_\Omega \frac{|u|^{2^*_\mu}}{|x-y|^\mu}dy \in L^\infty(\Omega)$$

and so

$$|g(x,u)| \le C_6(1+|u|^{2^*-1}).$$

From Theorem 1.16 of [7], we have the weak solution $u \in L^\infty(\Omega)$ and $u \in C^2(\overline{\Omega})$. □

Lemma 4.5. *For all $0 < \lambda < \Lambda$, there exists a positive solution for (1.11) which is a local minimum of the functional J_λ in the C^1-topology.*

Proof. Let $0 < \lambda' < \lambda < \lambda'' < \Lambda$ and $u_{\lambda'}$ and $u_{\lambda''}$ be the corresponding minimal solutions to (1.11), $\lambda = \lambda'$ and λ'' respectively. Denote $u := u_{\lambda''} - u_{\lambda'}$. Then, since minimal solutions is increasing with respect to λ, we have

$$\begin{cases} -\Delta u \ge 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In order to obtain the existence of positive solution, we may assume that

$$f_\lambda(u) = \Big(\int_\Omega \frac{|u|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|u|^{2^*_\mu-2}u + \lambda u^q + u^p,$$

for $u \ge 0$ and $f_\lambda(u) = 0$, for $u < 0$. We set

$$f_\lambda^*(u) = \begin{cases} f_\lambda(u_{\lambda'}) & \text{if } u \le u_{\lambda'}, \\ f_\lambda(u) & \text{if } u_{\lambda'} < u < u_{\lambda''}, \\ f_\lambda(u_{\lambda''}) & \text{if } u \ge u_{\lambda''}, \end{cases}$$

$$F_\lambda^*(u) = \int\limits_0^u f_\lambda^*(s)ds$$

and

$$J_\lambda^*(u) = \frac{1}{2}\int\limits_\Omega |\nabla u|^2 dx - \int\limits_\Omega F_\lambda^*(u)dx.$$

Standard calculation shows that J_λ^* achieves its global minimum at some $u_0 \in H_0^1(\Omega)$ satisfying

$$\begin{cases} -\Delta u_0 = f_\lambda^*(u_0) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$J_\lambda^*(u_0) \le J_\lambda^*(u), \quad \forall u \in H_0^1(\Omega).$$

By the Maximum Principle, we get that $u_{\lambda'} < u_0 < u_{\lambda''}$ in Ω, as well as

$$\frac{\partial}{\partial\nu}(u_0 - u_{\lambda'}) < 0, \quad \frac{\partial}{\partial\nu}(u_0 - u_{\lambda''}) > 0, \quad x \in \partial\Omega,$$

where ν is the outer unit normal at $\partial\Omega$. If $\|u - u_0\|_{C_0^1(\Omega)} \le \varepsilon$ for ε small enough, then $u_{\lambda'} \le u \le u_{\lambda''}$ and so we have $J_\lambda^*(u) = J_\lambda(u)$. Then

$$J_\lambda(u) = J_\lambda^*(u) \ge J_\lambda^*(u_0) = J_\lambda(u_0)$$

for any $u \in C_0^1(\Omega)$ with $\|u-u_0\|_{C_0^1(\Omega)} \le \varepsilon$ and so u_0 is a local minimum for J_λ in the sense of C^1 topology. □

Lemma 4.6. *Let $u_0 \in H_0^1(\Omega)$ be a local minimum of the functional J_λ in $C_0^1(\Omega)$, by this we mean that there exists $r > 0$ such that*

$$J_\lambda(u_0) \le J_\lambda(u_0 + u), \quad \forall u \in C_0^1(\Omega) \quad \textit{with} \quad \|u\|_{C_0^1(\Omega)} \le r.$$

Then u_0 is a local minimum of J_λ in $H_0^1(\Omega)$, that is, there exists $\varepsilon_0 > 0$ such that

$$J_\lambda(u_0) \le J_\lambda(u_0 + u), \quad \forall u \in H_0^1(\Omega) \quad \textit{with} \quad \|u\| \le \varepsilon_0.$$

Proof. Arguing by contradiction, we may suppose that for any $\varepsilon > 0$ small we have

$$\min_{u \in B_\varepsilon(u_0)} J_\lambda(u) < J_\lambda(u_0),$$

where $B_\varepsilon(u_0) = \{u \in H_0^1(\Omega) : \|u - u_0\| \le \varepsilon\}$.

Following Brezis and Nirenberg [15], we sketch the proof here. Applying a standard argument of weak lower semi-continuity, we may take $u_{0,\varepsilon}$ such that $\min_{u \in B_\varepsilon(u_0)} J_\lambda(u) = J_\lambda(u_{0,\varepsilon})$. We need to prove that

$u_{0,\varepsilon} \to u_0$ in $C_0^1(\Omega)$ as $\varepsilon \searrow 0$. Note that the Euler–Lagrange equation satisfied by $u_{0,\varepsilon}$ involves a Lagrange multiplier ζ_ε such that

$$\langle J'_\lambda(u_{0,\varepsilon}), \varphi\rangle = \zeta_\varepsilon \langle u_{0,\varepsilon}, \varphi\rangle_{H_0^1(\Omega)}, \qquad \forall \varphi \in H_0^1(\Omega). \tag{4.3}$$

From $u_{0,\varepsilon}$ is a minimum of J_λ in $B_\varepsilon(u_0)$, we have

$$\zeta_\varepsilon = \frac{\langle J'_\lambda(u_{0,\varepsilon}), u_{0,\varepsilon}\rangle}{\|u_{0,\varepsilon}\|^2} \leq 0. \tag{4.4}$$

By (4.3) we easily get that $u_{0,\varepsilon}$ satisfies

$$\begin{cases} -\Delta u_{0,\varepsilon} = \dfrac{1}{1-\zeta_\varepsilon} f_\lambda(u_{0,\varepsilon}) := f_{\lambda,\varepsilon}(u_{0,\varepsilon}) & \text{in } \Omega, \\ u_{0,\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u_{0,\varepsilon} \in H_0^1(\Omega)$ and (4.4), by Lemma 4.4, there exists a constant C independent of ε such that $\|u_{0,\varepsilon}\|_{C^2(\overline{\Omega})} < C$. By Ascoli–Arzelá Theorem there exists a subsequence, still denoted by $u_{0,\varepsilon}$, such that $u_{0,\varepsilon} \to u_0$ uniformly in $C_0^1(\Omega)$ as $\varepsilon \searrow 0$. This implies that for ε small enough,

$$J_\lambda(u_{0,\varepsilon}) < J_\lambda(u_0)$$

for any $u_{0,\varepsilon}$ with $\|u_{0,\varepsilon} - u_0\|_{C_0^1(\Omega)} < \varepsilon$. This contradicts our hypothesis. $\square$

From Lemmas 4.5 and 4.6, we know there exists a local minimum u_0 in $H_0^1(\Omega)$. For $0 < \lambda < \Lambda$, we consider the translated nonlinearity defined by

$$g(u) = \begin{cases} \mathbb{F}(u) & \text{if } u \geq 0, \\ 0 & \text{if } u < 0, \end{cases}$$

where

$$\begin{aligned} \mathbb{F}(u) &= \lambda(u_0+u)^q - \lambda u_0^q + (u_0+u)^p - u_0^p \\ &\quad + \Big(\int_\Omega \frac{|u_0+u|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|u_0+u|^{2^*_\mu-2}(u_0+u) - \Big(\int_\Omega \frac{|u_0|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|u_0|^{2^*_\mu-2}u_0. \end{aligned}$$

We consider the translated problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

The Hardy–Littlewood–Sobolev inequality implies that

$$\bar{J}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(u) dx,$$

is well defined, where

$$G(u) = \int_0^u g(s) ds.$$

Therefore if $\bar{u} \not\equiv 0$ is a critical point of $\bar{J}_\lambda$ then it is a solution of (4.5), by the Maximum Principle, $\bar{u} > 0$ and $u = u_0 + \bar{u}$ will be a second solution of (1.11). In order to finish the proof the Theorem 1.6, we are going to investigate the existence of nontrivial critical points for $\bar{J}_\lambda$.

Lemma 4.7. *$u = 0$ is a local minimum of $\bar{J}_\lambda$ in $H_0^1(\Omega)$.*

Proof. We only need to show that $u = 0$ is a local minimum of $\bar{J}_\lambda$ in C^1 topology. Let $u \in C_0^1(\Omega)$, by direct computation, we know

$$\begin{aligned}
\bar{J}_\lambda(u) &= \frac{1}{2}\|u\|^2 - \int\limits_\Omega G(u)dx \\
&= \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int\limits_\Omega |u_0+u|^{q+1}dx + \frac{\lambda}{q+1}\int\limits_\Omega |u_0|^{q+1}dx + \lambda\int\limits_\Omega u_0^q u dx - \frac{1}{p+1}\int\limits_\Omega |u_0+u|^{p+1}dx \\
&\quad + \frac{1}{p+1}\int\limits_\Omega |u_0|^{p+1}dx + \int\limits_\Omega u_0^p u dx - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|(u_0+u)(x)|^{2_\mu^*}|(u_0+u)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad + \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + \int\limits_\Omega \Big(\int\limits_\Omega \frac{|u_0|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_0|^{2_\mu^*-2}u_0 u dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&J_\lambda(u_0+u) \\
&= \frac{1}{2}\|u_0+u\|^2 - \frac{\lambda}{q+1}\int\limits_\Omega |u_0+u|^{q+1}dx - \frac{1}{p+1}\int\limits_\Omega |u_0+u|^{p+1}dx \\
&\quad - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|(u_0+u)(x)|^{2_\mu^*}|(u_0+u)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u\|^2 + \int\limits_\Omega \nabla u\nabla u_0 dx - \frac{\lambda}{q+1}\int\limits_\Omega |u_0+u|^{q+1}dx - \frac{1}{p+1}\int\limits_\Omega |u_0+u|^{p+1}dx \\
&\quad - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|(u_0+u)(x)|^{2_\mu^*}|(u_0+u)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int\limits_\Omega |u_0+u|^{q+1}dx - \frac{1}{p+1}\int\limits_\Omega |u_0+u|^{p+1}dx + \lambda\int\limits_\Omega u_0^q u dx + \int\limits_\Omega u_0^p u dx \\
&\quad - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|(u_0+u)(x)|^{2_\mu^*}|(u_0+u)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + \int\limits_\Omega \Big(\int\limits_\Omega \frac{|u_0|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_0|^{2_\mu^*-2}u_0 u dx.
\end{aligned}$$

Since u_0 is a local minimum of J_λ, we have that

$$\begin{aligned}
\bar{J}_\lambda(u) = J_\lambda(u_0+u) - \frac{1}{2}\|u_0\|^2 + \frac{\lambda}{q+1}\int\limits_\Omega |u_0|^{q+1}dx \\
+ \frac{1}{p+1}\int\limits_\Omega |u_0|^{p+1}dx + \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy
\end{aligned}$$

$$
\begin{aligned}
&= J_\lambda(u_0+u) - J_\lambda(u_0)\\
&\geq 0\\
&= \bar{J}_\lambda(0)
\end{aligned}
$$

provided $\|u\|_{C_0^1(\Omega)} < \varepsilon$. $\square$

Lemma 4.8. *If $u = 0$ is the only critical point of $\bar{J}_\lambda$ in $H_0^1(\Omega)$ then $\bar{J}_\lambda$ satisfies a local Palais–Smale condition below the critical level $\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.*

Proof. If $\{w_n\}$ is a $(PS)_c$ sequence of $\bar{J}_\lambda$, then

$$
\bar{J}_\lambda(w_n) \to c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \quad \bar{J}'_\lambda(w_n) \to 0. \tag{4.6}
$$

Since the fact that w_0 is a critical point implies $\bar{J}_\lambda(w_n) = J_\lambda(u_n) - J_\lambda(w_0)$, where $u_n = w_n + w_0$, we have that

$$
J_\lambda(u_n) \to c + J_\lambda(w_0), \quad J'_\lambda(u_n) \to 0. \tag{4.7}
$$

Similar to Lemma 2.2, we have, if $\{u_n\}$ is a $(PS)_c$ sequence of J_λ, then $\{u_n\}$ is bounded, if $u_\infty \in H_0^1(\Omega)$ is the weak limit of $\{u_n\}$, then u_∞ is a weak solution of problem (1.11). Let $v_n := u_n - u_\infty$, then we know $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_n \to 0$ a.e. in Ω. From by the proof of Lemma 2.4, we can assume there exists a nonnegative constant b such that

$$
\int\limits_\Omega |\nabla v_n|^2 dx \to b
$$

as $n \to +\infty$ and we obtain

$$
\lim_{n\to\infty} J_\lambda(u_n) \geq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + J_\lambda(u_\infty) \tag{4.8}
$$

or $b = 0$. If (4.8) holds, then by (4.6) and (4.7), we have

$$
\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + J_\lambda(u_\infty) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + J_\lambda(w_0).
$$

Note that $u = 0$ is the only critical point of $\bar{J}_\lambda$ in $H_0^1(\Omega)$ then, $u_\infty = w_0$. Thus we have

$$
\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
$$

This a contradiction. Then $b = 0$, this is

$$
\|u_n - u_\infty\| \to 0
$$

as $n \to +\infty$. This ends the proof of Lemma 4.8. $\square$

Lemma 4.9. *Let $0 < q < 1$, $1 < p < 2^* - 1$, $\lambda > 0$ and u_ε as defined in (2.6). Then, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{t\geq 0} \bar{J}_\lambda(tu_\varepsilon) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \tag{4.9}$$

Proof. Since

$$(a+b)^p \geq a^p + b^p + pa^{p-1}b$$

for every $a, b \geq 0$ and $p > 1$, then

$$\begin{aligned}
&\int\limits_\Omega\int\limits_\Omega \frac{|(u_\varepsilon+u_0)(x)|^{2_\mu^*}|(u_\varepsilon+u_0)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\geq \int\limits_\Omega\int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + \int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + 2_\mu^*\int\limits_\Omega\int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_0(y)||u_\varepsilon(y)|^{2_\mu^*-1}}{|x-y|^\mu}dxdy \\
&\quad + 2_\mu^*\int\limits_\Omega\int\limits_\Omega \frac{|u_0(y)|^{2_\mu^*}|u_0(x)|^{2_\mu^*-1}|u_\varepsilon(x)|}{|x-y|^\mu}dxdy + 2_\mu^*\int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*-1}|u_\varepsilon(y)|}{|x-y|^\mu}dxdy \\
&\geq \int\limits_\Omega\int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + \int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad + 2_\mu^* C_1\int\limits_{B_\delta}\int\limits_{B_\delta} \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*-1}}{|x-y|^\mu}dxdy + 2\cdot 2_\mu^*\int\limits_\Omega\int\limits_\Omega \frac{|u_0(y)|^{2_\mu^*}|u_0(x)|^{2_\mu^*-1}|u_\varepsilon(x)|}{|x-y|^\mu}dxdy,
\end{aligned}$$

thanks to $u_0 \geq C_1 > 0$ on B_δ. On the other hand, since for every $u \geq 0$

$$g(u) \geq \Big(\int\limits_\Omega \frac{|u_0+u|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_0+u|^{2_\mu^*-2}(u_0+u) - \Big(\int\limits_\Omega \frac{|u_0|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_0|^{2_\mu^*-2}u_0,$$

we have

$$\begin{aligned}
\bar{J}_\lambda(u_\varepsilon) &\leq \frac{1}{2}\int\limits_\Omega |\nabla u_\varepsilon|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|(u_\varepsilon+u_0)(x)|^{2_\mu^*}|(u_\varepsilon+u_0)(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\qquad + \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + \int\limits_\Omega\Big(\int\limits_\Omega \frac{|u_0|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|u_0|^{2_\mu^*-2}u_0u_\varepsilon dx \\
&\leq \frac{1}{2}\int\limits_\Omega |\nabla u_\varepsilon|^2 dx - \frac{1}{2\cdot 2_\mu^*}\int\limits_\Omega\int\limits_\Omega \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - C_2\int\limits_{B_\delta}\int\limits_{B_\delta} \frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*-1}}{|x-y|^\mu}dxdy.
\end{aligned}$$

By a direct computation, we know

$$
\begin{aligned}
&\int_{B_\delta}\int_{B_\delta}\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&\quad=\int_{B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_\mu}|U_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&\quad=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\int_{B_\delta}\int_{B_\delta}\frac{1}{(1+|\frac{x}{\varepsilon}|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|\frac{y}{\varepsilon}|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&\quad=\varepsilon^{\frac{2\mu-3N-2}{2}}[N(N-2)]^{\frac{3N-2\mu+2}{4}}\varepsilon^{2N-\mu}\int_{B_{\frac{\delta}{\varepsilon}}}\int_{B_{\frac{\delta}{\varepsilon}}}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&\quad\geq O(\varepsilon^{\frac{N-2}{2}})\int_{B_\delta}\int_{B_\delta}\frac{1}{(1+|x|^2)^{\frac{2N-\mu}{2}}|x-y|^\mu(1+|y|^2)^{\frac{N-\mu+2}{2}}}dxdy\\
&\quad=O(\varepsilon^{\frac{N-2}{2}})
\end{aligned}
\tag{4.10}
$$

provided $\varepsilon < 1$. Therefore, by (2.7), (2.8) and (4.10), we have

$$
\begin{aligned}
\bar{J}_\lambda(tu_\varepsilon) &\leq \frac{t^2}{2}\int_\Omega|\nabla u_\varepsilon|^2dx-\frac{t^{2\cdot 2^*_\mu}}{2\cdot 2^*_\mu}\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy-C_2t^{2\cdot 2^*_\mu-1}\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu-1}}{|x-y|^\mu}dxdy\\
&\leq\frac{t^2}{2}(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2}))-\frac{t^{2\cdot 2^*_\mu}}{2\cdot 2^*_\mu}(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))-t^{2\cdot 2^*_\mu-1}O(\varepsilon^{\frac{N-2}{2}})\\
&:=g(t).
\end{aligned}
$$

It is clear that $g(t)\to-\infty$ as $t\to+\infty$. It follows that there exists $t_\varepsilon>0$ such that $\sup_{t>0}g(t)$ is attained at t_ε. Differentiating $g(t)$ and equaling to zero, we obtain that

$$
t_\varepsilon(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2}))-t_\varepsilon^{2\cdot 2^*_\mu-1}(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))-t_\varepsilon^{2\cdot 2^*_\mu-2}O(\varepsilon^{\frac{N-2}{2}})=0
$$

and so

$$
t_\varepsilon<\Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2})}{C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}})}\Big)^{\frac{1}{22^*_\mu-2}}:=S_{H,L}(\varepsilon)
$$

and there exists $t_0>0$ such that for $\varepsilon>0$ small enough

$$
t_\varepsilon>t_0.
$$

Since the function

$$
t\mapsto\frac{t^2}{2}(C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O(\varepsilon^{N-2}))-\frac{t^{2\cdot 2^*_\mu}}{2\cdot 2^*_\mu}(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O(\varepsilon^{N-\frac{\mu}{2}}))
$$

is increasing on $[0,S_{H,L}(\varepsilon)]$, we have

$$
\begin{aligned}
\max_{t\geq 0} \bar{J}_\lambda(tu_\varepsilon) &\leq \frac{N+2-\mu}{4N-2\mu}\Big(\frac{C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{\Big(C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O(\varepsilon^{N-\frac{\mu}{2}})\Big)^{\frac{N-2}{2N-\mu}}}\Big)^{\frac{2N-\mu}{N+2-\mu}} - O(\varepsilon^{\frac{N-2}{2}}) \\
&\leq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + O(\varepsilon^{\min\{N-2, N-\frac{\mu}{2}\}}) - O(\varepsilon^{\frac{N-2}{2}}) \\
&< \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},
\end{aligned}
$$

thanks to $t_0 < t_\varepsilon < S_{H,L}(\varepsilon)$ and (4.10). $\square$

Proof of Theorem 1.6. For every $u_1 \in H_0^1(\Omega)\backslash\{0\}$, one easily checks that

$$
\bar{J}_\lambda(tu_1) < 0
$$

for $t > 0$ large enough. Combining with Lemma 4.7, we have $\bar{J}_\lambda$ has the Mountain Pass Geometry. Then there exists a (PS) sequence $\{u_n\}$ such that $\bar{J}_\lambda(u_n) \to c$ and $\bar{J}'_\lambda(u_n) \to 0$ in $H_0^1(\Omega)^{-1}$ at the minimax level

$$
c^* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \bar{J}_\lambda(\gamma(t)) > 0,
$$

where

$$
\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \bar{J}_\lambda(\gamma(1)) < 0\}.
$$

From Lemma 4.9, we know there exists $\varepsilon > 0$ small enough such that

$$
\sup_{t\geq 0} \bar{J}_\lambda(tu_\varepsilon) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
$$

Therefore, by the definition of c^*, we know $c^* < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. This estimate jointly with Lemma 4.8 and the Mountain Pass Theorem if the minimax energy level is positive, or the refinement of the Mountain Pass Theorem [31] if the minimax level is zero, gives the existence of a second solution to (1.11). $\square$

5. Infinitely many solutions in the case of a sublinear perturbation

In this Section, we will study the existence of infinitely many solutions for the critical Choquard equation with sublinear local perturbation, i.e. the equation (1.8) with the exponent of the perturbation satisfying $0 < q < 1$. By applying the Dual Fountain Theorem in [11], we are going to prove that the energy functional J_λ has infinitely many critical values.

We denote the sequence of eigenvalues of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary data by

$$
0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_j \leq \lambda_{j+1} \leq ...
$$

Moreover, $\{e_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ will be the sequence of eigenfunctions corresponding to λ_j which is also an orthogonal basis of $H_0^1(\Omega)$. Define $X_j := \mathbb{R}e_j$, we will use the following notations:

$$
Y_k := \oplus_{j=0}^k X_j, Z_k := \overline{\oplus_{j=k}^\infty X_j},
$$
$$
B_k := \{u \in Y_k : \|u\| \leq \rho_k\}, N_k := \{u \in Z_k : \|u\| \leq r_k\}
$$

where $\rho_k > r_k > 0$.

Definition 5.1. (See [48].) Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function I satisfies the $(PS)_c^*$ condition (with respect to (Y_n)) if any sequence $\{u_n\} \subset X$ such that

$$u_{n_j} \in Y_{n_j}, I(u_{n_j}) \to c, \text{ and } \quad I|'_{Y_{n_j}}(u_{n_j}) \to 0, \quad \text{as } \ n_j \to +\infty$$

contains a subsequence converging to a critical point of I.

Theorem 5.2 *(Dual Fountain Theorem). (See [11].) Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional. If, for every $k \geq k_0 \geq 2$, there exists $\rho_k > r_k > 0$ such that*

(B_1) $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} I(u) \geq 0$,
(B_2) $b_k := \max_{u \in Y_k, \|u\| = r_k} I(u) < 0$,
(B_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} I(u) \to 0$, $k \to \infty$,
(B_4) *I satisfies the $(PS)_c^*$ condition for every $[d_{k_0}, 0)$,*

then I has a sequence of negative critical values converging to 0.

We are ready to establish the following convergence criteria for the $(PS)_c$ sequences.

Lemma 5.3. *Let $0 < q < 1$. If there exists $M_0 > 0$ such that, for any $\lambda > 0$ and*

$$c < \frac{N + 2 - \mu}{4N - 2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} - M_0 \lambda^{\frac{2 \cdot 2_\mu^*}{2 \cdot 2_\mu^* - q - 1}}, \tag{5.1}$$

then J_λ satisfies the $(PS)_c^*$ condition.

Proof. Consider a sequence $\{u_{n_j}\} \subset H_0^1(\Omega)$ such that

$$u_{n_j} \in Y_{n_j}, J_\lambda(u_{n_j}) \to c, \text{ and } \quad J_\lambda|'_{Y_{n_j}}(u_{n_j}) \to 0, \quad \text{as } \ n_j \to +\infty,$$

where c satisfies (5.1). As in the proof of Lemma 2.2, we have u_0 is a weak solution of problem (1.8) with $0 < q < 1$, where $u_0 \in H_0^1(\Omega)$ is the weak limit of $\{u_{n_j}\}$. Taking $\varphi = u_0 \in H_0^1(\Omega)$ as a test function in (1.8), we have

$$\int_\Omega |\nabla u_0|^2 dx = \lambda \int_\Omega u_0^{q+1} dx + \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*} |u_0(y)|^{2_\mu^*}}{|x - y|^\mu} dx dy,$$

and so

$$J_\lambda(u_0) = \left(\frac{\lambda}{2} - \frac{\lambda}{q+1}\right) \int_\Omega u_0^{q+1} dx + \frac{N + 2 - \mu}{4N - 2\mu} \int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*} |u_0(y)|^{2_\mu^*}}{|x - y|^\mu} dx dy.$$

By Hölder inequality and $0 < q < 1$, we have

$$\left(\frac{\lambda}{2} - \frac{\lambda}{q+1}\right) \int_\Omega u_0^{q+1} dx \geq \left(\frac{\lambda}{2} - \frac{\lambda}{q+1}\right) |\Omega|^{\frac{2^* - q - 1}{2^*}} |u_0|_{2^*}^{q+1} \geq -C\lambda \Big(\int_\Omega \int_\Omega \frac{|u_0(x)|^{2_\mu^*} |u_0(y)|^{2_\mu^*}}{|x - y|^\mu} dx dy \Big)^{\frac{q+1}{2 \cdot 2_\mu^*}}$$

and so

$$J_\lambda(u_0) \geq \frac{N+2-\mu}{4N-2\mu}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - C\lambda\Big(\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\Big)^{\frac{q+1}{2\cdot 2_\mu^*}}. \tag{5.2}$$

We define $M_0 \geq 0$ by

$$\min_{t>0}\Big(\frac{N+2-\mu}{4N-2\mu}t^{2\cdot 2_\mu^*} - C\lambda t^{q+1}\Big) = -M_0\lambda^{\frac{2\cdot 2_\mu^*}{2\cdot 2_\mu^*-q-1}}. \tag{5.3}$$

Denote by $v_{n_j} := u_{n_j} - u_0$, then we know $v_{n_j} \rightharpoonup 0$ in $H_0^1(\Omega)$ and $v_{n_j} \to 0$ a.e. in Ω. From the proof of Lemma 2.4, we have

$$c \leftarrow J_\lambda(u_{n_j}) = J_\lambda(u_0) + \frac{1}{2}\int_\Omega |\nabla v_{n_j}|^2dx - \frac{1}{2\cdot 2_\mu^*}\int_\Omega\int_\Omega \frac{|v_{n_j}(x)|^{2_\mu^*}|v_{n_j}(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o_n(1) \tag{5.4}$$

and

$$o_n(1) = \int_\Omega |\nabla v_{n_j}|^2dx - \int_\Omega\int_\Omega \frac{|v_{n_j}(x)|^{2_\mu^*}|v_{n_j}(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy. \tag{5.5}$$

From (5.5), we know there exists a nonnegative constant b such that

$$\int_\Omega |\nabla v_{n_j}|^2dx \to b$$

and

$$\int_\Omega\int_\Omega \frac{|v_{n_j}(x)|^{2_\mu^*}|v_{n_j}(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \to b,$$

as $n_j \to +\infty$. By the definition of the best constant $S_{H,L}$ in (1.5), we have

$$S_{H,L}\Big(\int_\Omega\int_\Omega \frac{|v_{n_j}(x)|^{2_\mu^*}|v_{n_j}(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\Big)^{\frac{N-2}{2N-\mu}} \leq \int_\Omega |\nabla v_{n_j}|^2dx,$$

which yields $b \geq S_{H,L}b^{\frac{N-2}{2N-\mu}}$. Thus we have either $b = 0$ or $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$, then we obtain from (5.2), (5.3) and (5.4) that

$$\begin{aligned}
c &= \frac{N+2-\mu}{4N-2\mu}b + J_\lambda(u_0)\\
&\geq \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} + \frac{N+2-\mu}{4N-2\mu}\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy - C\lambda\Big(\int_\Omega\int_\Omega \frac{|u_0(x)|^{2_\mu^*}|u_0(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\Big)^{\frac{q+1}{2\cdot 2_\mu^*}}\\
&\geq \frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} - M_0\lambda^{\frac{2\cdot 2_\mu^*}{2\cdot 2_\mu^*-q-1}},
\end{aligned}$$

which contradicts with (5.1). Thus $b = 0$, and

$$\|u_{n_j} - u_0\| \to 0$$

as $n_j \to +\infty$. This ends the proof of Lemma 5.3. □

Proof of Theorem 1.7. Denote by

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} |u|_{q+1},$$

by Lemma 3.8 of [48], we know

$$\beta_k \to 0, k \to \infty. \tag{5.6}$$

From the Sobolev embedding theorem and the Hardy–Littlewood–Sobolev inequality, exists $R > 0$ small enough such that, for any $\|u\| \le R$,

$$\frac{1}{2 \cdot 2_\mu^*} \int_\Omega \int_\Omega \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \le \frac{1}{2 \cdot 2_\mu^*} C_1 \|u\|^{2(\frac{2N-\mu}{N-2})} \le \frac{1}{4}\|u\|^2.$$

Thus we get, for all $u \in Z_k \setminus \{0\}, \|u\| \le R$,

$$\begin{aligned} J_\lambda(u) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1} dx - \frac{1}{4}\|u\|^2 \\ &\ge \frac{1}{4}\|u\|^2 - \frac{\lambda}{q+1}\beta_k^{q+1}\|u\|^{q+1}. \end{aligned} \tag{5.7}$$

We choose $\rho_k := (\frac{4\lambda\beta_k^{q+1}}{q+1})^{\frac{1}{1-q}}$. From (5.6) and $0 < q < 1$, we have $\rho_k \to 0, k \to \infty$ and so there exists k_0 such that for every $\rho_k \le R$ when $k \ge k_0$. Thus, for every $k \ge k_0$, there exists $\rho_k > 0$ such that $a_k = \inf_{u \in Z_k, \|u\|=\rho_k} J_\lambda(u) \ge 0$. Condition (B_1) is thus proved.

Since Y_k is a finite dimensional subspace of $H_0^1(\Omega)$, we have all norms on Y_k are equivalent and so condition (B_2) is satisfied for every $r_k > 0$ small enough when $\lambda > 0$.

By (5.7), we know for all $u \in Z_k \setminus \{0\}, \|u\| \le \rho_k, k \ge k_0$,

$$J_\lambda(u) \ge -\frac{\lambda}{q+1}\beta_k^{q+1}\|u\|^{q+1} \ge -\frac{\lambda}{q+1}\beta_k^{q+1}\rho_k^{q+1}.$$

Then condition (B_3) is satisfied from $\beta_k \to 0$, $\rho_k \to 0$, $k \to \infty$.

We know that there exists $\lambda^* > 0$ such that, for every $0 < \lambda < \lambda^*$ and $c < 0$, J_λ satisfies the $(PS)_c^*$ condition from Lemma 5.3. By Theorem 5.2, we have there exists $\lambda^* > 0$ such that, for every $0 < \lambda < \lambda^*$, problem (1.8) has a sequence of solutions $\{u_n\} \subset H_0^1(\Omega)$ such that $J_\lambda(u_n) \to 0$, $n \to \infty$. □

To obtain the multiplicity results for the subcritical nonlocal case, we need to recall the famous Fountain Theorem in [10] which states as

Theorem 5.4. *Let X be a Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional. If, for every $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that*

(A_1) $a_k := \max_{u \in Y_k, \|u\|=\rho_k} I(u) \le 0$,
(A_2) $b_k := \inf_{u \in Z_k, \|u\|=r_k} I(u) \to \infty$, $k \to \infty$,
(A_3) *I satisfies the $(PS)_c$ condition for every $c > 0$,*

then I has an unbounded sequence of critical values converging to ∞.

Proof of Theorem 1.8. We denote the energy functional associated to equation (1.12) by

$$J_{\lambda,p}(u) = \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 dx - \frac{1}{2p}\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{\mu}} dxdy - \frac{\lambda}{2}\int\limits_{\Omega} |u|^{q+1} dx.$$

It is standard to prove the (PS) condition (A_3) holds everywhere and we only need to check the conditions (A_1) and (A_2) hold. In fact, similar to the proof Lemma 2.3 in [28], we have

$$\|\cdot\|_2 := \Big(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|\cdot|^p|\cdot|^p}{|x-y|^{\mu}} dxdy\Big)^{\frac{1}{2p}}$$

defines a norm on $L^{\frac{2Np}{2N-\mu}}(\Omega)$. Thus

$$J_{\lambda,p}(u) \le \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 dx - \frac{1}{2p}\|u\|_2^{2p}.$$

Since Y_k is a finite dimensional subspace of $H_0^1(\Omega)$, we have all norms on Y_k are equivalent and so relation (A_1) is satisfied for every $\rho_k > 0$ large enough when $\lambda > 0$.

We denote

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} |u|_{\frac{2Np}{2N-\mu}}$$

and have

$$\beta_k \to 0, k \to \infty \tag{5.8}$$

by Lemma 3.8 of [48]. By the Hardy–Littlewood–Sobolev inequality, we obtain

$$\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{\mu}} dxdy \le C_1|u|^{2p}_{\frac{2Np}{2N-\mu}}.$$

Thus we get, for all $u \in Z_k\setminus \{0\}$,

$$\begin{aligned} J_{\lambda}(u) &\ge \frac{1}{2}\int\limits_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{q+1}\int\limits_{\Omega} |u|^{q+1} dx - C_2|u|^{2p}_{\frac{2Np}{2N-\mu}} \\ &\ge \frac{1}{2}\|u\|^2 - C_2\beta_k^{2p}\|u\|^{2p} - \frac{\lambda}{q+1}\int\limits_{\Omega} |u|^{q+1} dx. \end{aligned} \tag{5.9}$$

We choose $r_k := (2pC_2\beta_k^{2p})^{\frac{1}{2-2p}}$ and get, for all $u \in Z_k\setminus \{0\}$ and $\|u\| = r_k$,

$$J_{\lambda}(u) \ge (\frac{1}{2} - \frac{1}{2p})(2pC_2\beta_k^{2p})^{\frac{1}{2-2p}} - \frac{\lambda}{q+1}\int\limits_{\Omega} |u|^{q+1} dx.$$

From (5.8), we have that relation (A_2) is proved. □

Acknowledgment

The authors would like to thank the anonymous referee for his/her useful comments and suggestions which help to improve the presentation of the paper greatly.

References

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 248 (2004) 423–443.
[2] C.O. Alves, D. Cassani, C. Tarsi, M. Yang, Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$, J. Differential Equations 261 (2016) 1933–1972.
[3] C.O. Alves, A.B. Nóbrega, M. Yang, Multi-bump solutions for Choquard equation with deepening potential well, Calc. Var. Partial Differential Equations 55 (2016) 48.
[4] C.O. Alves, M. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, J. Differential Equations 257 (2014) 4133–4164.
[5] C.O. Alves, M. Yang, Multiplicity and concentration behavior of solutions for a quasilinear Choquard equation via penalization method, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016) 23–58.
[6] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.
[7] A. Ambrosetti, A. Malchiodi, Nonlinear Analysis and Semilinear Elliptic Problems, Cambridge Stud. Adv. Math., vol. 104, Cambridge University Press, Cambridge, 2007.
[8] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations 252 (2012) 6133–6162.
[9] B. Barrios, E. Colorado, R. Servadei, F. Soria, A critical fractional equation with concave-convex power nonlinearities, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015) 875–900.
[10] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, Nonlinear Anal. 20 (1993) 1205–1216.
[11] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, Proc. Amer. Math. Soc. 123 (1995) 3555–3561.
[12] H. Brezis, T. Kato, Remarks on the Schrödinger operator with regular complex potentials, J. Math. Pures Appl. 58 (1979) 137–151.
[13] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[14] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.
[15] H. Brézis, L. Nirenberg, Louis H^1 versus C^1 local minimizers, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993) 465–472.
[16] B. Buffoni, L. Jeanjean, C.A. Stuart, Existence of a nontrivial solution to a strongly indefinite semilinear equation, Proc. Amer. Math. Soc. 119 (1993) 179–186.
[17] D. Cao, S. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, J. Differential Equations 193 (2003) 424–434.
[18] A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985) 463–470.
[19] D. Castorina, M. Sanchón, Regularity of stable solutions to semilinear elliptic equations on Riemannian models, Adv. Nonlinear Anal. 4 (2015) 295–309.
[20] G. Cerami, D. Fortunato, M. Struwe, Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 341–350.
[21] G. Cerami, S. Solimini, M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, J. Funct. Anal. 69 (1986) 289–306.
[22] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, Z. Angew. Math. Phys. 63 (2012) 233–248.
[23] M. Clapp, D. Salazar, Positive and sign changing solutions to a nonlinear Choquard equation, J. Math. Anal. Appl. 407 (2013) 1–15.
[24] S. Dipierro, M. Medina, I. Peral, E. Valdinoci, Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^n$, Manuscr. Math., http://dx.doi.org/10.1007/s00229-016-0878-3, online ISSN 1432-1785.
[25] S. Dipierro, E. Valdinoci, A density property for fractional weighted Sobolev spaces, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 (2015) 397–422.
[26] P. Drábek, Y. Huang, Multiplicity of positive solutions for some quasilinear elliptic equation in $\mathbb{R}^N$ with critical Sobolev exponent, J. Differential Equations 140 (1997) 106–132.
[27] M. Ferrara, G. Molica Bisci, Existence results for elliptic problems with Hardy potential, Bull. Sci. Math. 138 (2014) 846–859.
[28] F. Gao, M. Yang, On the Brezis–Nirenberg type critical problem for nonlinear Choquard equation, arXiv:1604.00826v4.
[29] J. García Azorero, I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, Trans. Amer. Math. Soc. 323 (1991) 877–895.
[30] M. Ghimenti, J. Van Schaftingen, Nodal solutions for the Choquard equation, J. Funct. Anal. 271 (2016) 107–135.
[31] N. Ghoussoub, D. Preiss, A general mountain pass principle for locating and classifying critical points, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989) 321–330.

[32] N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponent, Trans. Amer. Math. Soc. 352 (2000) 5703–5743.

[33] E. Janneli, The role played by space dimension in elliptic critical problems, J. Differential Equations 156 (1999) 407–426.

[34] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976/1977) 93–105.

[35] E. Lieb, M. Loss, Analysis, Grad. Stud. Math., AMS, Providence, Rhode Island, 2001.

[36] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980) 1063–1072.

[37] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455–467.

[38] V. Moroz, J. Van Schaftingen, Ground states of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.

[39] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Amer. Math. Soc. 367 (2015) 6557–6579.

[40] V. Moroz, J. Van Schaftingen, Semi-classical states for the Choquard equation, Calc. Var. Partial Differential Equations 52 (2015) 199–235.

[41] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent, Commun. Contemp. Math. 17 (5) (2015) 1550005.

[42] S. Pekar, Untersuchungüber die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.

[43] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativity Gravitation 28 (1996) 581–600.

[44] V. Radulescu, D. Smets, M. Willem, Hardy–Sobolev inequalities with remainder terms, Topol. Methods Nonlinear Anal. 20 (2002) 145–149.

[45] R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, J. Math. Anal. Appl. 389 (2012) 887–898.

[46] J. Tan, The Brezis–Nirenberg type problem involving the square root of the Laplacian, Calc. Var. Partial Differential Equations 42 (2011) 21–41.

[47] J. Wei, M. Winter, Strongly interacting bumps for the Schrödinger–Newton equations, J. Math. Phys. 50 (2009) 012905.

[48] M. Willem, Minimax Theorems, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.