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Wave propagation in an infectious disease model ☆

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A B S T R A C T

This paper is devoted to the study of the wave propagation in a reaction-convection infectious disease model with a spatio-temporal delay. Previous numerical studies have demonstrated the existence of traveling wave fronts for the system and obtained a critical value c^*, which is the minimal wave speed of the traveling waves. In the present paper, we provide a complete and rigorous proof. To overcome the difficulty due to the lack of monotonicity for the system, we construct a pair of upper and lower solutions, and then apply the Schauder fixed point theorem to establish the existence of a nonnegative solution for the wave equation on a bounded interval. Moreover, we use a limiting argument and in turn generate the solution on the unbounded interval $\mathbb{R}$. In particular, by constructing a suitable Lyapunov functional, we further show that the traveling wave solution converges to the epidemic equilibrium point as $t = +\infty$.

1. Introduction

In 2009, Li and Zou [8] used an SIR model to derive the following reaction-convection infectious disease model with a spatio-temporal delay

$$\begin{cases} S_t(t,x) = D_S S_{xx}(t,x) + \mu - dS(t,x) - rI(t,x)S(t,x), \\ I_t(t,x) = D_I I_{xx}(t,x) - \beta I(t,x) + \epsilon r \displaystyle\int_{-\infty}^{+\infty} f_\alpha(x-y) I(t-\tau,y) S(t-\tau,y)dy, \end{cases} \tag{1.1}$$

where S and I represent the densities of the susceptible and infective individuals at time t and position $x \in \mathbb{R}$, respectively, D_S and D_I are the corresponding diffusion rates. $\mu > 0$ is a constant recruiting rate, d is the natural death rate, $r > 0$ denotes the infection rate, ϵ measures the proportion of infected individuals that can survive the latent period, and the delay τ represents the latency length of the infective

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disease. $\beta = \sigma + \gamma + d$, σ, and γ are the disease-induced mortality rate and the recovery rate, respectively, $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$. Readers may refer to [8] for a precise interpretation of the biological implications for system (1.1).

Based on an abstract treatment, Li and Zou [8] addressed the existence, uniqueness and positivity of solutions of system (1.1). In addition, in view of the numeric simulations, they explored the existence of traveling wave fronts for system (1.1), and obtained a critical value c^*, which is the minimal wave speed c of the traveling wave fronts, i.e., system (1.1) admits traveling waves with wave speed $c \geq c^*$ but no such traveling waves with wave speed $c < c^*$. In this paper, we provide the first rigorous mathematical proof of the existence of traveling wave solutions for system (1.1).

Since system (1.1) does not satisfy the comparison principle and possess monotone properties, it is difficult to apply the general theory regarding the existence of traveling wave solutions for monotone system developed by Huang and Zou [7], Liang and Zhao [10], Ma [11], Wang, Li and Ruan [14], Wu and Zou [15], and the references cited therein. In this paper, motivated by previous works [1–4,9,13,16], we use an iteration process [1] (see also [2–4,9,13,16]) to construct a pair of upper and lower solutions $(\overline{S}, \overline{I})$ and $(\underline{S}, \underline{I})$. Using the constructed pair of upper and lower solutions, we build an appropriately invariant cone Γ_X of initial functions defined on a bounded interval, and we then apply the Schauder fixed point theorem for this cone to establish the existence of a nonnegative solution of (2.2) on the bounded interval, which serves as a candidate for the traveling wave solution for (2.2) on the unbounded interval $\mathbb{R}$. Furthermore, following the idea proposed in [17] (also see [2–4,13]), we employ a limiting argument to generate the solution on $\mathbb{R}$.

We should stress that according to the construction of the upper and lower solutions, the obtained traveling wave (S, I) is a nonnegative solution for (2.2) on $\mathbb{R}$ with $(S, I)(-\infty) = (1, 0)$. To demonstrate the existence of a traveling wave connecting the disease-free and endemic equilibrium, we need to prove that $(S, I)(+\infty) = (S^*, I^*)$. It is well known that the method of Lyapunov functionals [5] is a direct and effective approach for studying the global stability of delayed differential systems. However, it is challenging and difficult to construct a suitable Lyapouov functional for the differential systems with delay. In the present paper, inspired by the ideas proposed by [2–4,9], we successfully construct a Lyapunov functional and then show that $(S, I)(+\infty) = (S^*, I^*)$. We also comment that the construction of the Lyapunov functional is nontrivial and difficult because the corresponding wave profile system (2.2) is a second order functional differential system of mixed type (i.e., with both advanced and non-local delayed arguments).

The remainder of this paper is organized as follows. In Section 2, we give an important lemma and state the main results. In Section 3, we derive the preliminary results, including the construction of the upper and lower solutions, and the existence of the solution to (2.2) on a bounded interval. The proof of Theorem 2.1 is given in Section 4. Finally, we provide a brief discussion.

2. Main results

In this section, we state the main results. For simplicity, let

$$\widetilde{S}(t, x) = \frac{d}{\mu} S(t, x\sqrt{D_I}), \ \ \widetilde{I}(t, x) = \frac{d}{\mu} S(t, x\sqrt{D_I}),$$

and

$$\widetilde{d} = \frac{D_S}{D_I}, \ \ \widetilde{r} = \frac{r\mu}{d}, \ \ k = \frac{\epsilon r \mu}{d}.$$

By dropping the tilde for convenience, we then consider the following system

$$\begin{cases} S_t(t, x) = dS_{xx}(t, x) + \mu(1 - S(t, x)) - rI(t, x)S(t, x), \\ I_t(t, x) = I_{xx}(t, x) - \beta I(t, x) + k \displaystyle\int_{-\infty}^{+\infty} f_\alpha(x - y) I(t - \tau, y) S(t - \tau, y) dy. \end{cases} \tag{2.1}$$

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By a simple calculation, (2.1) always has a disease-free equilibrium $E^0(1,0)$. In addition, if $\mathcal{R}_0 := \frac{k}{\beta} > 1$, then (2.1) admits a unique endemic equilibrium $E^*(S^*, I^*)$ with

$$S^* = \frac{\beta}{k}, \quad I^* = \frac{\mu}{r}\left(\frac{k}{\beta} - 1\right).$$

Motivated by the work [8], we mainly consider the existence of traveling waves for system (2.1) that connect the disease-free equilibrium $E^0(1,0)$ and endemic equilibrium $E^*(S^*, I^*)$. A traveling wave solution of (2.1) is a special type of the solution of system (2.1) with the form $(S(t,x), I(t,x)) = (S(x+ct), I(x+ct))$, where $c > 0$ is the wave speed, and we let $x + ct$ by t, which satisfies the following wave equation

$$\begin{cases} dS''(t) - cS'(t) + \mu(1 - S(t)) - rS(t)I(t) = 0, \\ I''(t) - cI'(t) - \beta I(t) + k \int\limits_{-\infty}^{+\infty} f_\alpha(y)S(t-y-c\tau)I(t-y-c\tau)dy = 0, \end{cases} \tag{2.2}$$

on $\mathbb{R}$ with the boundary conditions

$$S(-\infty) = 1, \ I(-\infty) = 0, \ S(+\infty) = S^*, \ I(+\infty) = I^*. \tag{2.3}$$

Consequently, showing the existence of a traveling wave solution of system (2.1) that connects the equilibria E^0 and E^* is equivalent to showing the existence of a nonnegative solution of systems (2.2) and (2.3).

Before stating our main results, let us give an important lemma. Linearizing the second equation of (2.2) at $E^0(1,0)$ gives

$$I''(t) - cI'(t) - \beta I(t) + k \int\limits_{-\infty}^{+\infty} f_\alpha(y)I(t-y-c\tau)dy = 0.$$

If we let $J(t) = e^{\lambda t}$, then we obtain a characteristic equation

$$\Delta(\lambda, c) = \lambda^2 - c\lambda - \beta + ke^{\alpha\lambda^2 - \lambda c\tau} = 0.$$

It is easy to show the following lemma (see [12, Lemma 4.4]).

Lemma 2.1. *Assume that $\mathcal{R}_0 := \frac{k}{\beta} > 1$. Then there are two constants $c^* > 0$ and $\lambda^* > 0$ such that*

$$\Delta(\lambda^*, c^*) = 0 \quad \textit{and} \quad \frac{\partial}{\partial\lambda}\Delta(\lambda, c)|_{(\lambda^*, c^*)} = 0.$$

Furthermore,

(i) *if $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in [0, \infty)$;*
(ii) *if $c > c^*$, then the characteristic equation $\Delta(\lambda, c) = 0$ has two positive roots $\lambda_1(c)$ and $\lambda_2(c)$ with $0 < \lambda_1(c) < \lambda^* < \lambda_2(c)$ such that*

$$\Delta(\lambda, c) \begin{cases} > 0 & \textit{for all } \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), \infty), \\ < 0 & \textit{for all } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

In the sequel, we always assume that $\mathcal{R}_0 > 1$. In addition, we fix $c > c^*$ and always denote $\lambda_i(c)$ by λ_i, $i = 1, 2$. Then, the main results regarding the existence of traveling waves for system (2.1) can be stated by the following theorem.

Theorem 2.1. *For $c > c^*$, system (2.1) has a nonnegative traveling wave solution $(S(x + ct), I(x + ct))$ such that*

(i) $0 < S(t) < 1$ *and* $I(t) > 0$ *on* $\mathbb{R}$;
(ii) $\lim_{t\to-\infty}(S(t), I(t)) = (1, 0)$, $\lim_{t\to+\infty}(S(t), I(t)) = (S^*, I^*)$, *i.e., the traveling wave solution connects the disease-free equilibrium* $E^0(1, 0)$ *and the endemic equilibrium* $E^*(S^*, I^*)$;
(iii) $\lim_{t\to-\infty} I(t)e^{-\lambda_1 t} = 1$ *and* $\lim_{t\to-\infty}(S'(t), I'(t)) = \lim_{t\to+\infty}(S'(t), I'(t)) = (0, 0)$.

The proof of Theorem 2.1 is completed by Propositions 4.1 and 4.2, which are provided in Section 4.

3. System (2.2) on a bounded interval

In this section, we use an iteration process [1] (also see [2–4,9,13,16]) to construct a pair of upper and lower solutions of (2.1) with $c > c^*$. Using the constructed pair of upper and lower solutions, we build an appropriately invariant cone Γ_X of initial functions defined on a bounded interval $[-X, X]$ for any $X > 0$, and we apply the Schauder fixed point theorem to this cone Γ_X to establish the existence of a nonnegative solution of (2.2) on the interval $[-X, X]$, which serves as a candidate of the traveling wave solution for (2.2) on $\mathbb{R}$.

3.1. Upper and lower solutions

Since λ_1 is the root of $\Delta(\lambda, c) = 0$, it is natural to obtain the following lemma.

Lemma 3.1. *The function $\bar{I}(t) = e^{\lambda_1 t}$ satisfies the equation*

$$I''(t) - cI'(t) - \beta I(t) + k \int_{-\infty}^{+\infty} f_\alpha(y) I(t - y - c\tau)dy = 0.$$

We select $\sigma \in (0, \min\{c/(2d), \lambda_1\})$, then $d\sigma^2 - c\sigma - \mu < 0$ and $\lambda_1 - \sigma > 0$. Since $e^{(\lambda_1-\sigma)t} \to 0$ as $t \to -\infty$, then there is a $t_1 < 0$ such that

$$e^{(\lambda_1-\sigma)t} < \frac{1}{r}(c\sigma + \mu - d\sigma^2), \quad t < t_1,$$

which gives

$$(c\sigma + \mu - d\sigma^2)e^{\sigma t} > re^{\lambda_1 t}, \quad t < t_1. \tag{3.1}$$

Then, we choose $M := e^{-\sigma t_1} > 1$ since $\sigma > 0$ and $t_1 < 0$.

Lemma 3.2. *Let $\sigma \in (0, \min\{c/(2d), \lambda_1\})$ and $M > 1$. Then the function $\underline{S}(t) = \max\{0, 1 - Me^{\sigma t}\}$ satisfies the inequality*

$$dS''(t) - cS'(t) + \mu(1 - S(t)) - \gamma S(t)\bar{I}(t) \geq 0 \tag{3.2}$$

for all $t \neq t_1 := -\frac{1}{\sigma}\ln M$.

Proof. For $t > t_1$, the inequality (3.2) holds immediately since $\underline{S}(t) = 0$ in $[t_1, \infty)$. For $t < t_1$, $\underline{S}(t) = 1 - Me^{\sigma t}$ and $0 \leq \underline{S}(t) < 1$. Thus, by (3.1), a simple computation yields,

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$$\begin{aligned} d\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) &= M(c\sigma + \mu - d\sigma^2)e^{\sigma t} \\ &> re^{\lambda_1 t} > r(1 - Me^{\sigma t})\bar{I}(t) = r\underline{S}(t)\bar{I}(t). \end{aligned}$$

Then, (3.2) holds, which completes the proof. □

Let $0 < \eta < \min\{\sigma, \lambda_1 - \lambda_2\}$. Then, $\Delta(\lambda_1 + \eta, c) < 0$. Let

$$K > M \max\Big\{1, -\frac{ke^{\alpha(\lambda_1+\sigma)^2 - c\tau(\lambda_1+\sigma)}}{\Delta(\lambda_1 + \eta, c)}\Big\}, \tag{3.3}$$

and $t_2 := -\frac{1}{\eta}\ln K$. Then $t_2 < t_1 < 0$ since $K > M$.

Lemma 3.3. *Let $\eta \in (0, \min\{\sigma, \lambda_1 - \lambda_2\})$ and K satisfy (3.3). Then the function $\underline{I}(t) = \max\{0, e^{\lambda_1 t}(1 - Ke^{\eta t})\}$ satisfies the inequality*

$$I''(t) - cI'(t) - \beta I(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\underline{S}(t - y - c\tau)I(t - y - c\tau)dy \geq 0 \tag{3.4}$$

for all $t \neq t_2$.

Proof. Obviously, we can see that the inequality (3.4) holds since $\underline{I}(t) = 0$ in $[t_2, \infty)$. For $t < t_2$, $\underline{I}(t) = e^{\lambda_1 t}(1 - Ke^{\eta t}) > 0$ and $\underline{S}(t) = 1 - Me^{\sigma t} > 0$. Then, recalling that $\underline{I}(t) \geq e^{\lambda_1 t}(1 - Ke^{\eta t})$ for $t \in \mathbb{R}$, we have

$$\begin{aligned} &\underline{I}''(t) - cI'(t) - \beta\underline{I}(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\underline{I}(t - y - c\tau)dy \\ &\geq \Delta(\lambda_1, c)e^{\lambda_1 t} - K\Delta(\lambda_1 + \eta, c)e^{(\lambda_1+\eta)t} \\ &= -K\Delta(\lambda_1 + \eta, c)e^{(\lambda_1+\eta)t}. \end{aligned}$$

In view of the fact that $0 \geq \underline{S}(t) - 1 \geq -Me^{\sigma t}$ and $0 \leq \underline{I}(t) \leq e^{\lambda_1 t}$ for all $t \in \mathbb{R}$, then

$$\begin{aligned} &\int_{-\infty}^{+\infty} f_\alpha(y)\underline{I}(t - y - c\tau)(\underline{S}(t - y - c\tau) - 1)dy \\ &\geq -M\int_{-\infty}^{+\infty} f_\alpha(y)e^{\lambda_1(t-y-c\tau)}e^{\sigma(t-y-c\tau)}dy \\ &= -Me^{\alpha(\lambda_1+\sigma)^2 - c\tau(\lambda_1+\sigma)}e^{(\lambda_1+\sigma)t} \\ &= -Me^{\alpha(\lambda_1+\sigma)^2 - c\tau(\lambda_1+\sigma)}e^{(\sigma-\eta)t}e^{(\lambda_1+\eta)t} \\ &\geq -Me^{\alpha(\lambda_1+\sigma)^2 - c\tau(\lambda_1+\sigma)}e^{(\lambda_1+\eta)t}, \end{aligned}$$

since $e^{(\sigma-\eta)t} < 1$ for $t < t_2 < 0$. Therefore, for $t < t_2$,

$$\underline{I}''(t) - cI'(t) - \beta\underline{I}(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\underline{S}(t - y - c\tau)\underline{I}(t - y - c\tau)dy$$

$$
\begin{aligned}
&= \underline{I}''(t) + c\underline{I}'(t) - \beta\underline{I}(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\underline{I}(t-y-c\tau)dy \\
&\quad + k\int_{-\infty}^{+\infty} f_\alpha(y)\underline{I}(t-y-c\tau)(\underline{S}(t-y-c\tau)-1)dy \\
&\geq e^{(\lambda_1+\eta)t}\big(-K\Delta(\lambda_1+\eta,c) - kMe^{\alpha(\lambda_1+\sigma)^2-c\tau(\lambda_1+\sigma)}\big) \\
&> 0,
\end{aligned}
$$

and thus (3.4) holds. The proof is complete. □

3.2. *Verification of the Schauder fixed point theorem*

In this subsection, we consider system (2.2) in a bounded domain $[-X, X]$ for any $X > 0$. First, we give the following two lemmas.

Lemma 3.4 *(See [4, Lemma A.1]). Let A be a positive constant and let f and h be continuous functions on $[a, b]$. Suppose that $w \in C([a, b]) \cap C^2((a, b))$ satisfies the differential equation*

$$w''(t) - Aw'(t) + f(t)w(t) = h(t) \tag{3.5}$$

in (a, b) and $w(a) = w(b) = 0$. If

$$-C_1 \leq f \leq 0 \quad \text{and} \quad |h| \leq C_2 \ \text{ on } [a, b],$$

for some constants C_1, C_2, then there exists a positive constant C_3 that depends only on A, C_1, and the length of the interval $[a, b]$ such that

$$||w||_{C([a,b])} := \max\{|w(t)| : t \in [a, b]\} \leq C_2C_3.$$

Lemma 3.5 *(See [4, Lemma A.2]). Let A, f, and h be as given in Lemma 3.4. Suppose that $w \in C([a, b]) \cap C^2((a, b))$ satisfies (3.5) in (a, b). If $||w||_{C([a,b])} \leq C_0$ for some constant C_0, then there exists a positive constant C_4 that depends only on A, C_0, C_1, C_2, and the length of the interval $[a, b]$ such that*

$$||w'||_{C([a,b])} \leq C_4.$$

Let $X > \frac{1}{\eta}\ln K$, and

$$\Gamma_X = \left\{ (S_0, I_0) \in C([-X, X]) : \begin{array}{l} S_0(\pm X) = \underline{S}(\pm X),\ \ I_0(\pm X) = \underline{I}(\pm X), \\ \underline{S}(t) \leq S_0(t) \leq 1,\ \ \underline{I}(t) \leq I_0(t) \leq \overline{I}(t), \\ \text{for all } t \in [-X, X]. \end{array} \right\}.$$

Then, Γ_X is a nonempty closed and convex set in $C([-X, X])$ with the norm

$$||(\varphi_1, \varphi_2)||_X = ||\varphi_1||_{C([-X,X])} + ||\varphi_2||_{C([-X,X])}.$$

Now, for any $(S_0, I_0) \in \Gamma_X$, define

$$\phi(t) = \begin{cases} S_0(t), & |t| \leq X, \\ \underline{S}(t), & |t| > X, \end{cases} \qquad \psi(t) = \begin{cases} I_0(t), & |t| \leq X, \\ \underline{I}(t), & |t| > X. \end{cases}$$

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Consider the following two-point boundary value problem

$$\begin{cases} dS''(t) - cS'(t) + \mu(1 - S(t)) - rS(t)I_0(t) = 0, \\ I''(t) - cI'(t) - \beta I(t) + k \int\limits_{-\infty}^{+\infty} f_\alpha(y)\phi(t - y - c\tau)\psi(t - y - c\tau)dy = 0, \end{cases} \tag{3.6}$$

in $[-X, X]$ with

$$(S(-X), I(-X)) = (\underline{S}(-X), \underline{I}(-X)), \quad (S(X), I(X)) = (\underline{S}(X), \underline{I}(X)). \tag{3.7}$$

Note that system (3.6) comprises two non-coupled inhomogeneous linear equations since $S_0(t)$, $I_0(t)$, and thus $\phi(t)$ and $\varphi(t)$ are given, so the existence and uniqueness of the solutions of (3.6) for the boundary value problem (3.7) can be obtained easily by standard fundamental theory (e.g., see Theorem 3.1 on page 419 in [6]). Then, the problem (3.6)–(3.7) admits a unique solution $(S, I)(t)$ that satisfies $(S, I) \in C^2([-X, X])$. Consequently, we define an operator $F = (F_1, F_2) : \Gamma_X \to C([-X, X])$ by, for any $(S_0, I_0) \in \Gamma_X$,

$$F(S_0, I_0) = (S, I),$$

where the pair of functions $(S, I) \in C^2([-X, X])$ satisfies (3.6)–(3.7).

In the following, we verify that the mapping F satisfies the conditions of the Schauder fixed point theorem.

Lemma 3.6. *The mapping F maps Γ_X into Γ_X.*

Proof. First, we claim that $0 \le S(t) \le 1$ on $[-X, X]$. Obviously, 0 is a lower solution of the first equation of (3.6), which follows from the maximum principle that $S(t) \ge 0$ for all $t \in [-X, X]$. Similarly, we obtain $S(t) \le 1$ for all $t \in [-X, X]$. Note that $\underline{S}(t) = 1 - Me^{\sigma t}$ for all $t \in [-X, X_1]$ with $X_1 = -\frac{1}{\sigma}\ln M$ and satisfies (3.2). It follows that

$$\begin{aligned} 0 &\le d\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - r\underline{S}(t)\bar{I}(t) \\ &\le d\underline{S}''(t) - c\underline{S}'(t) + \mu(1 - \underline{S}(t)) - r\underline{S}(t)I_0(t) \end{aligned}$$

for all $t \in [-X, X_1]$. Thus, $\underline{S}(t)$ is a lower solution of the first equation of (3.6). Note that $S(-X) = \underline{S}(-X)$ and $S(X_1) \ge \underline{S}(X_1) = 0$. Then, the comparison principle implies that $\underline{S}(t) \le S(t)$ for all $t \in [-X, X_1]$. Hence, $\underline{S}(t) \le S(t) \le 1$ for all $t \in [-X, X]$.

Next, we consider $I(t)$. We can easily see that $I(t) \ge 0$ for all $t \in [-X, X]$, and

$$\underline{S}(t) \le \phi(t) \le 1, \quad \underline{I}(t) \le \psi(t) \le \bar{I}(t), \ \forall\, t \in \mathbb{R}.$$

Now, let us show that $\underline{I}(t) \le I(t)$ for all $t \in [-X, X]$. Indeed, since $\underline{I}(t) = e^{\lambda_1 t}(1 - Me^{\eta t})$ for all $t \in [-X, X_2]$ with $X_2 = -\frac{1}{\eta}\ln\frac{1}{M}$, then by Lemma 3.3, we have

$$\begin{aligned} 0 &\le \underline{I}''(t) - c\underline{I}'(t) - \beta\underline{I}'(t) + k\int\limits_{-\infty}^{\infty} f_\alpha(y)\underline{S}(t - y - c\tau)\underline{I}(t - y - c\tau)dy \\ &\le \underline{I}''(t) - c\underline{I}'(t) - \beta\underline{I}'(t) + k\int\limits_{-\infty}^{\infty} f_\alpha(y)\phi(t - y - c\tau)\psi(t - y - c\tau)dy. \end{aligned}$$

In view of the fact that $I(X_2) \geq \underline{I}(X_2) = 0$ and $I(-X) = \underline{I}(-X)$, it follows from the comparison principle that $\underline{I}(t) \leq I(t)$ for all $t \in [-X, X_2]$, which implies that $\underline{I}(t) \leq I(t)$ for all $t \in [-X, X]$.

Next, we need to verify that $I(t) \leq \overline{I}(t)$ for all $t \in [-X, X]$. In fact, noting that $\overline{I}(t) = e^{\lambda_1 t}$ for all $t \in \mathbb{R}$, by Lemma 3.1, we get

$$\begin{aligned} 0 &= \overline{I}''(t) - c\overline{I}'(t) - \beta\overline{I}(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\overline{S}(t-y-c\tau)\overline{I}(t-y-c\tau)dy \\ &\geq \overline{I}''(t) - c\overline{I}'(t) - \beta\overline{I}(t) + k\int_{-\infty}^{+\infty} f_\alpha(y)\phi(t-y-c\tau)\varphi(t-y-c\tau)dy. \end{aligned}$$

Since $I(-X) = \underline{I}(-X) \leq \overline{I}(-X)$ and $I(X) = \underline{I}(X) \leq \overline{I}(X)$, then by using the comparison principle again, we have $I(t) \leq \overline{I}(t)$ for all $t \in [-X, X]$. Thus, we deduce that $\underline{I}(t) \leq I(t) \leq \overline{I}(t)$ for all $t \in [-X, X]$. This completes the proof. □

Lemma 3.7. *The mapping $F : \Gamma_X \to \Gamma_X$ is a continuous mapping.*

Proof. For the given (S_0, I_0) and $(\widetilde{S}_0, \widetilde{I}_0)$ in Γ_X, let

$$(S, I) = F(S_0, I_0) \quad \text{and} \quad (\widetilde{S}, \widetilde{I}) = F(\widetilde{S}_0, \widetilde{I}_0).$$

First, we consider the function $w_1 := S - \widetilde{S}$. It is easy to see that $w_1(-X) = w_1(X) = 0$, and

$$w_1''(t) - \frac{c}{d}w_1'(t) + f(t)w_1(t) = h_1(t),\ t \in [-X, X],$$

where

$$f(t) = -\frac{1}{d}(\mu + rI_0(t)) \quad \text{and} \quad h_1(t) = \frac{r}{d}\widetilde{S}(t)(I_0(t) - \widetilde{I}_0(t)).$$

Note that

$$-\frac{1}{d}(\mu + re^{\lambda_1 X}) \leq f(t) \leq 0 \quad \text{and} \quad |h_1(t)| \leq \frac{\gamma}{d}||I_0 - \widetilde{I}_0||_{C([-X,X])}.$$

In addition, from the definition of λ_1, we know that the value of λ_1 depends only on c. Then, Lemma 3.4 asserts that there exists a positive constant C_1 (which depends only on d, c, μ, r, and X) such that

$$||w_1||_{C([-X,X])} \leq C_1||I_0 - \widetilde{I}_0||_{C([-X,X])},$$

which, together with definition of w_1, implies that

$$||S - \widetilde{S}||_{C([-X,X])} \leq C_1||I_0 - \bar{I}_0||_{C([-X,X])}. \tag{3.8}$$

Next, we consider the function $w_2 := I - \widetilde{I}$. It is easy to verify that $w_2(-X) = w_2(X) = 0$, and

$$w_2''(t) - cw_2'(t) - \beta w_2(t) = h_2(t),\ t \in [-X, X],$$

where

$$h_2(t) = -k\Big(\int_{-\infty}^{+\infty} f_\alpha(y)\phi(t-y-c\tau)\psi(t-y-c\tau)dy - \int_{-\infty}^{+\infty} f_\alpha(y)\widetilde{\phi}(t-y-c\tau)\widetilde{\psi}(t-y-c\tau)dy\Big).$$

Note that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} f_\alpha(y)\phi(t-y-c\tau)\psi(t-y-c\tau)dy \\
&\quad = \int_{-\infty}^{+\infty} f_\alpha(t-z-c\tau)\phi(z)\psi(z)dz \\
&\quad = \int_{-\infty}^{-X} f_\alpha(t-z-c\tau)\underline{S}(z)\underline{I}(z)dz + \int_{-X}^{X} f_\alpha(t-z-c\tau)S_0(z)I_0(z)dz,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} f_\alpha(y)\widetilde{\phi}(t-y-c\tau)\widetilde{\psi}(t-y-c\tau)dy \\
&\quad = \int_{-\infty}^{-X} f_\alpha(t-z-c\tau)\underline{S}(z)\underline{I}(z)dz + \int_{-X}^{X} f_\alpha(t-z-c\tau)\widetilde{S}_0(z)\widetilde{I}_0(z)dz.
\end{aligned}$$

Therefore,

$$h_2(t) = -k\int_{-X}^{X} f_\alpha(t-z-c\tau)(S_0(z)I_0(z) - \widetilde{S}_0(z)\widetilde{I}_0(z))dz.$$

Consequently,

$$|h_2| \le ke^{\lambda_1 X}\big(||S_0 - \widetilde{S}_0||_{C([-X,X])} + ||I_0 - \widetilde{I}_0||_{C([-X,X])}\big).$$

Then, Lemma 3.4 asserts that there is a positive constant C_2 that depends only on c, k, β, and X such that

$$||w_2||_{C([-X,X])} \le C_2(||S - \widetilde{S}||_{C([-X,X])} + ||I_0 - \widetilde{I}_0||_{C([-X,X])}),$$

and thus

$$||I - \widetilde{I}||_{C([-X,X])} \le C_2(||S_0 - \widetilde{S}_0||_{C([-X,X])} + ||I_0 - \widetilde{I}_0||_{C([-X,X])}). \tag{3.9}$$

Finally, we use (3.8) and (3.9) to deduce that

$$\begin{aligned}
||F(S_0, I_0) - F(\widetilde{S}_0, \widetilde{I}_0)||_X &= ||(S, I) - (\widetilde{S}, \widetilde{I})||_X \\
&= ||S - \widetilde{S}||_{C([-X,X])} + ||I - \widetilde{I}||_{C([-X,X])} \\
&\le C_3||(S_0, I_0) - (\widetilde{S}_0, \widetilde{I}_0)||_X,
\end{aligned}$$

where $C_3 = C_1 + C_2$. This shows that F is a continuous mapping, which completes the proof. □

Lemma 3.8. *The mapping $F : \Gamma_X \to \Gamma_X$ is precompact.*

Proof. For a given $\{(S_{0,n}, I_{0,n})\}_{n\in\mathbb{N}}$ in Γ_X, let $(S_n, I_n) = F(S_{0,n}, I_{0,n})$. Since $0 \le \underline{S}(t) \le S(t) \le \overline{S}(t) = 1$ and $0 \le \underline{I}(t) \le I(t) \le \overline{I}(t) \le e^{\lambda_1 t}$ on $[-X, X]$, then from the definition of the set Γ_X and Lemma 3.6, we can easily see that the sequences $\{S_{0,n}\}$, $\{I_{0,n}\}$, $\{S_n\}$, and $\{I_n\}$ are uniformly bounded in $[-X, X]$. Then, by Lemma 3.5, we know that $\{S'_n\}$ and $\{I'_n\}$ are also uniformly bounded in $[-X, X]$. Therefore, we use the Arzelá–Ascoli theorem and a nested subsequence argument to obtain a subsequence (still denoted) $\{(S_n, I_n)\}$, which tends toward a function $(S, I) \in \Gamma_X$ such that $(S_n, I_n) \to (S, I)$ uniformly in $[-X, X]$ as $n \to \infty$. Hence, the mapping $F : \Gamma_X \to \Gamma_X$ is precompact. This completes the proof. □

By Lemmas 3.6–3.8, according to the Schauder fixed point theorem, we can see that F has a fixed point, which is a non-negative solution for system (3.6)–(3.7). Hence, we have the following lemma.

Lemma 3.9. *If $c > c^*$, then there exist a pair of functions $(S, I) \in C([-X, X])$ that satisfy (2.2) on $[-X, X]$. Moreover,*

$$0 \le \underline{S}(t) \le S(t) \le 1 \quad \text{and} \quad 0 \le \underline{I}(t) \le I(t) \le \overline{I}(t) \tag{3.10}$$

on $[-X, X]$.

4. Proof of Theorem 2.1

In this section, we present a proof of Theorem 2.1. First, we give the following results.

Proposition 4.1. *If $c > c^*$, then system (2.2) admits a solution (S, I) that satisfies*

$$\lim_{t\to-\infty} (S(t), I(t)) = (1, 0), \quad \lim_{t\to-\infty} (S'(t), I'(t)) = (0, 0).$$

Moreover, $\lim_{t\to-\infty} I(t)e^{-\lambda_1 t} = 1$, and

$$0 < S(t) < 1, \quad I(t) > 0, \quad t \in \mathbb{R}. \tag{4.1}$$

Proof. Let $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence with $X_n > \frac{1}{\sigma} \ln K$ and $\lim_{n\to\infty} X_n = +\infty$, and let (S_n, I_n), $n \in \mathbb{N}$, be a solution of the following system with $X = X_n$,

$$\begin{cases} dS''(t) - cS'(t) + \mu(1 - S(t)) - rS(t)I(t) = 0, \\ I''(t) - cI'(t) - \beta I(t) + k \displaystyle\int_{-\infty}^{+\infty} f_\alpha(y)\phi(t - y - c\tau)\psi(t - y - c\tau)dy = 0 \end{cases} \tag{4.2}$$

in $[-X, X]$ with

$$\phi(t) = \begin{cases} S(t), & |t| \le X, \\ \underline{S}(t), & |t| > X, \end{cases} \qquad \psi(t) = \begin{cases} I(t), & t \le X, \\ \underline{I}(t), & |t| > X. \end{cases}$$

For any fixed $N \in \mathbb{N}$, since $\overline{I}(t)$ is bounded above in $[-X_N, X_N]$, from (3.10), it follows that the sequences

$$\{S_n\}_{n\ge N}, \quad \{I_n\}_{n\ge N}, \quad \{S_n I_n\}_{n\ge N}$$

are uniformly bounded in $[-X_N, X_N]$. Then, we can use Lemma 3.5 to infer that the sequences $\{S'_n\}_{n\geq N}$ and $\{I'_n\}_{n\geq N}$ are uniformly bounded in $[-X_N, X_N]$. Therefore, we can see that $\{S_n\}_{n\geq N}$ and $\{I_n\}_{n\geq N}$ are equicontinuous in $[-X_N, X_N]$.

Using (4.2), we can express $\{S''_n\}$ and $\{I''_n\}$ in terms of $\{S_n\}$, $\{I_n\}$, $\{S'_n\}$, and $\{S'_n\}$. Thus, the sequences $\{S''_n\}$ and $\{I''_n\}$ are also uniformly bounded in $[-X_N, X_N]$. Therefore, we get that the sequences $\{S'_n\}_{n\geq N}$ and $\{I'_n\}_{n\geq N}$ are also equicontinuous in $[-X_N, X_N]$. Let

$$M := \max_{t\in[-X,X]} \big\{S(t),\ I(t),\ |S'(t)|,\ |I'(t)|,\ |S''(t)|,\ |I''(t)|\big\}.$$

Next we claim that the sequences $\{S''_n\}_{n\geq N}$ and $\{I''_n\}_{n\geq N}$ are equicontinuous in $[-X_N, X_N]$. In fact, it follows from the first equation of (4.2), that for any $\xi, \eta \in [-X_N, X_N]$,

$$\begin{aligned} d|S''(\xi) - S''(\eta)| \leq\ & c|S'(\xi) - S'(\eta)| + \mu|S(\xi) - S(\eta)| \\ & + r\big(S(\xi)|I(\xi) - I(\eta)| + I(\eta)|S(\xi) - S(\eta)|\big) \\ \leq\ & M(c + \mu + 2rM)|\xi - \eta|. \end{aligned} \tag{4.3}$$

In addition, from the second equation of (4.2), it follows that for any $\xi, \eta \in [-X_N, X_N]$,

$$\begin{aligned} |I''(\xi) - I''(\eta)| \leq\ & c|I'(\xi) - I'(\eta)| + \beta|I(\xi) - I(\eta)| \\ & + k\Big|\int_{-\infty}^{+\infty} f_\alpha(y)\phi(\xi - y - c\tau)\psi(\xi - y - c\tau)dy \\ & - \int_{-\infty}^{+\infty} f_\alpha(y)\phi(\eta - y - c\tau)\psi(\eta - y - c\tau)dy\Big|. \end{aligned} \tag{4.4}$$

We note that

$$\begin{aligned} & \int_{-\infty}^{+\infty} f_\alpha(y)\phi(\xi - y - c\tau)\psi(\xi - y - c\tau)dy \\ & \quad - \int_{-\infty}^{+\infty} f_\alpha(y)\phi(\eta - y - c\tau)\psi(\eta - y - c\tau)dy \\ & = \int_{-\infty}^{+\infty} [f_\alpha(\xi - z - c\tau) - f_\alpha(\eta - z - c\tau)]\phi(z)\psi(z)dz \\ & = \int_{-\infty}^{-X} [f_\alpha(\xi - z - c\tau) - f_\alpha(\eta - z - c\tau)]\underline{S}(z)\underline{I}(z)dz \\ & \quad + \int_{-X}^{X} [f_\alpha(\xi - z - c\tau) - f_\alpha(\eta - z - c\tau)]S(z)I(z)dz. \end{aligned} \tag{4.5}$$

A direct computations leads to

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$$\Big|\int_{-\infty}^{-X}[f_\alpha(\xi-z-c\tau)-f_\alpha(\eta-z-c\tau)]\underline{S}(z)\underline{I}(z)dz\Big|$$
$$\leq L_f|\xi-\eta|\int_{-\infty}^{-X}e^{\lambda_1 z}dz\leq\frac{L_f}{\lambda_1}|\xi-\eta|, \tag{4.6}$$

where $L_f>0$ is the Lipschitz constant of the function $f_\alpha(y)$ and

$$\begin{aligned}
&\Big|\int_{-X}^{X}[f_\alpha(\xi-z-c\tau)-f_\alpha(\eta-z-c\tau)]S(z)I(z)dz\Big|\\
&=\Big|\int_{\xi-X-c\tau}^{\xi+X-c\tau}f_\alpha(y)S(\xi-y-c\tau)I(\xi-y-c\tau)dy\\
&\quad-\int_{\eta-X-c\tau}^{\eta+X-c\tau}f_\alpha(y)S(\eta-y-c\tau)I(\eta-y-c\tau)dy\Big|\\
&=\Big|\int_{\xi-X-c\tau}^{\eta-X-c\tau}f_\alpha(y)S(\xi-y-c\tau)I(\xi-y-c\tau)dy\\
&\quad-\int_{\eta+X-c\tau}^{\xi+X-c\tau}f_\alpha(y)S(\eta-y-c\tau)I(\eta-y-c\tau)dy\\
&\quad+\int_{\eta-X-c\tau}^{\eta+X-c\tau}f_\alpha(y)\big[S(\xi-y-c\tau)I(\xi-y-c\tau)-S(\eta-y-c\tau)I(\eta-y-c\tau)\big]dy\Big|\\
&\leq 2M^2\big(1+\frac{1}{\sqrt{2\pi\alpha}}\big)|\xi-\eta|.
\end{aligned} \tag{4.7}$$

Therefore, by (4.4)–(4.7),

$$|I''(\xi)-I''(\eta)|\leq M_0|\xi-\eta|, \tag{4.8}$$

where

$$M_0:=cM+\beta M+2M^2\big(1+\frac{1}{\sqrt{2\pi\alpha}}\big)+\frac{kL_f}{\lambda_1}.$$

Hence, by (4.3) and (4.8), we know that the claim is true. Consequently, we get that the sequences

$$\{S_n\}_{n\geq N},\quad \{I_n\}_{n\geq N},\quad \{S_n'\}_{n\geq N},\quad \{I_n'\}_{n\geq N},\quad \{S_n''\}_{n\geq N},\quad \{I_n''\}_{n\geq N},$$

are uniformly bounded and equicontinuous in $[-X_N,X_N]$. Using the Arzelá–Ascoli theorem and the standard diagonal method, we can obtain a subsequence (S_{n_k},I_{n_k}) of (S_n,I_n) such that

$$(S_{n_k},I_{n_k})\to(S,I),\quad (S_{n_k}',I_{n_k}')\to(S',I')\quad (S_{n_k}'',I_{n_k}'')\to(S'',I''),\quad k\to+\infty,$$

uniformly in any compact interval of $\mathbb{R}$ for some functions S and I in $C^2(\mathbb{R})$. By applying Lebesgue's dominated convergence theorem, we get

$$\int_{-\infty}^{+\infty} f_\alpha(y)S_{n_k}(t-y-c\tau)I_{n_k}(t-y-c\tau)dy \to \int_{-\infty}^{+\infty} f_\alpha(y)S(t-y-c\tau)I(t-y-c\tau)dy$$

for any $t \in \mathbb{R}$ and $k \to \infty$. Then, it is easy to see that (S, I) is a nonnegative solution of system (2.2) on $\mathbb{R}$ and

$$\underline{S}(t) \le S(t) \le 1, \quad \underline{I}(t) \le I(t) \le \overline{I}(t),\ t \in \mathbb{R}.$$

Together with the definitions of $\underline{S}(t)$ and $\underline{I}(t)$, it follows that $(S, I)(-\infty) = (1, 0)$ and $\lim_{t\to-\infty} I(t)e^{-\lambda t} = 1$.

To show that $\lim_{t\to-\infty}(S'(t), I'(t)) = (0, 0)$, we use the first equation of (2.2) to deduce that

$$S'(t) = e^{\frac{c}{d}(t-\eta)}S'(\eta) + \frac{1}{d}e^{\frac{c}{d}t}\int_t^\eta e^{-\frac{c}{d}\theta}\big[\mu(1-S(\theta)) - rS(\theta)I(\theta)\big]d\theta.$$

By fixing η and letting $t \to -\infty$ in the equation above, we immediately obtain

$$\begin{aligned}\limsup_{t\to-\infty}|S'(t)| &\le \frac{1}{d}\max_{\theta\le\eta}\big|\mu(1-S(\theta)) - rS(\theta)I(\theta)\big|\limsup_{t\to-\infty} e^{\frac{c}{d}t}\int_t^\eta e^{-\frac{c}{d}\theta}d\theta \\ &\le \frac{1}{c}\max_{\theta\le\eta}\big|\mu(1-S(\theta)) - rS(\theta)I(\theta)\big|\end{aligned}$$

for any fixed $\eta \in \mathbb{R}$. Recalling the fact that $\lim_{t\to-\infty}(\mu(1-S(t)-rS(t)I(t)) = 0$, we get $\lim_{t\to-\infty} S'(t) = 0$. Similar to the arguments employed above and using the second equation of (2.2), we also find that $\lim_{t\to-\infty} I'(t) = 0$.

Next, we claim that $0 < S(t) < 1$ and $I(t) > 0$ for all $t \in \mathbb{R}$. Indeed, by using the strong maximum principle applied to the first equation in (2.2), we get that $S(t) < 1$ for all $t \in \mathbb{R}$. Since $rS(t)I(t) \ge 0$, we know that $S(t) = 0$ cannot be a minimal value for $S(t)$ such that $S(t) > 0$ for all $t \in \mathbb{R}$. Applying the strong maximum principle applied to the second equation of (2.2), we get $I(t) > 0$ for all $t \in \mathbb{R}$. This completes the proof. □

In order to show the convergence of the traveling waves toward the endemic equilibrium $E^*(S^*, I^*)$ at $t = +\infty$, we construct a suitable Lyapunov functional. First, we need to derive the boundedness property of the solution $(S(t), I(t))$ of system (2.2).

Using the ideas in [4], we give the following lemma.

Lemma 4.1. *Let (S, I) be a positive bounded solution of (2.2) such that (4.1) holds. Then,*

$$-L_1S(t) < S'(t) < L_2S(t), \quad -L_3I(t) < I'(t) < L_4I(t), \quad t \ge 0,$$

where L_i $(i = 1, 2, 3, 4)$ are sufficiently large positive constants such that $-L_1S(0) < S'(0)$ and $cL \ge 2rI^0$, $S'(0) < L_2S(0)$ and $dL_2^2 - cL_2 - \mu - rI^0 > 0$, $-L_3I(0) < I'(0)$ and $cL \ge 2\beta$, $I'(0) < L_4I(0)$ and $L_4^2 - cL_4 - \beta > 0$, $I^0 = \sup_{t\ge0} I(t)$.

Proof. We only show that the inequality $-L_1S(t) < S'(t) < L_2S(t)$ holds for all $t \ge 0$ because the proof of the second inequality is similar. First, we show that $-L_1S(t) < S'(t)$ for all $t \ge 0$. Let

$$\Phi_1(t) := S'(t) + L_1 S(t),\ t \geq 0.$$

Clearly, it is sufficient to show that $\Phi_1(t) > 0$ for all $t \geq 0$. If not, note that $\Phi_1(0) > 0$, then we may assume that there exists a $\bar{t}_1 > 0$ such that $\Phi_1(\bar{t}_1) > 0$ and $\Phi_1'(\bar{t}_1) \leq 0$. Then, we have the following two cases, i.e.,

$$\Phi_1(t) \leq 0,\ t \geq \bar{t}_1, \tag{4.9}$$

or there is a $\bar{t}_2 \geq \bar{t}_1$ such that

$$\Phi_1(\bar{t}_2) = 0, \quad \Phi_1'(\bar{t}_2) \leq 0. \tag{4.10}$$

For the first case, (4.9) and the fact that $cL_1 \geq 2rI^0$ imply that $cS'(t) \leq -2rI^0 S(t)$ for $t \geq \bar{t}_1$. Since $0 < S(t) < 1$ and $0 < I(t) \leq I^0$ (by (4.1)) for all $t \geq 0$, then from the first equation of (2.2), we get

$$dS''(t) = cS'(t) - \mu(1 - S(t)) + rS(t)I(t) \leq -rI^0 S(t) < 0,\ t \geq \bar{t}_1,$$

and thus $S'(t)$ is decreasing in $[\bar{t}_1, \infty)$. Hence, $S'(t) \leq S'(\bar{t}_1) \leq -L_1 S(\bar{t}_1) < 0$ for all $t > \bar{t}_1$, which contradicts the boundedness of S.

For the second case, (4.10) implies that

$$S'(\bar{t}_2) = -L_1 S(\bar{t}_2) < 0 \ \text{ and } \ S''(\bar{t}_2) \geq -L_1 S'(\bar{t}_2) > 0. \tag{4.11}$$

Again from the first equation of (2.2), by (4.11), it follows that

$$\begin{aligned} 0 &= dS''(\bar{t}_2) - cS'(\bar{t}_2) + \mu(1 - S(\bar{t}_2)) - rS(\bar{t}_2)I(\bar{t}_2) \\ &\geq cL_1 S(\bar{t}_2) - rI^0 S(\bar{t}_2) \\ &\geq rI^0 S(\bar{t}_2) \\ &> 0, \end{aligned}$$

which is a contradiction.

Next, we show that $S'(t) < L_2 S(t)$ for all $t \geq 0$. Let

$$\Phi_2(t) := S'(t) - L_2 S(t),\ t \geq 0.$$

Obviously, it is sufficient to show that $\Phi_2(t) < 0$ for all $t \geq 0$. By contrast, if we suppose that $\Phi_2(0) < 0$, then we may assume that there is a $\bar{t}_3 > 0$ such that $\Phi_2(\bar{t}_3) = 0$ and $\Phi_2'(\bar{t}_3) \geq 0$. Then,

$$\Phi_2(\bar{t}_3) = L_2(\bar{t}_3) \ \text{ and } \ S''(\bar{t}_3) \geq L_2 S'(\bar{t}_3) = L_2^2 S(\bar{t}_3).$$

Thus, from the first equation of (2.2), we can deduce that

$$\begin{aligned} 0 &= dS''(\bar{t}_3) - cS'(\bar{t}_3) + \mu(1 - S(\bar{t}_3)) - rS(\bar{t}_3)I(\bar{t}_3) \\ &\geq (dL_2^2 - cL_2 - \mu - rI^0)S(\bar{t}_3) > 0, \end{aligned}$$

which is again a contradiction. The proof is complete. □

Second, we construct the Lyapunov functional. To simplify the notation, let

$$g(x) = x - 1 - \ln x, \quad x > 0,$$

and

$$\alpha_1(y) = \int_y^{+\infty} f_\alpha(x)dx \quad \text{for} \quad y \geq 0, \qquad \alpha_2(y) = \int_{-\infty}^{y} f_\alpha(x)dx \quad \text{for} \quad y \leq 0.$$

It is easy to see that

$$g(x) = x - 1 - \ln x \geq 0, \ x > 0,$$

and

$$g(x) = x - 1 - \ln x = 0 \text{ if and only if } x = 1.$$

In addition,

$$\alpha_1(0) = \alpha_2(0) = \frac{1}{2}, \quad \frac{d\alpha_1(y)}{dy} = -f_\alpha(y), \quad \frac{d\alpha_2(y)}{dy} = f_\alpha(y). \tag{4.12}$$

Define

$$\mathcal{D} = \big\{(S, I) : 0 < S < 1, \ 0 < I \leq I^0, \ -L_1 S < S' < L_2 S, \ -L_3 I < I' < L_4 I\big\}.$$

For each $(S, I) \in \mathcal{D}$, we consider the Lyapunov functional $V(S, I) : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$V(S, I)(t) = V_1(S, I)(t) + V_2(S, I)(t) + V_3(S, I)(t),$$

where

$$\begin{aligned}
V_1(S, I)(t) &= \frac{k}{r} S^* \Big(cg\big(\frac{S}{S^*}\big) - dS'\big(\frac{1}{S^*} - \frac{1}{S}\big)\Big) + I^* \Big(cg\big(\frac{I}{I^*}\big) - I'\big(\frac{1}{I^*} - \frac{1}{I}\big)\Big), \\
V_2(S, I)(t) &= \beta I^* \int_{t-c\tau}^{t} g\big(\frac{S(\theta)I(\theta)}{S^* I^*}\big) d\theta, \\
V_3(S, I)(t) &= \beta I^* \Big(\int_0^{+\infty} \alpha_1(y) g\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^* I^*}\Big) dy \\
&\quad - \int_{-\infty}^{0} \alpha_2(y) g\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^* I^*}\Big) dy \Big).
\end{aligned}$$

Then, we obtain the following result.

Proposition 4.2. *Let (S, I) be a positive solution of system (2.2) with $(S, I) \in \mathcal{D}$. Then, there exists a constant $m \in \mathbb{R}$ such that $V(S, I)(t) > m$ for all $t \in \mathbb{R}^+$ and the map $t \to V(S, I)(t)$ is non-increasing. Moreover,*

$$\lim_{t \to +\infty} (S(t), I(t)) = (S^*, I^*), \quad \lim_{t \to +\infty} (S'(t), I'(t)) = (0, 0).$$

Proof. By Lemma 4.1, we can see that there is a constant m such that $V(S, I)(t) > m$ for all $t \in \mathbb{R}^+$. Next, we show that $t \to V(S, I)(t)$ is non-increasing. Indeed, it is easy to see that

$$\frac{dV_1}{dt}=-\frac{k}{r}\big(1-\frac{S^*}{S}\big)(dS''-cS')-S^*\frac{kd}{r}\big(\frac{S'}{S}\big)^2-\big(1-\frac{I^*}{I}\big)(I''-cI')-I^*\big(\frac{I'}{I}\big)^2.$$

From the fact that $\mu(1-S^*)=rI^*S^*$, it follows that

$$\begin{aligned}\frac{dV_1}{dt}&=-\frac{k}{r}\frac{S-S^*}{S}(\mu(S-S^*)+rSI-rS^*I^*)-S^*\frac{kd}{r}\big(\frac{S'}{S}\big)^2\\&\quad-\frac{I-I^*}{I}\Big(\beta I-k\int_{-\infty}^{+\infty}f_\alpha(y)S(t-y-c\tau)I(t-y-c\tau)dy\Big)-I^*\big(\frac{I'}{I}\big)^2\\&=-\frac{k\mu}{r}\frac{(S-S^*)^2}{S}-S^*\frac{kd}{r}\big(\frac{S'}{S}\big)^2-I^*\big(\frac{I'}{I}\big)^2-kSI+\beta I^*+\beta I^*\big(1-\frac{S^*}{S}\big)\\&\quad+k\int_{-\infty}^{+\infty}f_\alpha(y)S(t-y-c\tau)I(t-y-c\tau)dy\\&\quad-\beta I^*\int_{-\infty}^{+\infty}f_\alpha(y)\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I(t)}dy,\end{aligned}\tag{4.13}$$

and

$$\frac{dV_2}{dt}=\beta I^*\Big(g\big(\frac{S(t)I(t)}{S^*I^*}\big)-g\big(\frac{S(t-c\tau)I(t-c\tau)}{S^*I^*}\big)\Big).\tag{4.14}$$

Using integration by parts and by (4.12), we then obtain

$$\begin{aligned}\frac{dV_3}{dt}&=\beta I^*\Big(-\int_0^{+\infty}\alpha_1(y)\frac{d}{dy}g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)dy\\&\quad+\int_{-\infty}^{0}\alpha_2(y)\frac{d}{dy}g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)dy\Big)\\&=\beta I^*\Big(-\alpha_1(y)g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)\Big|_{y=0}^{+\infty}\\&\quad+\int_0^{+\infty}\frac{d\alpha_1(y)}{dy}g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)dy\\&\quad+\alpha_2(y)g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)\Big|_{y=-\infty}^{0}\\&\quad-\int_{-\infty}^{0}\frac{d\alpha_2(y)}{dy}g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)dy\Big).\end{aligned}$$

Note that $(S,I)\in\mathcal{D}$, we get

$$\lim_{y\to+\infty}\alpha_1(y)g\big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\big)=0,$$

and

$$\lim_{y\to-\infty} \alpha_2(y) g\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\Big) = 0.$$

Recalling that $kS^* = \beta$, we get

$$\begin{aligned}
\frac{dV_3}{dt} &= \beta I^* \Big(g\Big(\frac{S(t-c\tau)I(t-c\tau)}{S^*I^*}\Big) - \int_{-\infty}^{+\infty} f_\alpha(y) g\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\Big) dy \Big) \\
&= \beta I^* g\Big(\frac{S(t-c\tau)I(t-c\tau)}{S^*I^*}\Big) - k \int_{-\infty}^{+\infty} f_\alpha(y) S(t-y-c\tau)I(t-y-c\tau) dy \\
&\quad + \beta I^* + \beta I^* \int_{-\infty}^{+\infty} f_\alpha(y) \ln\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I^*}\Big) dy \\
&= \beta I^* g\Big(\frac{S(t-c\tau)I(t-c\tau)}{S^*I^*}\Big) - k \int_{-\infty}^{+\infty} f_\alpha(y) S(t-y-c\tau)I(t-y-c\tau) dy \\
&\quad + \beta I^* \int_{-\infty}^{+\infty} f_\alpha(y) \Big(1 + \ln\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I(t)}\Big)\Big) dy + \beta I^* \ln \frac{I(t)}{I^*}.
\end{aligned} \tag{4.15}$$

Combining (4.13)–(4.15) yields

$$\begin{aligned}
\frac{dV}{dt} = &-\frac{kd}{r}\frac{(S-S^*)^2}{S} - S^*\frac{kd}{r}\Big(\frac{S'}{S}\Big)^2 - I^*\Big(\frac{I'}{I}\Big)^2 - \beta I^* g\Big(\frac{S^*}{S}\Big) \\
&- \beta I^* \int_{-\infty}^{+\infty} f_\alpha(y) g\Big(\frac{S(t-y-c\tau)I(t-y-c\tau)}{S^*I(t)}\Big) dy.
\end{aligned}$$

Thus, $V(t)$ is non-increasing and

$$\frac{dV(t)}{dt} = 0 \ \text{ if and only if } \ S(t) \equiv S^*,\ I(t) \equiv I^*,\ S'(t) = 0,\ I'(t) = 0 \ \text{ for } t \in \mathbb{R}. \tag{4.16}$$

Choose an increasing constant sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = +\infty$ and denote

$$\{S_n(t)\}_{n=1}^\infty = \{S(t+t_n)\}_{n=1}^\infty, \quad \{I_n(t)\}_{n=1}^\infty = \{I(t+t_n)\}_{n=1}^\infty.$$

Note that $\{S_n(t)\}_{n=1}^\infty$ and $\{I_n(t)\}_{n=1}^\infty$ are bounded, then there exists a subsequence of the functions (still denoted by S_n and I_n) such that $\lim_{n\to\infty} S_n(t) := \tilde{S}(t)$ and $\lim_{n\to\infty} I_n(t) := \tilde{I}(t)$. Since $V(t)$ is decreasing and bounded below, then for any $n \in \mathbb{N}$,

$$m \le V(S_n, I_n)(t) = V(S, I)(t+t_n) \le V(S, I)(t),$$

which implies that there is $\tilde{V}_0 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} V(S_n, I_n)(t) = \lim_{t+t_n\to\infty} V(S_n, I_n)(t+t_n) = \tilde{V}_0,$$

for all $t \in \mathbb{R}$. Thus, Lebesgue's dominated convergence theorem follows that

$$\lim_{n\to\infty} V(S_n, I_n)(t) = V(\tilde{S}, \tilde{I})(t).$$

Thus, $V(\tilde{S}, \tilde{I})(t) = \tilde{V}_0$. Consequently,

$$\frac{dV(\tilde{S}, \tilde{I})(t)}{dt} = 0. \tag{4.17}$$

Thus, by combining (4.16) and (4.17), we can see that $\tilde{S}(t) = S^*$ and $\tilde{I}(t) = I^*$, i.e., $\lim_{t\to+\infty}(S(t), I(t)) = (S^*, I^*)$. Hence, by (4.16), $\lim_{t\to+\infty}(S'(t), I'(t)) = (0, 0)$. This completes the proof. □

Using Propositions 4.1 and 4.2, we can complete the proof of Theorem 2.1.

5. Discussion

In this paper, we considered the reaction-convection infectious disease model (1.1) introduced by [8] for describing the dynamics of diseases with a fixed latent period in a spatially continuous environment. Li and Zou [8] obtained the necessary condition ($\mu\epsilon r - \beta d > 0$) for system (1.1) that determines a critical value c^*, which serves at least as a lower bound for the wave speed in the sense that when $c < c^*$, no traveling wave front connects the disease-free equilibrium $E^0(\mu/d, 0)$ and the endemic equilibrium $E^*(\beta/r\epsilon, \mu\epsilon/\beta - d/r)$ with speed c. Furthermore, numeric simulations demonstrated the existence of traveling wave solutions of system (1.1) for $c > c^*$, where the value c^* is the minimal wave speed as well as the spatial spread speed of the disease. In the present paper, we provided a rigorous mathematical proof by establishing the existence of positive traveling wave solutions for (2.1) that connect the two equilibria $E^0(1, 0)$ and $E^*(S^*, I^*)$ for $c > c^*$. Biologically, a traveling wave solution that connects the two equilibria $E^0(1, 0)$ and $E^*(S^*, I^*)$ accounts for the transition from the disease-free equilibrium $E^0(1, 0)$ to the endemic equilibrium $E^*(S^*, I^*)$ over time, where the wave speed c may explain the spatial spread speed of the disease, which may measure how fast the disease invades geographically. Hence, the study of the traveling waves is very important for disease models with spatial heterogeneity.

As mentioned in [8], all roots of the characteristic equation $\Delta(\lambda, c) = 0$ are complex for $0 < c < c^*$, so the solution of (2.2) oscillates about $(1, 0)$, and thus the I component of the solution of (2.2) will take negative values. Hence, for $0 < c < c^*$, (2.2) cannot have a positive solution that satisfies (2.3). Therefore, Li and Zou [8] have showed the non-existence of the traveling waves for $c < c^*$ (see [8] on page 2060 for details). As a final remark, it should be noted that in the present paper, we did not derive the existence or non-existence of the traveling waves for the minimal wave speed c^* (in the case where $\mathcal{R}_0 > 1$). A detailed analysis of this problem will be challenging and we leave this as a further project.

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