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$\varphi-(h,e)$-concave operators and applications

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A B S T R A C T

In this article, by introducing a new set and a new concept of $\varphi-(h,e)$-concave operators, and by using the cone theory and monotone iterative method, we present some new existence and uniqueness theorems of fixed points for increasing $\varphi-(h,e)$-concave operators without requiring the existence of upper and lower solutions. As an application, we establish the existence and uniqueness of a nontrivial solution for a new form of fractional differential equation with integral boundary conditions. The main results of this paper improve and extend some known results, and present a new method to study nonlinear equation problems.

1. Introduction

In [24], we introduced the definition of generalized convex operator and considered the nonlinear operator equation on ordered Banach spaces $x = Ax + x_0$, where A is an increasing, generalized concave operator, and we established some existence and uniqueness results of positive solutions for such equations. Further, we also studied the operator equation $Ax = \lambda x$, where A is an increasing, generalized concave operator and the parameter $\lambda > 0$, and we give an existence and uniqueness result of positive solutions for any given $\lambda > 0$. Moreover, we present some clear properties of positive solutions for the operator equation, see [23]. These results can be applied to study nonlinear differential and integral equations, see [8,10,12,16,18,19,21,22,26,28,29] for example. By using the fixed point results of generalized concave operators, [8] presented the existence and uniqueness of positive solutions for a singular Lane–Emden–Fowler equation; [12] gave the existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions; [19] studied the existence and uniqueness of positive solutions for a nonlinear perturbed fractional two-point boundary value problem; [18,22,26,28,29] also get the existence and uniqueness of positive solutions for a three-point boundary value problem of second order impulsive differential equations ([18]), a Neumann boundary value problem of second order impulsive differential equations ([22]), an optimal

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control problem of second order impulsive differential equations ([26]), two classes of nonlinear perturbed Neumann boundary value problems for second-order differential equations ([28]) and two-point boundary value problems for second-order impulsive differential equations ([29]). In [10], the authors also considered the operator equation $x = Ax + x_0$ and gave a fixed point theorem for decreasing and convex operators in a probabilistic Banach space partially ordered by a normal cone, and then disscussed the existence and uniqueness of positive solutions for a two-point boundary value problem. Based upon the results in [24], we studied the operator equation $Ax + Bx + Cx = x$ on ordered Banach spaces, and then utilized the fixed point results to get the existence and uniqueness of positive solutions for two classes nonlinear problems: fourth-order two-point boundary value problems for elastic beam equations and elliptic value problems for Lane–Emden–Fowler equations. Similarly, [16] studied a class of operator equation $A(x, x) + Bx = x$, and proved the existence and uniqueness of positive solutions for a nonlinear integral equation of second-order on time scales.

In fact, in [24,23], the existence and uniqueness property is local, and the operators which we considered are defined in a special set P_h, where P_h is a subset of a cone P. Stimulated by our works [24,23], in this paper, we introduce a new set $P_{h,e}$ which includes the set P_h and needs not be a subset of a cone P. Further, we define a new concept of $\varphi-(h, e)$-concave operators which extend generalized concave operators. Without requiring the existence of upper and lower solutions, we prove the existence and uniqueness of solutions for $\varphi - (h, e)$-concave operators. So our results in essence extend the corresponding ones in [24,23]. And then, we use our main fixed point theorem to study the following new form of fractional differential equation with integral boundary conditions:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = b, \ 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \beta \int_0^1 u(s)ds, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$, $0 < \beta < \alpha$, $b > 0$ is a constant, $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous, D_{0+}^{α} is the Riemann–Liouville fractional derivative of order α which is given as the following:

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(s)}{(t-s)^{\alpha+1-n}} ds, \ n = [\alpha] + 1,$$

here Γ denotes the Euler gamma function, $[\alpha]$ denotes the integer part of number α provided that the right side is point-wise defined on $(0, +\infty)$, the concept of fractional derivative can be seen in [15]. And we get a new result on the existence and uniqueness of solutions for the problem (1.1), which is not a consequence of the corresponding fixed point theorems in [24,23]. Here we should point out that this paper presents a new method to study nonlinear equation problems.

For the discussion of our main results, we list some notations, concepts in ordered Banach spaces. For the convenience of readers, we refer them to [7,11,24] for details.

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. θ denotes the zero element of E. P is called normal if there exists $M > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$; in this case M is the infimum of such constants, it is called the normality constant of P. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For any $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we define the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 1. (See [24].) Let $A : P \to P$ be a given operator. For any $x \in P$ and $r \in (0, 1)$, there exists $\varphi(r) \in (r, 1)$ such that $A(rx) \geq \varphi(r)Ax$. Then, A is called a generalized concave operator.

2. Main results

Let $e \in P$ with $\theta \le e \le h$. We define a new set

$$P_{h,e} = \{x \in E | x + e \in P_h\}.$$

Then we can see that $h \in P_{h,e}$ and

$$P_{h,e} = \{x \in E | \text{ there exist } \mu = \mu(h, e, x) > 0,\ \nu = \nu(h, e, x) > 0 \text{ such that } \mu h \le x + e \le \nu h\}.$$

Remark 1. If $e = \theta$, then $P_{h,\theta} = P_h$. Moreover, $P_h \subseteq P_{h,e}$. But $P_{h,e}$ is not a subset of P for some e, so P_h and $P_{h,e}$ are different.

Lemma 1. *If $x \in P_{h,e}$, then $\lambda x + (\lambda - 1)e \in P_{h,e}$ for $\lambda > 0$.*

Proof. If $x \in P_{h,e}$, then there are $\mu, \nu > 0$ such that

$$\mu h \le x + e \le \nu h.$$

For $\lambda > 0$, we get $\lambda x + (\lambda - 1)e + e = \lambda x + \lambda e = \lambda(x + e)$ and then

$$\lambda \mu h \le \lambda x + (\lambda - 1)e + e \le \lambda \nu h.$$

Hence, $\lambda x + (\lambda - 1)e \in P_{h,e}$. □

Lemma 2. *If $x \in P_{h,e}$, then $\lambda x \in P_{h,e}$ for $0 < \lambda < 2$.*

Proof. For $0 < \lambda < 2$, we have

$$\lambda x + e = \lambda x + (\lambda - 1)e + (2 - \lambda)e.$$

From Lemma 1, we have $\lambda x + (\lambda - 1)e \in P_{h,e}$. Note that

$$\theta \le (2 - \lambda)e \le (2 - \lambda)h,$$

we can get

$$\lambda x + (\lambda - 1)e + (2 - \lambda)e \in P_h.$$

That is, $\lambda x + e \in P_h$. So $\lambda x \in P_{h,e}$. □

Lemma 3. *If $x, y \in P_{h,e}$, then there exist $0 < \mu < 1$, $\nu > 1$ such that*

$$\mu y + (\mu - 1)e \le x \le \nu y + (\nu - 1)e.$$

Further, we can choose a small $r \in (0, 1)$ such that

$$ry + (r - 1)e \le x \le r^{-1} y + (r^{-1} - 1)e.$$

Proof. If $x, y \in P_{h,e}$, there exist $0 < \mu_1, \mu_2 < 1$, $\nu_1, \nu_2 > 1$ such that

$$\mu_1 h \leq x + e \leq \nu_1 h, \quad \mu_2 h \leq y + e \leq \nu_2 h.$$

So

$$x + e \geq \mu_1 h = \frac{\mu_1}{\nu_2}\nu_2 h \geq \frac{\mu_1}{\nu_2}(y + e),$$
$$x + e \leq \nu_1 h = \frac{\nu_1}{\mu_2}\mu_2 h \leq \frac{\nu_1}{\mu_2}(y + e).$$

Let $\mu = \frac{\mu_1}{\nu_2}$, $\nu = \frac{\nu_1}{\mu_2}$, then $0 < \mu < 1$, $\nu > 1$. So

$$\mu(y + e) \leq x + e \leq \nu(y + e),$$

and thus $\mu y + (\mu - 1)e \leq x \leq \nu y + (\nu - 1)e$. Further, we can take $r \in (0, 1)$ such that $r < \mu < \nu < r^{-1}$, and then

$$r(y + e) \leq \mu(y + e) \leq x + e \leq \nu(y + e) \leq r^{-1}(y + e),$$

and in consequence, $ry + (r - 1)e \leq x \leq r^{-1}y + (r^{-1} - 1)e$. $\square$

To obtain our main results, we first give the definition of $\varphi - (h, e)$-concave operators.

Definition 2. Let $A : P_{h,e} \to E$ be a given operator. For any $x \in P_{h,e}$ and $\lambda \in (0, 1)$, there exists $\varphi(\lambda) > \lambda$ such that

$$A(\lambda x + (\lambda - 1)e) \geq \varphi(\lambda)Ax + (\varphi(\lambda) - 1)e. \tag{2.1}$$

Then A is called a $\varphi - (h, e)$-concave operator.

Remark 2. If $e = \theta$ in (2.1), then $A(\lambda x) \geq \varphi(\lambda)Ax$. That is, A is a generalized concave operator. So a generalized concave operator can be said to be a $\varphi - (h, \theta)$-concave operator. In addition, from (2.1), we obtain

$$Ax \leq \frac{1}{\varphi(\lambda)}A(\lambda x + (\lambda - 1)e) + \left(\frac{1}{\varphi(\lambda)} - 1\right)e. \tag{2.2}$$

Theorem 1. *Let P be normal and A be an increasing $\varphi - (h, e)$-concave operator with $Ah \in P_{h,e}$. Then A has a unique fixed point x^* in $P_{h,e}$. Moreover, for any $w_0 \in P_{h,e}$, making the sequence $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$, then we obtain $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

Proof. Because $Ah \in P_{h,e}$, $h \in P_{h,e}$, from Lemma 3, we can choose a small $t_0 \in (0, 1)$ such that

$$t_0 h + (t_0 - 1)e \leq Ah \leq t_0^{-1}h + (t_0^{-1} - 1)e. \tag{2.3}$$

Since $\varphi(t_0) > t_0$, we can find a positive integer k such that

$$\left(\frac{\varphi(t_0)}{t_0}\right)^k \geq \frac{1}{t_0}. \tag{2.4}$$

Put

$$x_n = t_0^n h + (t_0^n - 1)e, \quad y_n = t_0^{-n} h + (t_0^{-n} - 1)e, \quad n = 1, 2, \ldots.$$

Then we easily get

$$x_n = t_0 x_{n-1} + (t_0 - 1)e, \quad y_n = t_0^{-1} y_{n-1} + (t_0^{-1} - 1)e, \quad n = 1, 2, \ldots. \tag{2.5}$$

Set $u_0 := x_k$, $v_0 := y_k$, then $u_0, v_0 \in P_{h,e}$ and $u_0 = t_0^{2k} v_0 < v_0$. Further, by the definition of A and (2.3)–(2.5), we have

$$\begin{aligned}
Au_0 &= Ax_k = A(t_0 x_{k-1} + (t_0 - 1)e) \\
&\geq \varphi(t_0) Ax_{k-1} + (\varphi(t_0) - 1)e \\
&= \varphi(t_0) A(t_0 x_{k-2} + (t_0 - 1)e) + (\varphi(t_0) - 1)e \\
&\geq \varphi(t_0)[\varphi(t_0) Ax_{k-2} + (\varphi(t_0) - 1)e] + (\varphi(t_0) - 1)e \\
&= (\varphi(t_0))^2 Ax_{k-2} + [(\varphi(t_0))^2 - 1]e \\
&\geq \cdots \geq (\varphi(t_0))^k Ah + [(\varphi(t_0))^k - 1]e \\
&\geq (\varphi(t_0))^k [t_0 h + (t_0 - 1)e] + [(\varphi(t_0))^k - 1]e \\
&\geq t_0^{k-1}[t_0 h + (t_0 - 1)e] + (t_0^{k-1} - 1)e \\
&= t_0^k h + (t_0^k - 1)e = u_0.
\end{aligned}$$

Also, from (2.2)–(2.5), we obtain

$$\begin{aligned}
Av_0 &= Ay_k = A(t_0^{-1} y_{k-1} + (t_0^{-1} - 1)e) \\
&\leq \frac{1}{\varphi(t_0)} Ay_{k-1} + \left(\frac{1}{\varphi(t_0)} - 1\right) e \\
&= \frac{1}{\varphi(t_0)} A(t_0^{-1} y_{k-2} + (t_0^{-1} - 1)e) + \left(\frac{1}{\varphi(t_0)} - 1\right) e \\
&\leq \frac{1}{\varphi(t_0)} \left[\frac{1}{\varphi(t_0)} Ay_{k-2} + \left(\frac{1}{\varphi(t_0)} - 1\right) e\right] + \left(\frac{1}{\varphi(t_0)} - 1\right) e \\
&= \frac{1}{(\varphi(t_0))^2} Ay_{k-2} + \left(\frac{1}{(\varphi(t_0))^2} - 1\right) e \\
&\leq \cdots \leq \frac{1}{(\varphi(t_0))^k} Ah + \left(\frac{1}{(\varphi(t_0))^k} - 1\right) e \\
&\leq \frac{1}{(\varphi(t_0))^k} [t_0^{-1} h + (t_0^{-1} - 1)e] + \left(\frac{1}{\varphi(t_0)^k} - 1\right) e \\
&\leq \frac{1}{t_0^{k-1}} [t_0^{-1} h + (t_0^{-1} - 1)e] + \left(\frac{1}{t_0^{k-1}} - 1\right) e \\
&= \frac{1}{t_0^k} h + \left(\frac{1}{t_0^k} - 1\right) e = v_0.
\end{aligned}$$

So we have

$$u_0 < v_0, \ Au_0 \geq u_0, \ Av_0 \leq v_0. \tag{2.6}$$

Let $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1, 2, \ldots$. From the monotonicity of A and (2.6), we can obtain that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_1 \le v_0. \tag{2.7}$$

Since $u_0, v_0 \in P_{h,e}$, from Lemma 3, we also choose a $\tau_1 > 0$ such that

$$u_0 \ge \tau_1 v_0 + (\tau_1 - 1)e,$$

and from $u_0 < v_0$, we get $\tau_1 \in (0,1)$. By using (2.7), we have

$$u_n \ge u_0 \ge \tau_1 v_0 + (\tau_1 - 1)e \ge \tau_1 v_n + (\tau_1 - 1)e, \quad n = 1, 2 \dots.$$

Denote

$$t_n = \sup\{t > 0 : u_n \ge t v_n + (t-1)e\}.$$

Then from (2.7), we get

$$t_n \in (0,1), \quad u_n \ge t_n v_n + (t_n - 1)e. \tag{2.8}$$

So, by (2.7), (2.8),

$$u_{n+1} \ge u_n \ge t_n v_n + (t_n - 1)e \ge t_n v_{n+1} + (t_n - 1)e$$

and in consequence, $t_{n+1} \ge t_n$. That is, $\{t_n\}$ is increasing. So we can set $t_n \to t^*$ as $n \to \infty$. Then $t^* \in [0,1]$. Next we prove $t^* = 1$. If $0 < t^* < 1$, we need to consider two cases:

Case one: there is an integer N such that $t_N = t^*$. So $t_n = t^*$ for all $n > N$. Then for $n \ge N$,

$$\begin{aligned} u_{n+1} &= Au_n \ge A(t_n v_n + (t_n - 1)e) \\ &= A(t^* v_n + (t^* - 1)e) \ge \varphi(t^*)Av_n + (\varphi(t^*) - 1)e \\ &= \varphi(t^*)v_{n+1} + (\varphi(t^*) - 1)e. \end{aligned}$$

From the definition of t_{n+1}, we get $t^* = t_{n+1} \ge \varphi(t^*) > t^*$, this is a contradiction.

Case two: for all integers n, $t_n < t^*$. Then we have

$$\begin{aligned} u_{n+1} &= Au_n \ge A(t_n v_n + (t_n - 1)e) \\ &= A\left(\frac{t_n}{t^*}(t^* v_n + (t^* - 1)e) + \left(\frac{t_n}{t^*} - 1\right)e\right) \\ &\ge \varphi\left(\frac{t_n}{t^*}\right) A(t^* v_n + (t^* - 1)e) + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\ &\ge \varphi\left(\frac{t_n}{t^*}\right)[\varphi(t^*)Av_n + (\varphi(t^*) - 1)e] + \left(\varphi\left(\frac{t_n}{t^*}\right) - 1\right)e \\ &= \varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*)v_{n+1} + \left(\varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) - 1\right)e. \end{aligned}$$

Also, from the definition of t_{n+1}, we get

$$t_{n+1} \ge \varphi\left(\frac{t_n}{t^*}\right)\varphi(t^*) > \frac{t_n}{t^*}\varphi(t^*).$$

Let $n \to \infty$, we obtain $t^* \ge \varphi(t^*) > t^*$, this is also a contradiction. Hence, $t^* = 1$ and thus $\lim_{n\to\infty} t_n = 1$.

Next, we prove the sequences $\{u_n\}, \{v_n\}$ are Cauchy sequences. For any natural number p we get

$$\begin{aligned}\theta &\leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n - (t_n - 1)e \\ &= (1-t_n)v_n + (1-t_n)e \leq (1-t_n)v_0 + (1-t_n)e.\end{aligned}$$

So

$$\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq (1-t_n)v_0 + (1-t_n)e.$$

Note that P is normal, we also get

$$\|u_{n+p} - u_n\| \leq M(1-t_n)\|v_0 + e\|, \quad \|v_n - v_{n+p}\| \leq M(1-t_n)\|v_0 + e\|, \tag{2.9}$$

where M is the normality constant. Let $n \to \infty$ in (2.9), we obtain

$$\|u_{n+p} - u_n\| \to 0, \quad \|v_n - v_{n+p}\| \to 0. \tag{2.10}$$

From (2.10), we can claim that $\{u_n\}, \{v_n\}$ are Cauchy sequences. Since E is complete, there are $u^*, v^* \in E$ such that $u_n \to u^*, v_n \to v^*$ as $n \to \infty$. From (2.7),

$$u_0 \leq u_n \leq u^* \leq v^* \leq v_n \leq v_0.$$

Thus, $u^*, v^* \in P_{h,e}$ and

$$\theta \leq v^* - u^* \leq v_n - u_n \leq (1-t_n)(v_0 + e).$$

Also, by the normality of P,

$$\|v^* - u^*\| \leq M(1-t_n)\|v_0 + e\| \to 0 \quad (n \to \infty).$$

Therefore, $u^* = v^*$. Set $x^* = u^* = v^*$ and we get

$$u_{n+1} = Au_n \leq Ax^* \leq Av_n = v_{n+1}.$$

Let $n \to \infty$, we obtain $Ax^* = x^*$. That is, x^* is a fixed point of A in $P_{h,e}$.

Now we show that x^* is the unique fixed point of A in $P_{h,e}$. Suppose that y^* is any fixed point of A in $P_{h,e}$. By Lemma 3, there exists $\tau_2 > 0$ such that $x^* \geq \tau_2 y^* + (\tau_2 - 1)e$. Set

$$\bar{t} = \sup\{t > 0 | x^* \geq ty^* + (t-1)e\}.$$

Next we prove $\bar{t} \geq 1$. If $0 < \bar{t} < 1$, then $x^* \geq \bar{t}y^* + (\bar{t} - 1)e$ and thus

$$\begin{aligned}x^* &= Ax^* \geq A(\bar{t}y^* + (\bar{t}-1)e) \\ &\geq \varphi(\bar{t})Ay^* + (\varphi(\bar{t}) - 1)e \\ &= \varphi(\bar{t})y^* + (\varphi(\bar{t}) - 1)e.\end{aligned}$$

By the definition of $\bar{t}$, we get $\bar{t} \geq \varphi(\bar{t}) > \bar{t}$, this is a contradiction. So $\bar{t} \geq 1$ and thus

$$x^* \geq \bar{t}y^* + (\bar{t} - 1)e \geq \bar{t}y^* \geq y^*.$$

Similarly, we also obtain $y^* \geq x^*$. Therefore, $x^* = y^*$.

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Finally, for any given $w_0 \in P_{h,e}$, let $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$, we prove $w_n \to x^*$ as $n \to \infty$. Since $x^*, w_0 \in P_{h,e}$, from Lemma 3, there exists $\tau_3 \in (0, 1)$ such that

$$\tau_3 x^* + (\tau_3 - 1)e \le w_0 \le \tau_3^{-1} x^* + (\tau_3^{-1} - 1)e. \tag{2.11}$$

Let

$$u_0' = \tau_3 x^* + (\tau_3 - 1)e, v_0' = \tau_3^{-1} x^* + (\tau_3^{-1} - 1)e,$$
$$u_n' = Au_{n-1}',\ v_n' = Av_{n-1}',\ n = 1, 2, \ldots.$$

Then,

$$u_0' \le w_0 \le v_0',\ u_0' \le x^* \le v_0',\ v_0' \ge \tau_3^{-1} x^* \ge \tau_3^{-2} u_0' > u_0'.$$

From the monotonicity of A, we have

$$u_n' \le w_n \le v_n', u_n' \le x^* \le v_n',\ n = 1, 2, \ldots. \tag{2.12}$$

Further,

$$\begin{aligned} u_1' = Au_0' = A(\tau_3 x^* + (\tau_3 - 1)e) &\ge \varphi(\tau_3) Ax^* + (\varphi(\tau_3) - 1)e \\ &\ge \tau_3 x^* + (\tau_3 - 1)e = u_0'. \end{aligned}$$

By (2.2), we obtain

$$\begin{aligned} v_1' = Av_0' = A(\tau_3^{-1} x^* + (\tau_3^{-1} - 1)e) &\le \frac{1}{\varphi(\tau_3)} Ax^* + \left(\frac{1}{\varphi(\tau_3)} - 1\right) e \\ &\le \frac{1}{\tau_3} x^* + \left(\frac{1}{\tau_3} - 1\right) e = v_0'. \end{aligned}$$

In a general, we have

$$u_0' \le u_1' \le \cdots \le u_n' \le \cdots \le v_n' \le \cdots \le v_1' \le v_0'.$$

Similar to the above proof, we can prove that $\{u_n'\}, \{v_n'\}$ have the same limitation. From (2.12), the limitation is x^*, and thus $w_n \to x^*$ as $n \to \infty$. □

Corollary 1. *(Theorem 2.1 of [23]) Let P be normal and A be an increasing $\varphi - (h, \theta)$-concave operator with $Ah \in P_h$. Then A has a unique fixed point x^* in P_h. Moreover, for any $w_0 \in P_h$, making the sequence $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$, we get $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

Corollary 2. *(Theorem 2.1 of [24]) Let P be a normal cone and $h > \theta$. Assume that:*
(D_1) $A : P \to P$ is increasing and $Ah + x_0 \in P_h$ with $x_0 \in P$;
(D_2) for $x \in P$ and $\lambda \in (0, 1)$, there exists $\varphi(\lambda) \in (\lambda, 1)$ such that $A(\lambda x) \ge \varphi(\lambda) Ax$.
Then the operator equation $x = Ax + x_0$ has a unique solution in P_h.

Proof. Let $Bx = Ax + x_0$, then $B : P \to P$ is increasing and $Bh \in P_h$. Morover, for $\lambda > 0$, we can get

$$B(\lambda x) = A(\lambda x) + x_0 \ge \varphi(\lambda) Ax + x_0 \ge \varphi(\lambda)(Ax + x_0) = \varphi(\lambda) Bx.$$

That is, B is a $\varphi - (h, \theta)$-concave operator. By Theorem 1, the conclusion holds. □

Remark 3. Under the conditions of Theorem 1, from the proof, we can obtain the existence of upper and lower solutions for the $\varphi-(h,e)$-concave operator A. If we assume that $A: P_{h,e} \to P_{h,e}$, then $Ah \in P_{h,e}$ is automatically satisfied. So we also get the following conclusion.

Corollary 3. *Let P be normal and $A: P_{h,e} \to P_{h,e}$ be an increasing $\varphi-(h,e)$-concave operator. Then A has a unique fixed point x^* in $P_{h,e}$. Moreover, for any $w_0 \in P_{h,e}$, making the sequence $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$, we obtain $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

3. Application

Integral boundary value problems have various applications in applied fields which include chemical engineering, blood flow problems, thermo-elasticity, underground water flow and population dynamics, see [5]. Recently, fractional integral boundary value problems have been extensively studied, see [1–6,9,13,14,17, 20,25,27,30] for example. The existence of solutions for integral boundary value problems is an important problem. In literature, most of the authors have studied the existence and multiplicity of solutions. In [3], Cabada and Hamdi considered a class of fractional equations involving the Riemann–Liouville fractional derivative with integral boundary value conditions

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \beta \int_0^1 u(s)ds. \end{cases} \tag{3.1}$$

The authors established the existence of one positive solution for problem (3.1) under sublinear case or superlinear case. The method used there is Guo–Krasnosel'skii fixed point theorem. However, there are few papers reported on the uniqueness of solutions for fractional differential equations with integral boundary conditions. In this section, we will use Theorem 1 to study the fractional integral boundary value problem (1.1).

Lemma 4. *(Theorem 2.1 of [3]) Let $2 < \alpha \leq 3$ and $\alpha \neq \beta$. Assume $y \in C[0,1]$, then the following fractional differential equation with integral boundary conditions*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = \beta \int_0^1 u(s)ds, \end{cases}$$

has a unique solution $u \in C^1[0,1]$, given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s)-(\alpha-\beta)(t-s)^{\alpha-1}}{(\alpha-\beta)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.3}$$

Lemma 5. *(Lemma 3.2 of [23]) Let $2 < \alpha \leq 3$ and $0 < \beta < \alpha$. The function $G(t,s)$ given as in (3.3) has the following properties:*

$$\frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)} t^{\alpha-1} \leq G(t,s) \leq \frac{(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)} t^{\alpha-1}, \ t,s \in [0,1].$$

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In the following, we will work in the Banach space $C[0,1]$, the space of all continuous functions on $[0,1]$, the norm is $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Evidently, this space can be equipped with a partial order

$$x, y \in C[0,1],\ x \le y \Leftrightarrow x(t) \le y(t) \text{ for } t \in [0,1].$$

Set $P = \{x \in C[0,1] | x(t) \ge 0,\ t \in [0,1]\}$, the standard cone. We know that P is a normal cone in $C[0,1]$. Let

$$e(t) = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left[\frac{\alpha-\beta}{\alpha}t^{\alpha-1} + \frac{\beta}{\alpha(\alpha+1)}t^{\alpha-1} - \frac{\alpha-\beta}{\alpha}t^{\alpha}\right],\ t \in [0,1].$$

Theorem 2. *Suppose that*
(H_1) $f : [0,1] \times [-e^*, +\infty) \to (-\infty, +\infty)$ is increasing with respect to the second variable, where $e^* = \max\{e(t) : t \in [0,1]\}$;
(H_2) for any $\lambda \in (0,1)$, there is $\varphi(\lambda) > \lambda$ such that

$$f(t, \lambda x + (\lambda-1)y) \ge \varphi(\lambda) f(t,x),\ \forall\ t \in [0,1],\ x \in (-\infty,+\infty),\ y \in [0,e^*];$$

(H_3) $f(t,0) \ge 0$ with $f(t,0) \not\equiv 0$ for $t \in [0,1]$.
Then the problem (1.1) has a unique nontrivial solution u^* in $P_{h,e}$, where $h(t) = Ht^{\alpha-1}$, $t \in [0,1]$ with $H \ge \frac{b(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+1)\Gamma(\alpha)}$. Moreover, for any given $w_0 \in P_{h,e}$, making a sequence

$$w_n(t) = \int_0^1 G(t,s) f(s, w_{n-1}(s))ds - \frac{b(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+1)\Gamma(\alpha)}t^{\alpha-1} + \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha},\ n = 1,2,\ldots,$$

we have $w_n(t) \to u^*(t)$ as $n \to \infty$.

Proof. Firstly, for $t \in [0,1]$,

$$e(t) = \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left[\frac{\alpha-\beta}{\alpha}t^{\alpha-1}(1-t) + \frac{\beta}{\alpha(\alpha+1)}t^{\alpha-1}\right] \ge 0.$$

That is, $e \in P$. Further, for $t \in [0,1]$,

$$e(t) = \frac{b(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+1)\Gamma(\alpha)}t^{\alpha-1} - \frac{b}{\alpha\Gamma(\alpha)}t^{\alpha} \le \frac{b(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+1)\Gamma(\alpha)}t^{\alpha-1} \le Ht^{\alpha-1} = h(t).$$

Hence, $0 \le e(t) \le h(t)$. In addition, $P_{h,e} = \{u \in C[0,1] | u + e \in P_h\}$.

From Lemma 4, the problem (3.1) has an integral formulation given by

$$\begin{aligned} u(t) &= \int_0^1 G(t,s) f(s,u(s))ds - b\int_0^1 G(t,s)ds \\ &= \int_0^1 G(t,s) f(s,u(s))ds - \frac{b}{(\alpha-\beta)\Gamma(\alpha)}\left[\frac{\alpha-\beta}{\alpha}t^{\alpha-1} + \frac{\beta}{\alpha(\alpha+1)}t^{\alpha-1} - \frac{\alpha-\beta}{\alpha}t^{\alpha}\right] \\ &= \int_0^1 G(t,s) f(s,u(s))ds - e(t). \end{aligned}$$

For any $u \in P_{h,e}$, we consider the following operator of the form

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds - e(t), \ t \in [0,1].$$

So $u(t)$ is the solution of the problem (1.1) if and only if $u(t) = Au(t)$.

First, we show that $A : P_{h,e} \to E$ is a $\varphi - (h,e)$-concave operator. For $u \in P_{h,e}$, $\lambda \in (0,1)$, from (H_2) we have

$$\begin{aligned}
A(\lambda u + (\lambda - 1)e)(t) &= \int_0^1 G(t,s)f(s,\lambda u(s) + (\lambda-1)e(s))ds - e(t) \\
&\geq \varphi(\lambda)\int_0^1 G(t,s)f(s,u(s))ds - e(t) \\
&= \varphi(\lambda)[\int_0^1 G(t,s)f(s,u(s))ds - e(t)] + [\varphi(\lambda)-1]e(t) \\
&= \varphi(\lambda)Au(t) + [\varphi(\lambda)-1]e(t).
\end{aligned}$$

Hence, we obtain

$$A(\lambda u + (\lambda-1)e) \geq \varphi(\lambda)Au + [\varphi(\lambda)-1]e, \ u \in P_{h,e}, \ \lambda \in (0,1).$$

Therefore, A is $\varphi - (h,e)$-concave operator.

Secondly, we prove that $A : P_{h,e} \to E$ is increasing. For $u \in P_{h,e}$, we have $u + e \in P_h$. So there exists $\mu > 0$ such that $u(t) + e(t) \geq \mu h(t)$, $t \in [0,1]$. Hence,

$$u(t) \geq \mu h(t) - e(t) \geq -e(t) \geq -e^*.$$

Therefore, from (H_1), $A : P_{h,e} \to E$ is increasing.

Next, we prove that $Ah \in P_{h,e}$. So we need to prove $Ah + e \in P_h$. By Lemma 5 and (H_1), (H_3),

$$\begin{aligned}
Ah(t) + e(t) &= \int_0^1 G(t,s)f(s,h(s))ds = \int_0^1 G(t,s)f(s,Hs^{\alpha-1})ds \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-1}(\alpha-\beta+\beta s)}{(\alpha-\beta)\Gamma(\alpha)} t^{\alpha-1} f(s,H)ds \\
&\leq \frac{\alpha}{(\alpha-\beta)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,H)ds \cdot t^{\alpha-1} \\
&= \frac{\alpha}{(\alpha-\beta)H\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,H)ds \cdot h(t)
\end{aligned}$$

and

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$$\begin{aligned} Ah(t)+e(t) &= \int_0^1 G(t,s)f(s,Hs^{\alpha-1})ds \\ &\geq \int_0^1 \frac{(1-s)^{\alpha-1}\beta s}{(\alpha-\beta)\Gamma(\alpha)}t^{\alpha-1}f(s,0)ds \\ &= \frac{\beta}{(\alpha-\beta)\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,0)ds\cdot t^{\alpha-1} \\ &= \frac{\beta}{(\alpha-\beta)H\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,0)ds\cdot h(t). \end{aligned}$$

Let

$$l_1=\frac{\beta}{(\alpha-\beta)H\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}f(s,0)ds,\ l_2=\frac{\alpha}{(\alpha-\beta)H\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,H)ds.$$

Because $\alpha>\beta$, $\Gamma(\alpha)>0$ and from (H_1), (H_3),

$$\int_0^1(1-s)^{\alpha-1}f(s,H)ds\geq\int_0^1 s(1-s)^{\alpha-1}f(s,0)ds>0$$

and thus $l_2\geq l_1>0$. So this shows that $Ah+e\in P_h$.

Consequently, by using Theorem 1, the operator A has a unique fixed point u^* in $P_{h,e}$ and thus

$$u^*(t)=\int_0^1 G(t,s)f(s,u^*(s))ds-e(t),\ t\in[0,1].$$

Evidently, $u^*(t)\not\equiv 0$, $t\in[0,1]$. Therefore, $u^*(t)$ is a nontrivial solution. Moreover, for any $w_0\in P_{h,e}$, the sequence $w_n=Aw_{n-1}$, $n=1,2,\ldots$ satisfies $w_n\to u^*$ as $n\to\infty$. That is,

$$w_n(t)=\int_0^1 G(t,s)f(s,w_{n-1}(s))ds-\frac{b(\alpha-\beta+1)}{(\alpha-\beta)(\alpha+1)\Gamma(\alpha)}t^{\alpha-1}+\frac{b}{\alpha\Gamma(\alpha)}t^{\alpha},\ n=1,2,\ldots,$$

and $w_n(t)\to u^*(t)$ as $n\to\infty$. $\square$

Remark 4. For some differential equation boundary value problems, we can find two functions $e(t),h(t)$ and we can also construct functions which satisfy the conditions of Theorem 2. For example, let $f(t,x)=[\frac{e(t)}{e^*}x+e(t)]^{\frac{1}{3}}$, where $0\leq e(t)\leq h(t)$, $e^*=\max\{e(t):t\in[0,1]\}>0$. Then $f:[0,1]\times(-\infty,+\infty)\to(-\infty,+\infty)$ is continuous and increasing with respect to the second variable, $f(t,0)=[e(t)]^{\frac{1}{3}}\geq 0$ with $f(t,0)\not\equiv 0$. Next we show that the condition (H_2) is satisfied. Let $\varphi(\lambda)=\lambda^{\frac{1}{3}}$, $\lambda\in(0,1)$. We have $\varphi(\lambda)>\lambda$ for $\lambda\in(0,1)$. For $\lambda\in(0,1)$, $x\in(-\infty,+\infty)$, $y\in[0,e^*]$,

$$\begin{aligned} f(t,\lambda x+(\lambda-1)y) &= \left\{\frac{e(t)}{e^*}\left[\lambda x+(\lambda-1)y\right]+e(t)\right\}^{\frac{1}{3}} \\ &= \lambda^{\frac{1}{3}}\left\{\frac{e(t)}{e^*}\left[x+(1-\frac{1}{\lambda})y\right]+\frac{1}{\lambda}e(t)\right\}^{\frac{1}{3}} \end{aligned}$$

$$= \lambda^{\frac{1}{3}} \left[\frac{e(t)}{e^*} x + (1 - \frac{1}{\lambda}) \frac{e(t)}{e^*} y + \frac{1}{\lambda} e(t) \right]^{\frac{1}{3}}$$
$$\geq \lambda^{\frac{1}{3}} \left[\frac{e(t)}{e^*} x + (1 - \frac{1}{\lambda}) \frac{e(t)}{e^*} e^* + \frac{1}{\lambda} e(t) \right]^{\frac{1}{3}}$$
$$= \lambda^{\frac{1}{3}} \left[\frac{e(t)}{e^*} x + e(t) \right]^{\frac{1}{3}} = \varphi(\lambda) f(t, x).$$

Remark 5. If $b = 0$, we can get the uniqueness of positive solutions for the problem (1.1) by using Corollary 1 (see Theorem 3.1 with $\lambda = 1$ in [23]). If $b > 0$, we can not obtain the similar results by using the corresponding fixed point theorems in [24,23].

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