



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



# Abelian integrals in unfoldings of codimension 3 singularities with nilpotent linear parts

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## ARTICLE INFO

### Article history:

Received 1 August 2016

Available online xxxx

Submitted by W. Sarlet

### Keywords:

Codimension 3 nilpotent singularities

Abelian integral

Limit cycle

Quadratic reversible system

Chebyshev criterion

Semi-algebraic system

## ABSTRACT

This paper is concerned with the upper bound of the number of limit cycles in unfolding of codimension 3 planar singularities with nilpotent linear parts. After making a central rescaling, the problem reduces to a perturbation problem of a one-parameter family of quadratic reversible systems. As the parameter  $a \in (-1, 1) \setminus \{0\}$  is rational, except the case  $a = -\frac{2}{3}$ , based on the Chebyshev criterion for Abelian integrals and a rationalizing transformation, the problem could be solved theoretically. To illustrate our approaches, two particular cases (corresponding to nilpotent codimension 3 saddle and elliptic case respectively) are proved where the upper bound of the number of limit cycles is two.

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## 1. Introduction and main results

The multi-parameters family of planar vector fields is the main theme of bifurcation theory and of some dynamical models arising in applications. In the pioneer work [2,30], Bogdanov and Takens independently studied the 2-parameters versal family of nilpotent cusp of codimension 2 with nilpotent linear part (it is usually called the Bogdanov–Takens bifurcation). After that many researchers have devoted to generalize Bogdanov and Takens' results to  $n$ -parameters generic family of nilpotent singularity of codimension  $n$  with  $n \geq 3$ , such as the 3-parameters family of nilpotent codimension three cusp in [7] (it is usually called Bogdanov–Takens bifurcations of codimension 3), the 3-parameters family of nilpotent codimension three saddle, focus and elliptic in [8,34] (it is usually called degenerate Bogdanov–Takens bifurcations of codimension 3), the 4-parameters family of nilpotent codimension four cusp and saddle in [6,25], the 5-parameters

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family of nilpotent codimension five cusp in [35], the  $n$ -parameters family of nilpotent codimension  $n$  cusp in [14,21], etc. On the other hand, bifurcations with high codimension in dynamical models arising at different disciplines, such as biological and epidemiological models, have important consequences for disease control or species management [3,9,15–18,24,28,29,36,37].

According to the classification proposed by Dumortier et al. [8], excluding the cusp case [7], the codimension three singularities with nilpotent linear part are as follows

$$y \frac{\partial}{\partial x} + (\alpha x^3 + xy \pm x^2 y) \frac{\partial}{\partial y}, \quad \alpha \neq 0, -\frac{1}{8}, \quad (1.1)$$

it is called saddle case as  $\alpha > 0$ , elliptic case as  $-\frac{1}{8} < \alpha < 0$ , focus case as  $\alpha < -\frac{1}{8}$ . Based on the hypothesis that the unfoldings of system (1.1) for all the three cases have at most two limit cycles, the bifurcation diagrams and topological phase portraits are given in [8]. Concerned with the saddle case, Xiao [33] completely studied the remaining problems in [8], such as the bifurcations from the singular trajectories, etc., and therefore confirmed the conjecture about the existence of universal unfolding.

By applying the central rescaling and transformations of the coordinates, Żoladek [38] reduced the three-parameters deformation of the vector field (1.1) for  $\alpha > 0$  and  $-\frac{1}{8} < \alpha < 0$  to the small polynomial perturbations of a one-parameter family of quadratic reversible systems

$$\begin{cases} \frac{dx}{dt} = -1 + y + ax^2 + \gamma_1 x + \gamma_2 x^3, \\ \frac{dy}{dt} = -2xy - \beta_0 - \beta_1 x^2 - \beta_2 x^4, \end{cases} \quad (1.2)$$

where  $(\gamma_1, \gamma_2, \beta_0, \beta_1, \beta_2) \triangleq \varepsilon$  is a small perturbation parameter vector, and parameter  $a \in (-1, 1) \setminus \{0\}$  is the solution of the following equation

$$a = 2\alpha(a-1)^2, \quad (1.3)$$

in which there exists a symmetry  $a \rightarrow \frac{1}{a}$ .

If the perturbation parameters vanish identically, i.e.,  $\varepsilon = 0$  in system (1.2), the unperturbed system  $(1.2)_{\varepsilon=0}$  is a quadratic reversible system, which is symmetric to the  $y$  axis and  $y = 0$  is an invariant line, moreover there always exists one simple center  $C(0, 1)$  for  $a \in (-1, 1) \setminus \{0\}$ .

System  $(1.2)_{\varepsilon=0}$  is integrable and non-Hamiltonian, the first integral is as follows

$$H(x, y) = \operatorname{sgn}(a)a(a+1)y^a(x^2 + \frac{y}{a+1} - \frac{1}{a}) \quad (1.4)$$

with integrating factor

$$M(x, y) = \operatorname{sgn}(a)a(a+1)y^{a-1},$$

here  $\operatorname{sgn}(a)$  represents the sign function of  $a$ , i.e.,  $\operatorname{sgn}(a) = 1$  as  $a$  is positive,  $\operatorname{sgn}(a) = -1$  as  $a$  is negative.

The Hamiltonian  $H(x, y)$  has a local minimum at  $C$  and there exists a punctured neighborhood  $\mathcal{P}$  of  $C$  foliated by ovals  $\gamma_h \subset \{H(x, y) = h\}$ , which is called period annulus. Then  $\gamma_h$  can be parameterized by the energy levels  $h \in (-1, 0)$  for  $0 < a < 1$  ( $h \in (1, +\infty)$  for  $-1 < a < 0$ ). In what follows, we shall denote the projection of period annulus  $\mathcal{P}$  on the  $y$ -axis by  $(y_l, y_r)$ . Moreover,

- Saddle case as  $a \in (0, 1)$ : the outer boundary of the period annulus is a heteroclinic loop connecting to hyperbolic saddles  $A(-\frac{1}{\sqrt{a}}, 0)$  and  $B(\frac{1}{\sqrt{a}}, 0)$ . The projection of  $\mathcal{P}$  onto  $y$  axis is  $(y_l, y_r) = (0, \frac{a+1}{a})$ . The phase portrait of system  $(1.2)_{\varepsilon=0}$  is shown as Fig. 1.1.

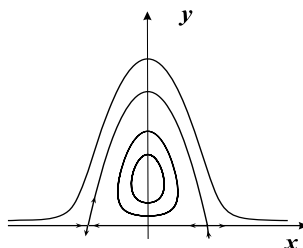


Fig. 1.1. The phase portrait of the unperturbed system (1.2) <sub>$\varepsilon=0$</sub>  as  $a \in (0, 1)$ .

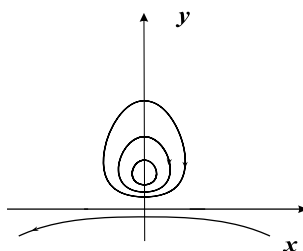


Fig. 1.2. The phase portrait of the unperturbed system (1.2) <sub>$\varepsilon=0$</sub>  as  $a \in (-1, 0)$ .

- Elliptic case as  $a \in (-1, 0)$ : the period annulus is the whole upper half plane. The projection of  $\mathcal{P}$  onto  $y$  axis is  $(y_l, y_r) = (0, +\infty)$ . The phase portrait of system (1.2) <sub>$\varepsilon=0$</sub>  in this case is shown as Fig. 1.2.

One of important problems for system (1.2) is to study the cyclicity of the period annulus  $\mathcal{P}$  in the unperturbed system (1.2) <sub>$\varepsilon=0$</sub> , which determines the maximum number of limit cycles for the generic unfolding of codimension 3 singularities with nilpotent linear part, and the key point is to estimate the number of zeros of the following Abelian integrals (see Proposition 2.1 of [38])

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h), \quad I_i(h) = \oint_{\gamma_h} y^{a-2} x^{2i+1} dy, \quad (1.5)$$

for  $i = 0, 1, 2$  and where the oval  $\gamma_h$  is the compact component of level curves determined by equation  $H(x, y) = h$ .

As mentioned in [22], H. Żoładek [38] studied this problem and claimed the existence of at most two limit cycles, while it seems that his main argument is not sufficient to draw the conclusion, concerning the order of contact and the relative position between a trajectory of a vector field and the hyperplane in the estimation of the zeros of Abelian integrals. F. Girard [12] tried to rectify the drawback, while the divergence of one Abelian integral invalidates his proof.

In the literature, there are many techniques and arguments to tackle the problem of bounding the number of zeros of Abelian integrals, lots of them are very long and non-trivial, see the part II of [5]. As a generalization of the criterion in [26], J. Villadelprat et al. [10,13,27] proposed an algebraic criterion to check whether the tuple of linear independent generating functions associated to the Abelian integral is Chebyshev or Chebyshev with accuracy  $k$ , namely, the number of isolated zeros of Abelian integral is no more than the dimension of linear span space minus one or plus  $k - 1$  counted with multiplicities. In some sense the approach reduces the estimation on the number of isolated zeros of (generalized) Abelian integral to solve a class of associated semi-algebraic (possibly transcendental) systems [31]. For semi-algebraic system, we mean that the system consists of polynomial equations, polynomial inequalities and inequations, see [32] for more details. Recall that an Abelian integral is the integral of a rational 1-form along an algebraic oval. If the

unperturbed system is integrable but non-Hamiltonian, one has to multiply a integrating factor  $M(x, y)$ , thus the first integral and perturbation functions are no longer polynomials, which may cause some new difficulties, the approach used for the Hamiltonian case does not work elsewhere.

By applying Theorem B of [13] and Theorem A of [27] to study the Chebyshev property of Abelian integral (1.5), we find that for irrational  $a$ , the involved functions in continuous Wronskian in general are non-algebraic, which invalidates the verification procedure based on Theorem B of [13] in polynomial setting [13,27,31], when  $a \in (-1, 1) \setminus \{0\}$  is rational, except the case  $a = -\frac{2}{3}$ , by performing a transformation of variables, we show that the deduced system could be reduced to the semi-algebraic system, thus in theory the problem could be done with aid of computers, however for system (1.2) with parameter  $a \in (-1, 1) \setminus \{0\}$ , it leads to a dead end even with the help of computer algebraic system due to the computational complexity. In this paper, two representative cases are stated, that is  $a = \frac{1}{2}$  (saddle case) and  $a = -\frac{1}{2}$  (elliptic case) concerned with system (1.2).

Concretely we have the following results.

**Theorem 1.1.** *Consider the perturbed quadratic reversible system (1.2), for parameter vector  $\varepsilon$  sufficiently small,*

- (a) *if  $a = \frac{1}{2}$ , Abelian integral (1.5) has at most two isolated zeros at any compact interval of  $(-1, 0)$ ,*
- (b) *if  $a = -\frac{1}{2}$ , Abelian integral (1.5) has at most two isolated zeros at any compact interval of  $(1, +\infty)$ .*

The paper is organized as follows, the backgrounds of the problem and main results are stated at Section 1. Some preliminary definitions and known results about the Chebyshev criterion for Abelian integral are given in Section 2. In Section 3, we give the proof of Theorem 1.1 by using Theorem B in [13] and Theorem A in [27], where an important step is the application of a rationalized transformation, under which the deduced non-algebraic system is changed to the semi-algebraic system. The drawbacks of our approaches are commented in Section 4.

## 2. Preliminary definitions and results

For the sake of convenience, we list some sufficient criteria and related definitions, the reader is referred to Theorem B of [13] and Theorem A of [27] for details about the following definition and lemma.

**Definition 2.1.** Given a family of analytic functions  $f_0, f_1, f_2, \dots, f_{n-1}$  defined on the open interval  $J \in \mathbb{R}$ .

- (i) The sets of functions  $\{f_0, f_1, f_2, \dots, f_{n-1}\}$  is a *Chebyshev system* on  $J$ , if any nontrivial linear combination

$$k_0 f_0(x) + k_1 f_1(x) + \dots + k_{n-1} f_{n-1}(x)$$

has at most  $n - 1$  isolated zeros on  $J$ .

- (ii) An ordered set of functions  $(f_0, f_1, f_2, \dots, f_{n-1})$  is called an *extended complete Chebyshev system* (ECT-system for short) on  $J$ , if for all  $i = 1, 2, \dots, n$ , any nontrivial linear combination

$$k_0 f_0(x) + k_1 f_1(x) + \dots + k_{i-1} f_{i-1}(x)$$

has at most  $i - 1$  isolated zeros on  $J$  counting the multiplicities of isolated zeros.

(iii) The continuous Wronskian of the tuple of functions  $(f_0, f_1, \dots, f_{k-1})$  at  $x \in \mathbb{R}$  is

$$W[f_0, f_1, \dots, f_{k-1}](x) = \det(f_i^{(j)}(x))_{0 \leq i, j \leq k-1} \\ = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1}(x) \\ f'_0(x) & f'_1(x) & \cdots & f'_{k-1}(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where  $f'(x)$  is the first order derivative of  $f(x)$  and  $f^{(i)}(x)$  is the  $i$ th order derivative of  $f(x)$ .

The following lemma constitutes the fundamental criteria to prove [Theorem 1.1](#).

**Lemma 2.1.** *Given the Abelian integral  $I_i(h) = \int_{\gamma_h} f_i(x)y^{2s-1}dx$ , suppose that  $f_i(x)$  is analytic on the interval  $(x_l, x_r)$ ,  $i = 0, 1, \dots, n-1$ , and the oval  $\gamma_h$  determined by Hamiltonian  $\{H(x, y) = A(x) + B(x)y^2 = h, h \in (0, h_s)\}$  surrounds the center  $(0, 0)$ ,  $H(0, 0) = 0$  and  $h_s \in (0, +\infty)$  is the energy corresponding to the outer boundary of the periodic annulus. Let  $\sigma$  be the involution associated to  $A(x)$  satisfying  $A(x) = A(\sigma(x))$ ,  $x \in (0, x_r)$ . For  $x \in (0, x_r)$ , denote the criterion functions as follows*

$$l_i(x) \triangleq \frac{f_i(x)}{A'(x)B(x)^{\frac{2s-1}{2}}} - \frac{f_i(\sigma(x))}{A'(\sigma(x))B(\sigma(x))^{\frac{2s-1}{2}}},$$

*then  $(I_0(h), I_1(h), \dots, I_{n-1}(h))$  is an ECT-system on  $(0, h_s)$  if  $s > n-2$  and none of the continuous Wronskian  $W[l_0, l_1, \dots, l_i](x)$  vanishes for all  $x \in (0, x_r)$  and  $i = 0, \dots, n-1$ .*

To fulfill the hypothesis  $s > n-2$  in [Lemma 2.1](#), we often use the following approach, which follows from Lemma 4.1 in [\[13\]](#) or [\[10\]](#), to get another set of Abelian integrals that we desired.

**Lemma 2.2.** *Let  $\gamma_h$  be an oval contained in the level curve  $\{(x, y) | A(x) + B(x)y^2 = h\}$ , the following statements hold*

(a) *Assume that  $F/A'$  is analytic at  $x = 0$ , then for any positive integer  $s$ ,*

$$\int_{\gamma_h} F(x)y^{2s-1}dx = \int_{\gamma_h} \mathcal{L}_F(x)y^{2s+1}dx$$

$$\text{where } \mathcal{L}_F(x) = \frac{2}{2s+1} \left( \frac{BF}{A'} \right)'(x) - \left( \frac{B'F}{A'} \right)(x).$$

(b) *Assume that  $G$  is analytic at  $x = 0$ ,*

$$\int_{\gamma_h} G(x)y^{2s+1}dx = \int_{\gamma_h} \mathcal{D}_G(x)y^{2s-1}dx,$$

$$\text{where } \mathcal{D}_G(x) = \frac{2s+1}{2} A'(x)B(x)^{\frac{2s-1}{2}} \int_0^x G(t)B(t)^{-\frac{2s+1}{2}} dt.$$

When  $F/A'$  in [Lemma 2.2](#) is not analytic at  $x = 0$ , we often firstly apply the property that Hamiltonian  $H(x, y) = A(x) + B(x)y^2$  is constant along the integral curves.

### 3. Proof of the main result

In this section, our goal is to bound the number of isolated zeros of Abelian integral (1.5) associated to system (1.2). By applying Lemma 2.1, as  $a = \frac{1}{2}$  (saddle case) or  $a = -\frac{1}{2}$  (elliptic case), we prove that the ordered set of functions  $(I_0, I_1, I_2)$  forms an extended complete Chebyshev system (ECT-system for short), namely it has at most two zeros for Abelian integral (1.5) on  $h \in (-1, 0)$  (or  $(1, +\infty)$ ) counted with multiplicities.

**Case (a).** As  $a \in (0, 1)$ , the level curves of the system  $(1.2)_{\varepsilon=0}$  in finite plane are shown as Fig. 1.1.

Firstly we shift the center  $(0, 1)$  of the system (1.2) to the origin by mapping  $u = y - 1$ , thus the Hamiltonian function is changed to

$$H(x, u) = A(u) + B(u)x^2$$

with

$$A(u) = (au - 1)(u + 1)^a + 1, \quad B(u) = a(a + 1)(u + 1)^a.$$

The period annulus is foliated by the ovals  $\gamma_h = \{(x, u) \in R^2 \mid H(x, u) = h, 0 < h < 1\}$ , and the projection of the period annulus on  $u$  axis is  $(-1, \frac{1}{a})$ . Here  $B(0) = a(a + 1) > 0$  and  $u A'(u) = a(a + 1)u^2(u + 1)^{a-1} > 0$ , therefore there exists an analytic involution  $\sigma$ , denote  $z = \sigma(u)$ , satisfying

$$A(u) - A(z) = (u + 1)^a(au - 1) - (z + 1)^a(az - 1) = 0.$$

We need to show that the tuple of functions  $\{I_i(h) = \oint_{\gamma_h} (u+1)^{a-2} x^{2i+1} du\}$  ( $i = 0, 1, 2$ ) is an ECT-system. Here  $n = 3$ , to use Lemma 2.1, we should lift the power of  $x$  in the integrand to 3. Furthermore, since function  $\frac{(u+1)^{a-2}}{A'(u)}$  is not analytic at  $u = 0$ , we can not apply Lemma 2.2 directly. To bypass this obstacle, however, by applying the property that Hamiltonian  $H(x, u) = A(u) + B(u)x^2$  is constant along the integral curves, we have

$$I_0(h) = \frac{1}{h} \oint_{\gamma_h} (u + 1)^{a-2} [A(u) + B(u)x^2] x du.$$

Setting  $s = 1$  and using Property (a) in Lemma 2.2, we have

$$I_0(h) = \frac{1}{h} \oint_{\gamma_h} f_0(u) x^3 du,$$

where

$$\begin{aligned} f_0(u) &= (1 + u)^{a-2} B(u) + \frac{2}{3} \left( \frac{B(u + 1)^{a-2} A}{A'} \right)' - \frac{B'(u + 1)^{a-2} A}{A'} \\ &= \frac{(u + 1)^{a-2} [(4a^2 u^2 + au^2 - au + 4u + 2)(u + 1)^a - au - 4u - 2]}{3u^2}. \end{aligned}$$

Similarly we have

$$I_1(h) = \frac{1}{h} \oint_{\gamma_h} (u + 1)^{a-2} [A(u) + B(u)x^2] x^3 du.$$

Setting  $s = 2$  and using (b) in Lemma 2.2, we have

$$I_1(h) = \frac{1}{h} \oint_{\gamma_h} f_1(u) x^3 du,$$

where

$$\begin{aligned} f_1(u) &= (1+u)^{a-2} A(u) + \frac{5}{2} A'(u) B(u)^{\frac{3}{2}} \int_0^u (s+1)^{a-2} B(s)^{-\frac{3}{2}} ds \\ &= (u+1)^{a-2} [(au-1)(u+1)^a + 1] + \frac{5a(a+1)u(u+1)^{2a-2} [(1+u)^{\frac{2+a}{2}} - 1]}{a+2}. \end{aligned}$$

Setting  $s = 2$  and using (b) in Lemma 2.2, we have

$$I_2(h) = \oint_{\gamma_h} (1+u)^{a-2} x^5 du = \oint_{\gamma_h} g(u) x^3 du,$$

where

$$g(u) = \frac{5}{2} A'(u) B(u)^{\frac{3}{2}} \int_0^u (s+1)^{a-2} B(s)^{-\frac{5}{2}} ds = \frac{5u(u+1)^{a-2} [(u+1)^{\frac{2+3a}{2}} - 1]}{3a+2}.$$

Similarly to the integral  $I_1(h)$ , we have

$$I_2(h) = \frac{1}{h} \oint_{\gamma_h} g(u) [A(u) + B(u)x^2] x^3 du = \frac{1}{h} \oint_{\gamma_h} f_2(u) x^3 du,$$

where

$$\begin{aligned} f_2(u) &= g(u) A(u) + \frac{5}{2} A'(u) B(u)^{\frac{3}{2}} \int_0^u g(s) B(s)^{-\frac{3}{2}} ds \\ &= -\frac{5u}{2(2+a)(2+3a)} [(1+u)^{a-2} (4+2a) - (1+u)^{2a-2} (24+22a+20u+26au+8a^2u) \\ &\quad + (1+u)^{\frac{5}{2}a-1} (6+13a) + (1+u)^{\frac{7}{2}a-2} (14+7a+14u-7au-7a^2u-14au^2-7a^2u^2)], \end{aligned}$$

which is a sum of functions having the forms  $g(a)(1+u)^{\frac{ka}{2}+l}$ ,  $k, l \in \mathbb{Z}$ .

If  $a$  is not rational, then  $f_0(u)$ ,  $f_1(u)$  and  $f_2(u)$  are all not algebraic and difficult to deal with, thus we have to consider only rational  $a$ . We set  $J_i = \oint_{\gamma_h} f_i(u) x^3 du$ ,  $i = 0, 1, 2$ . It is clear the tuple of functions  $(I_0(h), I_1(h), I_2(h))$  is an ECT-system on  $(-1, 0)$  if and only if so it is  $\{J_0(h), J_1(h), J_2(h)\}$  on  $(0, 1)$ . We now can use Lemma 2.1, for  $s = 2$  and the condition  $s > n - 2$  holds. Thus let

$$l_i(u) \triangleq \frac{f_i(u)}{A'(u) B(u)^{\frac{3}{2}}} - \frac{f_i(\sigma(u))}{A'(\sigma(u)) B(\sigma(u))^{\frac{3}{2}}}, \quad i = 0, 1, 2,$$

where  $z = \sigma(u)$  is implicitly determined by equation  $A(u) = A(z)$  as  $(u, z)$  is in the parameter domain

$$D_p = \{(u, z) \mid -1 < u < 0, 0 < z < \frac{1}{a}\}.$$

With the aid of the computer algebra system Maple, we compute the continuous Wronskians  $W[l_0](u)$ ,  $W[l_0, l_1](u)$ ,  $W[l_0, l_1, l_2](u)$  by following the standard procedure [10, 13, 27]. Our goal is to show that all of the

three Wronskians  $W[l_0](u)$ ,  $W[l_0, l_1](u)$  and  $W[l_0, l_1, l_2](u)$  do not vanish on  $(-1, 0)$ . While the Wronskians are not rational functions and the involved functions contain the fractional power of  $(u + 1)$ , it is followed the functions  $f_i(u)$ ,  $i = 0, 1, 2$  and  $B(u)^{\frac{2s-1}{2}}$  contain  $(1 + u)^{\frac{a}{2}}$ . Thus suppose that  $a = \frac{p}{q}$ , where  $p > 0$  and  $q$  are relatively prime integers, then we implement the following rationalizing transformation

$$u = m^{\frac{2q}{gcd(2,p)}} - 1, \quad z = n^{\frac{2q}{gcd(2,p)}} - 1, \quad (3.1)$$

where  $gcd(2, p)$  represents the greatest common divisor of 2 and  $p$ . For  $a \in (0, 1)$ , the parameter domain is changed to

$$\tilde{D}_p = \{(m, n) \mid 0 < m < 1, 1 < n < (1 + \frac{q}{p})^{\frac{gcd(2,p)}{2q}}\}.$$

Obviously,  $l_i$  ( $i = 0, 1, 2$ ) is changed to a rational function of  $m, n$  and the deduced system has been reduced to the semi-algebraic system.

When  $a = \frac{1}{2}$ , we have

$$\begin{aligned} f_0(u) &= \frac{(3u + 4)(u + 1)^{\frac{5}{2}} - (u + 1)(9u + 4)}{6(u + 1)^{\frac{5}{2}}u^2}, \\ f_1(u) &= \frac{3u(u + 1)^{\frac{11}{4}} - 2(u + 1)^{\frac{5}{2}} + 2(u + 1)}{2(u + 1)^{\frac{5}{2}}}, \\ f_2(u) &= \frac{5u[7(u - 2)(u + 1)^{\frac{9}{4}} - 10(u + 1)^{\frac{7}{4}} + 28(u + 1)^{\frac{3}{2}} - 4]}{14(u + 1)^{\frac{3}{2}}}. \end{aligned}$$

Note that here we take  $a = \frac{1}{2}$ , then the equation  $A(u) = A(z)$  for the involution  $z = \sigma(u)$  is changed to  $\frac{1}{2}(m - n)(m + n)(m^4 + n^2m^2 - 3 + n^4) = 0$ , in fact, it is

$$I(m, n) = m^4 + n^2m^2 + n^4 - 3 = 0. \quad (3.2)$$

Accordingly the continuous Wronskians are mapped to the rational function, and the problem reduces to solve a class of semi-algebraic systems consisting of the transformed Wronskians,  $I(m, n)$  and the parameter domain.

$W[l_0](u)$  is changed to

$$\tilde{W}[l_0](m) = \frac{-16\sqrt{3}(m - n)\omega_0(m, n)}{81m^7n^7(n - 1)(n + 1)(n^2 + 1)^3(m - 1)(m + 1)(m^2 + 1)^3},$$

where the function  $\omega_0(m, n)$  is a symmetric polynomial of degree 20 with respect to  $m, n$ . The resultant between  $\omega_0(m, n)$  and  $I(m, n)$  with respect to  $n$  is

$$r_0(m) = (m - 1)^2(m + 1)^2(m^2 + 1)^8\bar{r}_0(m),$$

where the degree of  $\bar{r}_0(m)$  is 52. Thanks to Sturm's Theorem, we assert that it has no zeros at  $(0, 1)$ , that follows  $W[l_0](u)$  does not vanish at  $(-1, 0)$ .

$W[l_0, l_1](u)$  is changed to

$$\tilde{W}[l_0, l_1](m) = \frac{-128(m - n)^2\omega_1(m, n)}{729m^{14}n^{12}(n - 1)^3(n + 1)^3(n^2 + 1)^6(m - 1)^2(m + 1)^2(m^2 + 1)^5},$$



where the function  $\omega_1(m, n)$  is a symmetric polynomial of degree 38 with respect to  $m, n$ . The resultant between  $\omega_1(m, n)$  and  $I(m, n)$  with respect to  $n$  is

$$r_1(m) = 16(m-1)^8(m+1)^8(m^2+1)^{16}\bar{r}_1(m),$$

where the degree of  $\bar{r}_1(m)$  is 96. With the help of Sturm's Theorem, we assert that it has no zeros at  $(0, 1)$ , that follows  $W[l_0, l_1](u)$  does not vanish at  $(-1, 0)$ .

Under the transformation (3.1), Wronskian  $W[l_0, l_1, l_2](u)$  is changed to

$$\widetilde{W}[l_0, l_1, l_2](m) = \frac{-640\sqrt{3}(m-n)^3\omega_2(m, n)}{137781m^{22}n^{16}(n-1)^6(n+1)^6(n^2+1)^{10}(m-1)^3(m+1)^3(m^2+1)^7},$$

here  $\omega_2(m, n)$  is a symmetric polynomial of degree 67 with respect to  $m, n$ , we get the resultant between  $\omega_2(m, n)$  and  $I(m, n)$  with respect to  $n$  is

$$r_2(m) = 1358954496(m-1)^{18}(m+1)^{18}(m^2+1)^{28}\bar{r}_2(m),$$

here  $\bar{r}_2(m)$  is a polynomial with degree 140, with the help of Sturm's Theorem, we assert that it does not vanish at  $(0, 1)$ , that follows  $W[l_0, l_1, l_2](u, z)$  has no common roots with the involution equation  $A(u) = A(z)$ ,  $z = \sigma(u)$ . That ends the proof of case (a).

**Case (b).** As  $a \in (-1, 0)$ , the unperturbed system  $(1.2)_{\varepsilon=0}$  has a non-degenerate center at  $C(0, 1)$ , enclosing which the period annulus is the upper half plane.

Firstly by shifting  $u = y - 1$ , and multiplying the integrating factor  $M(x, u) = -a(a+1)(u+1)^{a-1}$ , we get the transformed Hamiltonian is

$$H(x, u) = -a(a+1)\left[x^2(u+1)^a + \frac{(u+1)^a(-1+ua)}{a(a+1)} + \frac{1}{a(a+1)}\right],$$

with which  $H(0, 0) = 0$  and  $H(0, -1) = H(0, +\infty) = +\infty$ . The period annulus is foliated by the ovals  $\gamma_h = \{(x, u) \in R^2 \mid H(x, u) = h, 0 < h < +\infty\}$ , and the projection of period annulus on  $u$  axis is the interval  $(-1, +\infty)$ .

Similar to the case  $a \in (0, 1)$ , we have

$$\begin{aligned} I_0(h) &= \frac{1}{h} \oint_{\gamma_h} f_0(u)x^3 du, \\ I_1(h) &= \frac{1}{h} \oint_{\gamma_h} f_1(u)x^3 du, \\ I_2(h) &= \oint_{\gamma_h} g(u)x^3 du = \frac{1}{h} \oint_{\gamma_h} f_2(u)x^3 du, \end{aligned}$$

where  $f_0(u), f_1(u), g(u)$  and  $f_2(u)$  are obtained by the same way.

If  $a \neq -\frac{2}{3}$ , then similar to the case  $a \in (0, 1)$ , all the functions  $f_0(u), f_1(u), g(u)$  and  $f_2(u)$  are the sums of functions having the forms  $g(a)(1+u)^{ka+l}$ ,  $k, l \in \mathbb{Z}$ . After similar transformation as (3.1), we can transform them to rational functions, thus the deduced system has been reduced to the semi-algebraic system. But if  $a = -\frac{2}{3}$ , then in  $g(u)$  and  $f_2(u)$ , the item  $\ln(1+u)$  appears, which is not algebraic and cannot be changed to a rational function by our transformation.

When  $a = -\frac{1}{2}$ , the center  $(0, 1)$  of system  $(1.2)_{\varepsilon=0}$  is isochronous, the level curves in finite plane are shown as Fig. 1.2.

Following the notations of Lemma 2.1, we get

$$A(u) = -\frac{2\sqrt{u+1} - 2 - u}{2\sqrt{u+1}}, \quad B(u) = \frac{1}{4\sqrt{u+1}}, \quad (3.3)$$

which suffice  $B(0) > 0$  and  $uA'(u) = \frac{u^2}{4(u+1)^{\frac{3}{2}}} > 0$  as  $u \in (-1, +\infty) \setminus \{0\}$ . Thus there exists an analytic involution  $\sigma$  associated to  $A$  such that

$$A(u) - A(z) = -\frac{2(\sqrt{u+1} - \sqrt{z+1}) + z\sqrt{u+1} - u\sqrt{z+1}}{2\sqrt{u+1}\sqrt{z+1}} = 0, \quad (3.4)$$

where  $z = \sigma(u)$ , and  $(u, z)$  is in the parameter domain

$$\hat{D}_p = \{(u, z) \mid -1 < u < 0, 0 < z < +\infty\}.$$

We need to show that the tuple of functions  $(I_0, I_1, I_2)$  is an ECT-system on  $(0, +\infty)$ , here  $n = 3$ , to fulfill the condition  $s > n - 2$ , thus we should unify the power of  $x$  in 1-form to at least  $2s - 1 \geq 3$ .

It is easy to obtain

$$\begin{aligned} f_0(u) &= \frac{1}{6} \frac{(7u+4)(u+1)^{\frac{1}{2}} - u^2 - 9u - 4}{(u+1)^3 u^2}, \\ f_1(u) &= \frac{5u(u+1)^{\frac{15}{4}} - 6(u+1)^{\frac{7}{2}} - 2(u-3)(u+1)^3}{6(u+1)^6}, \\ f_2(u) &= \frac{5u[12(u+1)^{\frac{3}{4}} - 4(u+1)^{\frac{1}{4}}(9u+13) + 21(u+1)^{\frac{1}{2}}(u+2) - 2(u+1)]}{6(u+1)^{\frac{13}{4}}}. \end{aligned}$$

Denoting the criterion functions by

$$l_i(u) \triangleq \frac{f_i(u)}{A'(u)B(u)^{\frac{3}{2}}} - \frac{f_i(\sigma(u))}{A'(\sigma(u))B(u)^{\frac{3}{2}}},$$

and computing three continuous Wronskians of the tuple of functions  $(l_0, l_1, l_2)$ , we find that the involved functions consists of the factor  $(u+1)^{\frac{1}{4}}$ .

By implementing the following transformation

$$u = \frac{1}{m^4} - 1, \quad z = \frac{1}{n^4} - 1, \quad (3.5)$$

under which the parameter domain  $\hat{D}_p$  is mapped into

$$\hat{D}_p = \{(m, n) \mid 1 < m < +\infty, 0 < n < 1\},$$

and the involution equation (3.4) is changed to

$$I(m, n) = \frac{(mn+1)(m+n)(m-n)(mn-1)}{2m^2n^2} = 0,$$

therefore the involution associated to  $A$  with respect to  $m, n$  is reduced to  $mn = 1$ .

By using computer algebra system Maple, we get the transformed Wronskian

$$\widehat{W}[l_0](m, n) = \frac{-16(m-n)\omega_0(m, n)}{3(m^2+1)^3(m-1)(m+1)(n^2+1)^3(n-1)(n+1)},$$

here  $\omega_0(m, n)$  is a polynomial of degree 18. By applying resultant elimination, we get the resultant between  $\omega_0(m, n)$  and  $mn - 1$  with respect to  $m$  is

$$-(n^4 + 1)(4n^4 - 7n^2 + 4)(n^2 + 1)^6,$$

we assert it does not vanish at  $(0, 1)$ , that follows  $W[l_0](u, z) \neq 0$ , where  $(u, z) \in \hat{D}_p$  is confined by the involution equation (3.4), which is equivalent to that the Wronskian  $W[l_0, l_1](u)$  does not vanish at  $(0, 1)$ .

Under the transformation (3.5), the second Wronskian  $W[l_0, l_1](u, z)$  is changed to

$$\widehat{W}[l_0, l_1](m, n) = \frac{128(m-n)^2 m^2 \omega_1(m, n)}{9(n^2 + 1)^6 (n-1)^3 (n+1)^3 (m^2 + 1)^5 (m-1)^2 (m+1)^2},$$

where  $\omega_1(m, n)$  is a polynomial of degree 34. The resultant between  $\omega_1(m, n)$  and  $mn - 1$  with respect to  $m$  is

$$-4n^3(8n^4 - 7n^2 + 8)(n-1)^2(n+1)^2(n^2 + 1)^{12},$$

which has no zeros at  $(0, 1)$ , that implies  $W[l_0, l_1](m, n)$  and the involution equation  $mn - 1 = 0$  have no common roots at  $\hat{D}_p$ . Thus that follows  $W[l_0, l_1](u)$  does not vanish at  $(0, 1)$ .

Under the mapping (3.5), the third Wronskian  $W[l_0, l_1, l_2](u, z)$  is changed to

$$\widehat{W}[l_0, l_1, l_2](m, n) = \frac{-640(m-n)^3 m^4 \omega_2(m, n)}{3n^2(m^2 + 1)^7 (m-1)^3 (m+1)^3 (n^2 + 1)^{10} (n-1)^6 (n+1)^6},$$

where  $\omega_2(m, n)$  are the polynomials of degree 63. By using Maple, we get the resultant between  $\omega_2(m, n)$  and  $mn - 1$  with respect to  $m$  is

$$-512n^8(n-1)^6(n+1)^6(n^2 + 1)^{20},$$

which has no zeros at  $(0, 1)$ , that implies  $W[l_0, l_1, l_2](u, z)$  and the involution equation (3.4) have no common roots as  $(u, z) \in \hat{D}_p$ . That ends the proof of Theorem 1.1.

#### 4. Discussions

Bounding the number of isolated zeros of Abelian integrals is related to the famous Hilbert's 16th problem [20] or its weakened version proposed by Arnold [1]. Until now the problem still remains to be open even though a lot of work has been done in the past half century.

Over the past three decades, many scholars have extensively studied limit cycles which bifurcate from periodic annulus enclosing a center for a quadratic system [11]. It is well known that quadratic integrable systems with center can be classified into four classes: Hamiltonian  $Q_3^H$ , Reversible  $Q_3^R$ , codimension four  $Q_4$  and generalized Lotka–Volterra systems  $Q_3^{LV}$  [39]. Most of mathematicians working in this field believe that the perturbations of the class  $Q_3^R$  may give rise to the richest dynamical behaviors, but the study to which is also most difficult, for in general some known standard approaches and methods used for the Hamiltonian case does not work, thus the results considering integrable and non-Hamiltonian system are limited [4, 5, 19, 22].

Since the pioneer work by Bogdanov [2] and Takens [30], many researchers have devoted to generalize their results to  $n$ -parameters generic family of nilpotent singularity of codimension  $n$  with  $n \geq 3$ , such as the 3-parameters family of nilpotent codimension three saddle, focus and elliptic in [8, 34] (it is usually called degenerate Bogdanov–Takens bifurcations of codimension 3), but the study about degenerate Bogdanov–Takens bifurcations of codimension 3 is not complete, and some problems remain open until now, concerning

mainly the maximum number of limit cycles for the generic unfolding of codimension 3 singularities with nilpotent linear part [22,23]. Through a central rescaling, the unfolding of a codimension 3 Bogdanov–Takens singularity could be reduced to a perturbation from a one-parameter family of quadratic reversible system, and the maximum number of limit cycles is reduced to the number of isolated zeros of Abelian integrals (1.5) associated to system (1.2).

By applying the Chebyshev properties and criteria in [13] and [27], we can reduce the estimation to the number of isolated zeros of Abelian integrals to a pure algebraic problem, that is to solve a class of semi-algebraic systems consisting of equations and inequalities (maybe non-algebraic). With the help of computer algebraic software, we assert that, for almost all the non-zero rational parameter  $a \in (-1, 1)$ , the problem can be changed to solve semi-algebraic systems under the rationalized transformation (3.1). In order to illustrate our approaches, two particular cases are stated, that is  $a = \frac{1}{2}$  (corresponding to nilpotent codimension 3 saddle case) and  $a = -\frac{1}{2}$  (corresponding to nilpotent codimension 3 elliptic case) concerned with system (1.2), where the upper bound of the number of limit cycles for the generic unfolding in two cases is two. However, by checking the deduction of functions  $f_i(u)$  for  $i = 0, 1, 2$ , when  $a = -\frac{2}{3}$  we found that the function  $f_2(u)$  and the third Wronskian accordingly contain logarithmic function as we need to apply the property (b) of Lemma 2.2 to lower the power of  $x$  in 1-form, that invalidates our approaches consisting of resultant elimination, Sturm's Theorem and realroot isolation, which base on real algebraic geometry. Furthermore, how to deal with the problems when  $a$  is irrational is still challenging, we leave it for further research.

## Acknowledgments

The authors are grateful to Professor Dongmei Xiao and Professor Yulin Zhao for helpful discussions, and they would like to thank the referees for valuable comments and suggestions. J. Huang was partially supported by NSFC grant (No. 11471133), and Self-determined Research Funds of CCNU from the Colleges Basic Research and Operation of MOE (CCNU16A02009). C. Liu was partially supported by NSFC grant (No. 11371269) and sponsored by Qing Lan Project.

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