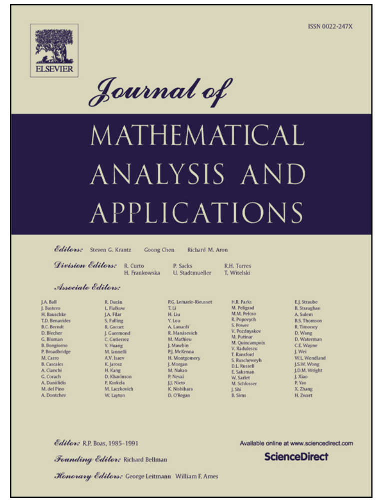

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A parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with logistic source

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Abstract

This paper deals with the parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with logistic source

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w) + ru - \mu u^2, & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \end{cases}$$

under no-flux boundary conditions in bounded domain with smooth boundary, where $\chi, \xi, \alpha, \beta, \gamma, \delta, r$ and μ are assumed to be positive.

When $\Omega\subseteq\mathbb{R}^3$, $D(u)$ is assumed to satisfy $D(0)>0,\ D(u)\geq c_D u^{m-1}$ with $m\geq 1$ and $c_D>0$, it is proved that if $\chi\alpha-\xi\gamma>0$ and $\mu=\frac{1}{3}(\chi\alpha-\xi\gamma)$, then for any given $u_0\in W^{1,\infty}(\Omega)$, the system possesses a global and bounded classical solution. For the case where $D(u)\equiv 1$ and $n\geq 3$, the convergence rate of the solution is established. When the random motion of the chemotactic species is neglected i.e. $(D(u)\equiv 0)$ and $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a convex domain, boundedness and the finite time blow up of the solution are investigated.

Keywords: Chemotaxis; Boundedness; Blow up; Asymptotic behavior; Logistic source

AMS(2010) Subject Classification: 92C17; 34K12; 35K55; 35B40; 35B44

1 Introduction

Chemotaxis is an interesting phenomenon which is used to measure the movement of cells in response to chemical substance. In 1970, Keller and Segel [8] (see also [7, 9]) introduced a model to describe the collective behavior of cells type which can be read as follows

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + f(u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v + u - v, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ \tau v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the density of cells and $v(x,t)$ represents the concentration of the chemoattractant. The function $f:[0,\infty)\to\mathbb{R}$ is smooth and fulfills $f(0)>0$. $\chi>0$ is a parameter referred to as chemosensitivity.

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Recently, there is an increasing interest in studying the Keller-Segel model and the qualitative analysis of the model is worthwhile and challenging. In the absence of the logistic source (i.e. $f(u) \equiv 0$) for system (1.1), the knowledge appears to be rather complete. For instance, if $D(u) \equiv 1$, it was shown in [31] that the system (1.1) admits a unique global solution under the condition $n = 1$. Nagai (see [28–30]) found a critical mass which determines the behavior of the solution when $n = 2$. More precisely, there is a threshold value $m_c > 0$ such that the solution to (1.1) exists globally and remains bounded if the initial mass $\|u_0\|_{L^1(\Omega)} < m_c$, whereas finite time blow up occurs when $\|u_0\|_{L^1(\Omega)} > m_c$. Additionally, when $n \geq 3$, relying on a Lyapunov function, Winkler [42] established the existence of radially symmetric solution blowing up in finite time with proper initial conditions.

In view of the underlying biological background, cell motility should be regarded as movement in a porous medium, accordingly, the cell movement can be described by a nonlinear function $D(u)$. There have been many results about whether the solutions are global bounded or blow up, the readers can refer to [2, 10, 14, 34, 38] and the references therein.

In fact, the blow up phenomenon of the solution is very extreme in practical applications. For this reason, many scholars investigate the chemotaxis model (1.1) with logistic source function $f(u)$ which is expected to prevent the blow up of the solution. For example, when the function $f(u)$ fulfills $f(0) \geq 0$ and $f(u) \leq a - \mu u^2$ for all $u \geq 0$ with some $a \geq 0$ and $\mu > 0$, for the case where $\tau = 0$ and $D(u) \equiv 1$, the main results in [36] showed the prevention of blow up under the conditions $n \leq 2$, $\mu > 0$ or $n \geq 3$ and $\mu > \frac{n-2}{n}\chi$ with $\chi > 0$. Moreover, when $\tau = 0$ and $D(u) \geq cu^m$ holds for all $u \geq 0$ with some $c > 0$ and $m \geq 1$, Wang et al. [45] established the boundedness and large time asymptotic behavior of the solution to system (1.1). Under the assumptions $\tau = 1$, $D(u) \equiv 1$ and Ω is a smooth bounded convex domain, Winkler [43] showed that sufficiently large μ ensures the global existence and boundedness of solutions when $n \geq 3$. Furthermore, in [39], Winkler investigated the following chemotaxis model without random motion of the cells

$$\begin{cases} u_t = -\nabla \cdot (u\nabla v) + ru - \mu u^2, & x \in \Omega, \ t > 0, \\ 0 = \Delta v + u - v, & x \in \Omega, \ t > 0 \end{cases}$$

in one dimensional case. For $\mu > 1$, the corresponding solution remains bounded. For $\mu < 1$, the solution blows up in finite time. Lankeit [25] considered the system in a ball for arbitrary spatial dimension n. Moreover, under the assumption of Ω being convex, Kang and Stevens [15] extended the recent results given by Winkler [39] and Lankeit [25]. For more results on the classical Keller-Segel model and its variants, we refer the readers to [1, 11, 12, 38, 40, 41, 46].

The main concern of the above Keller-Segel model is chemoattraction, however, in practical application, chemorepulsion is also involved in many biological processes, and can form various interesting biological patterns (see [26,32]). Let w be a secondary chemical substance as a chemorepellent which leads to the repulsion migration of cells, then system (1.1) can be directly generalized as the following attraction-repulsion chemotaxis model

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w) + f(u), & x\in\Omega,\ t>0, \\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0, \\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

$u = u(x,t)$ denotes the density of the cells population; $v = v(x,t)$ and $w = w(x,t)$ represent the concentration of the chemoattractant and chemorepellent, which lead to the attractive movement and the repulsion migration of cells. The function $f : [0,\infty) \to \mathbb{R}$ is smooth and fulfills $f(0) > 0$. $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta > 0$ are positive parameters. $\chi > 0$ and $\xi > 0$ measure the strength of the attraction and repulsion, respectively.

When $f(u) \equiv 0$ and $D(u) \equiv 1$, the main results in [6, 24] showed that the system (1.2) with $n = 1$ and $\tau = 1$ possesses a unique global bounded solution; under the assumptions $\Omega \subset \mathbb{R}^2$ and repulsion dominates (i.e. $\xi\gamma - \chi\alpha > 0$), Tao and Wang [33] proved that the model (1.2) admits a unique global solution, and Liu [21] improved the result of [33]; when $\Omega \subset \mathbb{R}^2$, $\tau = 0$ and attraction dominates (i.e. $\xi\gamma - \chi\alpha < 0$), all solutions to (1.2) are global in time if $\|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi\alpha-\xi\gamma}$ (see [3]), whereas finite time blow up occurs if $\|u_0\|_{L^1(\Omega)} > \frac{8\pi}{\chi\alpha-\xi\gamma}$ and $\delta \geq \beta$ (see [16]); under the assumption that repulsion cancels attraction (i.e. $\xi\gamma - \chi\alpha = 0$), Lin et al. [18] (see also [17]) proved the global existence of classical solution in two or three dimensions, as well as the large time asymptotic behavior.

In the case where $f(u)$ fulfills $f(u) = ru - \mu u^2$, where r and μ are positive constants. It is known that for $\tau = 0$ and $D(u) \equiv 1$, all solutions of problem (1.2) are bounded provided that $n \leq 2$, $\mu > 0$ or $n \geq 3$, $\mu > \frac{n-2}{n}(\chi\alpha - \xi\gamma)$ (see [47]); Wang [44] proved that if $\tau = 0$ and $D(u) \geq cu^m$ for all $u \geq 0$ holds with some $c > 0$ and $m \geq 1$, system (1.2) possesses a unique global bounded classical solution provided that $\mu > \mu^*$, where

$$\mu^* = \begin{cases} \frac{n-2}{n}(\chi\alpha - \xi\gamma), & \text{if } m \leq 2 - \frac{2}{n}, \\ 0, & \text{if } m > 2 - \frac{2}{n}. \end{cases}$$

Furthermore, for more results on the attraction-repulsion chemotaxis system with (without) logistic source, we refer the readers to [4, 5, 19, 20, 22, 23].

In this paper, we consider the following parabolic-elliptic-elliptic attraction-repulsion chemotaxis with logistic source

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w) + ru - \mu u^2, & x \in \Omega,\ \ t > 0, \\ 0 = \Delta v + \alpha u - \beta v, & x \in \Omega,\ \ t > 0, \\ 0 = \Delta w + \gamma u - \delta w, & x \in \Omega,\ \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

Our first aim is to discuss the effect of the diffusion exponent and logistic source for the solutions of model (1.3). To this end, we suppose that the diffusion function $D(u)$ satisfies

$$D(u) \in C^2([0,\infty)), \quad D(0) > 0, \tag{1.4}$$

and

$$D(u) \geq c_D u^{m-1} \ \text{ for all } \ u > 0, \tag{1.5}$$

where $c_D > 0$ and $m \geq 1$.

Motivated by the arguments in [44, 47] and the method in [12, 15], the present work focuses on the analysis of (1.3) under the assumptions $\Omega \subset \mathbb{R}^3$ and $\mu = \frac{1}{3}(\chi\alpha - \xi\gamma)$. Compared to [12, 15], the main obstacle in this paper is that we can not obtain the boundedness of $\|u\|_{L^{\frac{3}{2}}(\Omega)}$ directly. However, we can estimate $\|u\|_{L^{\frac{3}{2}}(\Omega)}$ based on the boundedness of $\|u\|_{L^{\frac{3}{2}-\theta}(\Omega)}$ (see Lemma 3.3).

Our first result reads as follows:

Theorem 1.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Suppose that χ, ξ, α, β, γ, δ and μ are given positive parameters satisfying

$$\chi\alpha - \xi\gamma > 0, \ \text{ and } \ \mu = \frac{1}{3}(\chi\alpha - \xi\gamma). \tag{1.6}$$

Assume that (1.4) and (1.5) hold, then for any nonnegative $u_0 \in W^{1,\infty}(\Omega)$, the system (1.3) possesses a unique global classical solution (u, v, w) which is uniformly bounded in $\Omega \times (0, \infty)$.

Remark 1.1. Our result in this paper is an improvement of the result in [44]. Moreover, for the special case $m = 1$, Theorem 1.1 extends the result in [47].

Remark 1.2. Our result in this paper, together with the previous results in [44], show that the system (1.3) possesses a unique global bounded classical solution for any $\mu \geq \mu^*$ when $n = 3$, where

$$\mu^* = \begin{cases} \frac{1}{3}(\chi\alpha - \xi\gamma), & \text{if } \ m \leq \frac{4}{3}, \\ 0, & \text{if } \ m > \frac{4}{3}. \end{cases}$$

Remark 1.3. We have to leave an open question whether the solution to (1.3) still remains bounded when $\chi\alpha - \xi\gamma > 0$ and $\mu = \frac{n-2}{n}(\chi\alpha - \xi\gamma)$ with $n \geq 4$.

For the case where $D(u) \equiv 1$, Zhang and Li [47] established the asymptotic behavior of the solution to system (1.3), but the convergence rate of the solution is still unknown. Thus, the second aim is to explore the convergence rate of the solution to (1.3) under the assumptions that $D(u) \equiv 1$ and $\mu > \max\left\{\frac{n-2}{n}(\chi\alpha - \xi\gamma), \frac{r\chi^2\alpha^2}{8\mu\beta} + \frac{r\gamma^2\xi^2}{8\mu\delta}\right\}$.

In order to prove our main result in this direction, similar to [13, 35], we construct

$$F(t) := \int_\Omega \left(u - \frac{r}{\mu} - \frac{r}{\mu}\ln\frac{\mu u}{r}\right)$$

which acts as a Lyapunov functional for (1.3). Relying on an estimate of the corresponding energy inequality, we can first obtain the convergence of (u, v, w) to $\left(\frac{r}{\mu}, \frac{\alpha r}{\beta\mu}, \frac{\gamma r}{\delta\mu}\right)$ in $L^2(\Omega)$ as well as in $L^\infty(\Omega)$ (Lemma 4.2 and Lemma 4.3). Finally, we establish the convergence rate of (u, v, w) by means of the Gagliardo-Nirenberg inequality.

The main result in this direction can be stated as follows.

Theorem 1.2. Let $n \geq 3$ and assume that $D(u) \equiv 1$ and $\mu > \max\left\{\frac{n-2}{n}(\chi\alpha - \xi\gamma), \frac{r\chi^2\alpha^2}{8\mu\beta} + \frac{r\gamma^2\xi^2}{8\mu\delta}\right\}$. Then for any initial data $u_0 \in C(\overline{\Omega})$, one can find two positive constants c and λ such that the classical solution of (1.3) satisfies

$$\left\|u - \frac{r}{\mu}\right\|_{L^\infty(\Omega)} + \left\|v - \frac{\alpha r}{\beta\mu}\right\|_{L^\infty(\Omega)} + \left\|w - \frac{\gamma r}{\delta\mu}\right\|_{L^\infty(\Omega)} \leq ce^{-\lambda t}, \quad t > 0. \tag{1.7}$$

To the best of our knowledge, very few results related to the system (1.3) without random motion of cells (i.e. $D(u) \equiv 0$) seem to be known. With regard to this, the third goal in this paper is to make a substantial step forward towards the behavior of the solution to (1.3) under the assumption $D(u) \equiv 0$.

When $D(u) \equiv 0$, we can not prove that there is a classical solution to system (1.3). However, we can prove that the model possesses at least one nonnegative regular solution which is defined as follows:

Definition 1.1. Let $T \in (0, \infty)$, nonnegative function (u, v, w) is called a regular solution of (1.3) with $D(u) \equiv 0$ if

(1) for any $q > n$,

$$\begin{aligned}
u &\in L^q((0,T), W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega),\\
v &\in L^q((0,T), W^{2,q}(\Omega)) \cap L^\infty((0,T)\times\Omega),\\
w &\in L^q((0,T), W^{2,q}(\Omega)) \cap L^\infty((0,T)\times\Omega);
\end{aligned}$$

(2) the integral equations

$$\begin{aligned}
-\int_0^T\int_\Omega u\varphi_t - \int_0^T\int_\Omega u_0\varphi(0) =& \chi\int_0^T\int_\Omega u\nabla v\cdot\nabla\varphi - \xi\int_0^T\int_\Omega u\nabla w\cdot\nabla\varphi\\
&+ r\int_0^T\int_\Omega u\varphi - \mu\int_0^T\int_\Omega u^2\varphi,
\end{aligned}$$

$$\int_0^T\int_\Omega \Delta v\zeta - \int_0^T\int_\Omega \beta v\zeta + \int_0^T\int_\Omega \alpha u\zeta = 0,$$

and

$$\int_0^T \int_\Omega \Delta w\eta - \int_0^T \int_\Omega \delta w\eta + \int_0^T \int_\Omega \gamma u\eta = 0$$

hold for all $\varphi \in C_0^\infty(\overline{\Omega} \times (0,T))$, $\zeta \in C_0^\infty(\overline{\Omega} \times (0,T))$ and $\eta \in C_0^\infty(\overline{\Omega} \times (0,T))$.

In view of the underlying biological background, we find it worthwhile to investigate the solution to system (1.3) when $n \geq 2$ and our results in this case can be stated as follows.

Theorem 1.3. Let $n \geq 2$, $D(u) \equiv 0$ and $\Omega \subset \mathbb{R}^n$ be a convex domain with smooth boundary. Suppose that $u_0 \in W^{1,\infty}(\Omega)$, then for any $q > n$, there is a maximal time $T_0 \in (0,\infty)$ such that the regular solution (u,v,w) to system (1.3) exists for any time $t < T_0$. Moreover:

(i) If $\mu > \mu_*$, where

$$\mu_* = \begin{cases} 0 & \text{if } \chi\alpha - \xi\gamma \leq 0, \\ \chi\alpha - \xi\gamma & \text{if } \chi\alpha - \xi\gamma > 0, \end{cases}$$

then the corresponding solution of system (1.3) is uniformly bounded.

(ii) If $\chi\alpha - \xi\gamma > 0$, $\mu < \chi\alpha - \xi\gamma$ and the initial data $\|u_0\|_{L^q(\Omega)}$ is sufficiently large, then the solution of (1.3) blows up in finite time.

Remark 1.4. Due to the lack of an effective way, it is not clear about the behavior of the solution when $\mu = \mu_*$ and we have to leave it as an open problem.

The rest of this paper is organized as follows. In the next section, we give some preliminary inequalities which are important for our proofs. Some estimates of the solution and the proof of Theorem 1.1 are shown in Section 3. In Section 4, we consider the case $D(u) \equiv 1$ and obtain the convergence rate of the solution. Finally, we give the proof of Theorem 1.3 in Section 5.

2 Preliminaries

In this section, we recall some preliminary estimates and some results which will be used in our proof. The following statement on local existence of classical solution to (1.3) has already been proven in [44].

Lemma 2.1. Suppose that $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary and $u_0 \in W^{1,\infty}(\Omega)$ is a non-negative function. Assume that $D(u)$ satisfies (1.4) and (1.5). Then problem (1.3) has a unique local-in-time non-negative classical solution

$$u \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$
$$v \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$
$$w \in C^0(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$

where T_{max} denotes the maximal existence time. Moreover, if $T_{max} < \infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \text{ as } t \to T_{max}.$$

In the proof of the main result, we will frequently use the following version of the Gagliardo-Nirenberg inequality, for details we refer the readers to [33].

Lemma 2.2. Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $r \in (0,p)$ and $\psi \in W^{1,2}(\Omega) \bigcap L^r(\Omega)$. Then there exists a positive constant C_{GN} depending on Ω, p, and r such that

$$\|\psi\|_{L^p(\Omega)} \le C_{GN}(\|\nabla\psi\|^a_{L^2(\Omega)}\|\psi\|^{1-a}_{L^r(\Omega)} + \|\psi\|_{L^r(\Omega)}) \tag{2.1}$$

holds with $a \in (0,1)$ fulfilling

$$\frac{1}{p} = a\left(\frac{1}{2} - \frac{1}{n}\right) + (1-a)\frac{1}{r},$$

that is

$$a = \frac{\frac{1}{r} - \frac{1}{p}}{\frac{1}{r} + \frac{1}{n} - \frac{1}{2}}.$$

Lemma 2.3. (see [47]) Let (u,v,w) be a nonnegative solution of (1.3). Then for any $\varepsilon > 0$, there is a constant $C := C(p,\varepsilon,\|u_0\|_{L^1(\Omega)}) > 0$ fulfilling

$$\int_\Omega w^{p+1} \le \varepsilon \int_\Omega u^{p+1} + C, \;\; p > 0 \tag{2.2}$$

for all $t \in (0, T_{max})$.

3 Proof of Theorem 1.1

The starting point of our analysis is the following inequality.

Lemma 3.1. Suppose that $D(u)$ satisfies (1.4) and (1.5) with $m \ge 1$, then for any $p \in [1, +\infty)$, the solution of (1.3) fulfills

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \le & r\int_\Omega u^p + \frac{p-1}{p}\xi\delta\int_\Omega u^p w \\ & - \left(\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)\right)\int_\Omega u^{p+1}\end{aligned} \tag{3.1}$$

for all $t \in (0, T_{max})$.

Proof. Multiplying both sides of the first equation in (1.3) by u^{p-1} and integrating by parts, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p = \int_\Omega u^{p-1}u_t = & \int_\Omega u^{p-1}\nabla\cdot(D(u)\nabla u) - \int_\Omega u^{p-1}\nabla\cdot(\chi u\nabla v) \\
& + \int_\Omega u^{p-1}\nabla\cdot(\xi u\nabla w) + r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
= & -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2 + (p-1)\int_\Omega \chi u^{p-1}\nabla u\cdot\nabla v \\
& -(p-1)\int_\Omega \xi u^{p-1}\nabla u\cdot\nabla w + r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
= & -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2 + \frac{(p-1)}{p}\int_\Omega \chi\nabla u^p\cdot\nabla v \\
& -\frac{(p-1)}{p}\int_\Omega \xi\nabla u^p\cdot\nabla w + r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \\
= & -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2 - \frac{(p-1)}{p}\int_\Omega u^p(\chi\Delta v - \xi\Delta w) \\
& + r\int_\Omega u^p - \mu\int_\Omega u^{p+1}
\end{aligned}$$

for all $t\in(0,T_{max})$. Invoking the second and the third equations in (1.3) and employing (1.4) and (1.5), we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le & -\frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 - \frac{(p-1)\beta\chi}{p}\int_\Omega u^p v + \frac{(p-1)}{p}(\chi\alpha-\xi\gamma)\int_\Omega u^{p+1} \\
& + \frac{(p-1)\xi\delta}{p}\int_\Omega u^p w + r\int_\Omega u^p - \mu\int_\Omega u^{p+1}
\end{aligned}$$

for all $t\in(0,T_{max})$. According to the fact that $v>0$, we deduce

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \le & -\left(\mu - \frac{(p-1)}{p}(\chi\alpha-\xi\gamma)\right)\int_\Omega u^{p+1} \\
& + \frac{(p-1)\xi\delta}{p}\int_\Omega u^p w + r\int_\Omega u^p
\end{aligned}$$

for all $t\in(0,T_{max})$. □

Lemma 3.2. Let $n=3$. Suppose that (1.4), (1.5) and (1.6) hold. Then for any $1\le p<\frac{3}{2}$ there exists a constant $C(p)>0$ such that

$$\int_\Omega u^p \le C(p) \ \text{ for all } \ t\in(0,T_{max}). \tag{3.2}$$

Proof. From (1.6), we deduce

$$\lambda := \mu - \frac{(p-1)}{p}(\chi\alpha-\xi\gamma) > 0 \ \text{ if } \ p\in\left[1,\frac{3}{2}\right).$$

According to the Young's inequality, there exist positive constants $\frac{\lambda}{2}$ and $C=C(\lambda)$ such that

$$\frac{(p-1)\xi\delta}{p}\int_\Omega u^p w \le \frac{\lambda}{2}\int_\Omega u^{p+1} + C\int_\Omega w^{p+1} \ \text{ for all } \ t\in(0,T_{max}). \tag{3.3}$$

Inserting (3.3) back into (3.1), we get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \le -\frac{\lambda}{2}\int_\Omega u^{p+1} + C\int_\Omega w^{p+1} + r\int_\Omega u^p$$

for all $t \in (0, T_{max})$. Employing (2.2) with $\varepsilon = \frac{\lambda}{4C}$, we find

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le -\frac{\lambda}{4}\int_\Omega u^{p+1} + r\int_\Omega u^p + C_1 \text{ for all } t \in (0, T_{max}), \tag{3.4}$$

where C_1 is a positive constant. In light of the Hölder's inequality, we discover

$$\frac{\left(\int_\Omega u^p\right)^{\frac{p+1}{p}}}{|\Omega|^{\frac{1}{p}}} \le \int_\Omega u^{p+1}. \tag{3.5}$$

A combination of (3.4) and (3.5) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le -\frac{\lambda}{4|\Omega|^{\frac{1}{p}}}\left(\int_\Omega u^p\right)^{\frac{p+1}{p}} + r\int_\Omega u^p + C_1 \text{ for all } t \in (0, T_{max}).$$

Thus a standard ODE comparison argument implies the boundedness of $\int_\Omega u^p$ on $(0, T_{max})$. $\square$

Lemma 3.3. Let $n = 3$. Assume that (1.4), (1.5) and (1.6) hold. Then there is a constant $C = C(\Omega, m, \mu, \xi, \delta)$ such that

$$\int_\Omega u^p \le C \text{ for all } t \in (0, T_{max}) \text{ with } p = \frac{3}{2}. \tag{3.6}$$

Proof. From (1.6) we see

$$\mu - \frac{(p-1)}{p}(\chi\alpha - \xi\gamma) = 0 \text{ if } p = \frac{3}{2}.$$

Thus we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \le (r+1)\int_\Omega u^p + \frac{p-1}{p}\xi\delta\int_\Omega u^p w$$

for all $t \in (0, T_{max})$. Using the Young's inequality once more, we conclude

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \\ \le &(r+1)\int_\Omega u^p + C_1\int_\Omega u^{\frac{3p+2}{3}} + C_2\int_\Omega w^{\frac{3p+2}{2}}\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{max})$ with some certain $C_1 > 0$ and $C_2 > 0$. In view of the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}(1+r)\int_\Omega u^p =& (1+r)\|u^{\frac{m+p-1}{2}}\|^{\frac{2p}{p+m-1}}_{L^{\frac{2p}{p+m-1}}(\Omega)} \\ \le& (1+r)C_{GN}\Big(\|\nabla u^{\frac{m+p-1}{2}}\|^{a_1\frac{2p}{p+m-1}}_{L^2(\Omega)}\|u^{\frac{m+p-1}{2}}\|^{(1-a_1)\frac{2p}{p+m-1}}_{L^{\frac{2\left(\frac{3}{2}-\theta\right)}{p+m-1}}(\Omega)} \\ &+ \|u^{\frac{m+p-1}{2}}\|^{\frac{2p}{p+m-1}}_{L^{\frac{2\left(\frac{3}{2}-\theta\right)}{p+m-1}}(\Omega)}\Big)\end{aligned} \tag{3.8}$$

for all $t \in (0, T_{max})$, where $a_1 = \frac{\frac{p+m-1}{3-2\theta} - \frac{p+m-1}{2p}}{\frac{1}{3} + \frac{p+m-1}{3-2\theta} - \frac{1}{2}} \in (0, 1)$ and $\theta > 0$ is sufficiently small. Due to $m \ge 1$, we have

$$\frac{2p}{p+m-1}a_1 = \frac{\frac{2p}{3-2\theta} - 1}{\frac{1}{3} + \frac{p+m-1}{3-2\theta} - \frac{1}{2}} = \frac{\frac{2p}{3-2\theta} - 1}{\frac{1}{3} + \frac{m-1}{3-2\theta} + \frac{p}{3-2\theta} - \frac{1}{2}} < 2. \tag{3.9}$$

In view of (3.2), there exists a positive constant C_3 satisfying

$$\|u^{\frac{m+p-1}{2}}\|^{\frac{2p}{p+m-1}}_{L^{\frac{2(\frac{3}{2}-\theta)}{p+m-1}}(\Omega)} < C_3. \tag{3.10}$$

Collecting (3.8)-(3.10) along with the Young's inequality, we can find $\varepsilon_1 > 0$ and $C_4 > 0$ such that

$$(1+r)\int_\Omega u^p \le \varepsilon_1\|\nabla u^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)} + C_4 \text{ for all } t\in(0,T_{max}). \tag{3.11}$$

For $C_1\int_\Omega u^{\frac{3p+2}{3}}$: employing the Gagliardo-Nirenberg inequality again, we arrive at

$$\begin{aligned} C_1\int_\Omega u^{\frac{3p+2}{3}} = \|u^{\frac{m+p-1}{2}}\|^{\frac{2(3p+2)}{3(m+p-1)}}_{L^{\frac{2(3p+2)}{3(m+p-1)}}(\Omega)} \le & C_1C_{GN}\|\nabla u^{\frac{m+p-1}{2}}\|^{\frac{2(3p+2)}{3(m+p-1)}a_2}_{L^2(\Omega)}\|u^{\frac{m+p-1}{2}}\|^{(1-a_2)\frac{2(3p+2)}{3(m+p-1)}}_{L^{\frac{2(\frac{3}{2}-\theta)}{m+p-1}}(\Omega)} \\ & + \|u^{\frac{m+p-1}{2}}\|^{\frac{2(3p+2)}{3(m+p-1)}}_{L^{\frac{2(\frac{3}{2}-\theta)}{m+p-1}}(\Omega)} \end{aligned} \tag{3.12}$$

for all $t\in(0,T_{max})$, where $a_2 = \frac{\frac{m+p-1}{3-2\theta}-\frac{3(m+p-1)}{2(3p+2)}}{\frac{m+p-1}{3-2\theta}+\frac{1}{3}-\frac{1}{2}} \in (0,1)$. We have

$$\frac{2(3p+2)}{3(m+p-1)}a_2 = \frac{\frac{2p}{3-2\theta}-1+\frac{4}{3(3-2\theta)}}{\frac{p}{3-2\theta}-\frac{1}{2}+\frac{1}{3}+\frac{m-1}{3-2\theta}} < 2$$

due to $m \ge 1$ and $\theta > 0$ sufficiently small. Thus, using the Young's inequality along with the boundedness of $\|u^{\frac{m+p-1}{2}}\|_{L^{\frac{2(\frac{3}{2}-\theta)}{m+p-1}}(\Omega)}$, we can find $\varepsilon_2 > 0$ and $C_5 > 0$ such that

$$C_1\int_\Omega u^{\frac{3p+2}{3}} \le \varepsilon_2\|\nabla u^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)} + C_5 \text{ for all } t\in(0,T_{max}). \tag{3.13}$$

For $C_2\int_\Omega w^{\frac{3p+2}{2}}$: after multiplying both sides of the third equation in (1.3) by $w^{\frac{3p}{2}}$ and integrating by parts over Ω, we have

$$\begin{aligned} \frac{24p}{(3p+2)^2}\int_\Omega|\nabla w^{\frac{3p+2}{4}}|^2 + \delta\int_\Omega w^{\frac{3p+2}{2}} =& \gamma\int_\Omega uw^{\frac{3p}{2}} \\ \le & C_6\int_\Omega u^{\frac{3p+2}{3}} + C_7\int_\Omega w^{\frac{3p(3p+2)}{2(3p-1)}} \end{aligned} \tag{3.14}$$

holds for all $t\in(0,T_{max})$ with certain $C_6>0$ and $C_7>0$. According to the Gagliardo-Nirenberg inequality, we calculate

$$\begin{aligned} C_7\int_\Omega w^{\frac{3p(3p+2)}{2(3p-1)}} = C_7\|w^{\frac{3p+2}{4}}\|^{\frac{6p}{3p-1}}_{L^{\frac{6p}{3p-1}}(\Omega)} \le & C_7C_{GN}\|\nabla w^{\frac{3p+2}{4}}\|^{\frac{6p}{3p-1}a_3}_{L^2(\Omega)}\|w^{\frac{3p+2}{4}}\|^{(1-a_3)\frac{6p}{3p-1}}_{L^{\frac{4(\frac{3}{2}-\theta)}{3p+2}}(\Omega)} \\ & + \|w^{\frac{3p+2}{4}}\|^{\frac{6p}{3p-1}}_{L^{\frac{4(\frac{3}{2}-\theta)}{3p+2}}(\Omega)} \end{aligned} \tag{3.15}$$

for all $t\in(0,T_{max})$, where $a_3 = \frac{\frac{3p+2}{6-4\theta}-\frac{3p-1}{6p}}{\frac{1}{3}+\frac{3p+2}{6-4\theta}-\frac{1}{2}} \in (0,1)$. Thanks to $p=\frac{3}{2}$ and $\theta>0$ is sufficiently small, we immediately obtain

$$\frac{6p}{3p-1}a_3 = \frac{\frac{(3p+2)6p}{(6-4\theta)(3p-1)}-1}{\frac{1}{3}+\frac{3p+2}{6-4\theta}-\frac{1}{2}} < 2. \tag{3.16}$$

Applying the classical elliptic L^p estimate, it follows that

$$\|w^{\frac{3p+2}{4}}\|_{L^{\frac{4(\frac{3}{2}-\theta)}{3p+2}}(\Omega)} \leq C\|u^{\frac{3p+2}{4}}\|_{L^{\frac{4(\frac{3}{2}-\theta)}{3p+2}}(\Omega)}. \tag{3.17}$$

Collecting (3.15)-(3.17) and applying Young's inequality again, one can find a positive constant $C_8 > 0$ such that

$$C_7 \int_\Omega w^{\frac{3p(3p+2)}{2(3p-1)}} \leq \frac{24p}{(3p+2)^2}\|\nabla w^{\frac{3p+2}{4}}\|^2_{L^2(\Omega)} + C_8 \text{ for all } t \in (0, T_{max}). \tag{3.18}$$

Thus a combination of (3.14) and (3.18) yields

$$\int_\Omega w^{\frac{3p+2}{2}} \leq \frac{C_6}{\delta}\int_\Omega u^{\frac{3p+2}{3}} + \frac{C_8}{\delta} \text{ for all } t \in (0, T_{max}). \tag{3.19}$$

Inserting (3.11), (3.13) and (3.19) into (3.7) with $\varepsilon_1 + (1 + \frac{C_6C_2}{\delta})\varepsilon_2 = \frac{4c_D(p-1)}{(m+p-1)^2}$, we get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \leq C_9 \text{ for all } t \in (0, T_{max})$$

with some constant $C_9 > 0$, this confirms (3.6). □

Lemma 3.4. Let the same assumptions as that in Lemma 3.3 hold. Then there exists $\sigma > 0$ sufficiently small such that for any $p \in \left(\frac{3}{2}, \frac{3}{2} + \sigma\right]$, we have $C(p) > 0$ satisfying

$$\int_\Omega u^p \leq C(p) \text{ for all } t \in (0, T_{max}). \tag{3.20}$$

Proof. Indeed, according to the definition of μ in (1.6), we find $\frac{(p-1)}{p}(\chi\alpha - \xi\gamma) - \mu > 0$ when $p > \frac{3}{2}$. Recalling (3.1) and employing the Young's inequality, one can find a positive constant C_1 satisfying

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \\ \leq& 2\left(\frac{p-1}{p}(\chi\alpha - \xi\gamma) - \mu\right)\int_\Omega u^{p+1} + C_1 \text{ for all } t \in (0, T_{max}). \end{aligned} \tag{3.21}$$

By the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \int_\Omega u^{p+1} = \|u^{\frac{m+p-1}{2}}\|^{\frac{2(p+1)}{m+p-1}}_{L^{\frac{2(p+1)}{m+p-1}}(\Omega)} \leq& C_{GN}\|\nabla u^{\frac{m+p-1}{2}}\|^{a\frac{2(p+1)}{m+p-1}}_{L^2(\Omega)}\|u^{\frac{m+p-1}{2}}\|^{(1-a)\frac{2(p+1)}{m+p-1}}_{L^{\frac{3}{m+p-1}}(\Omega)} \\ &+ \|u^{\frac{m+p-1}{2}}\|^{\frac{2(p+1)}{m+p-1}}_{L^{\frac{3}{m+p-1}}(\Omega)} \end{aligned} \tag{3.22}$$

hold for all $t \in (0, T_{max})$, where $a = \frac{\frac{m+p-1}{3} - \frac{m+p-1}{2(p+1)}}{\frac{m+p-1}{3} + \frac{1}{3} - \frac{1}{2}} \in (0, 1)$. We immediately get

$$\frac{2(p+1)}{m+p-1}a = \frac{\frac{2(p+1)}{3} - 1}{\frac{m+p}{3} - \frac{1}{2}} \leq 2 \tag{3.23}$$

thanks to $m \geq 1$.

• If $m = 1$, we have $\frac{2(p+1)}{m+p-1}a = \frac{\frac{2(p+1)}{3}-1}{\frac{m+p}{3}-\frac{1}{2}} = 2$. Using (3.22) along with the boundness of $\int_\Omega u^{\frac{3}{2}}$, it follows

$$\int_\Omega u^{p+1} = \|u^{\frac{m+p-1}{2}}\|^{\frac{2(p+1)}{m+p-1}}_{L^{\frac{2(p+1)}{m+p-1}}(\Omega)} \le C_2\|\nabla u^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)} + C_3 \tag{3.24}$$

for all $t \in (0, T_{max})$ with certain $C_2 > 0$ and $C_3 > 0$. Since

$$\lim_{p\to\frac{3}{2}} \frac{(p-1)}{p}(\chi\alpha - \xi\gamma) - \mu = 0,$$

one can find some $\sigma > 0$ sufficiently small satisfying

$$2\left(\frac{(p-1)}{p}(\chi\alpha - \xi\gamma) - \mu\right) C_2 \le \frac{4c_D(p-1)}{(m+p-1)^2} \text{ if } p \in \left(\frac{3}{2}, \frac{3}{2} + \sigma\right].$$

From this, (3.21) and (3.24), we infer that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le C_4 \text{ for all } t \in (0, T_{max}) \tag{3.25}$$

holds with some appropriate positive constant C_4.

• If $m > 1$, then $\frac{2(p+1)}{m+p-1}a = \frac{\frac{2(p+1)}{3}-1}{\frac{m+p}{3}-\frac{1}{2}} < 2$, in view of the Young's inequality and the boundedness of $\int_\Omega u^{\frac{3}{2}}$, we can find some positive constant C_5 satisfying

$$2\left(\frac{(p-1)}{p}(\chi\alpha - \xi\gamma) - \mu\right)\int_\Omega u^{p+1} \le \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 + C_5 \tag{3.26}$$

for all $t \in (0, T_{max})$. A combination of (3.21) and (3.26) yields a positive constant C_6 such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le C_6 \text{ for all } t \in (0, T_{max}). \tag{3.27}$$

From (3.25) and (3.27), we arrive at (3.20). □

Lemma 3.5. Let $n = 3$ and assume that (1.4), (1.5) and (1.6) hold. Then for any $\frac{3}{2} + \sigma < p < +\infty$ with σ provided by Lemma 3.4, there is a positive constant $C(p)$ independent of t such that the solution (u, v, w) of system (1.3) satisfies

$$\int_\Omega u^p \le C(p) \text{ for all } t \in (0, T_{max}). \tag{3.28}$$

Proof. The Young's inequality and (3.1) lead to

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 \\ \le& 2\left(\frac{p-1}{p}(\chi\alpha - \xi\gamma) - \mu\right)\int_\Omega u^{p+1} + C_1\end{aligned} \tag{3.29}$$

for $p > \frac{3}{2} + \sigma$ and all $t \in (0, T_{max})$ with some $C_1 > 0$. Using the Gagliardo-Nirenberg inequality once more, we discover

$$\begin{aligned}\int_\Omega u^{p+1} = \|u^{\frac{m+p-1}{2}}\|^{\frac{2(p+1)}{m+p-1}}_{L^{\frac{2(p+1)}{m+p-1}}(\Omega)} \le& C_{GN}\|\nabla u^{\frac{m+p-1}{2}}\|^{a\frac{2(p+1)}{m+p-1}}_{L^2(\Omega)}\|u^{\frac{m+p-1}{2}}\|^{(1-a)\frac{2(p+1)}{m+p-1}}_{L^{\frac{2p'}{m+p-1}}(\Omega)} \\ &+ \|u^{\frac{m+p-1}{2}}\|^{\frac{2(p+1)}{m+p-1}}_{L^{\frac{2p'}{m+p-1}}(\Omega)},\end{aligned} \tag{3.30}$$

where $p' := \frac{3}{2} + \sigma$ and $a = \frac{\frac{m+p-1}{2p'} - \frac{m+p-1}{2(p+1)}}{\frac{m+p-1}{2p'} + \frac{1}{3} - \frac{1}{2}} \in (0,1)$. It also holds that

$$\frac{2(p+1)}{m+p-1} a = \frac{\frac{2(p+1)}{2p'} - 1}{\frac{m+p}{2p'} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2p'}} < 2 \tag{3.31}$$

due to $m \geq 1$ and $p' > \frac{3}{2}$. Combining (3.30) with (3.31) and applying the Young's inequality along with the boundness of $\int_\Omega u^{p'}$, we can find a positive constant C_2 satisfying

$$2\left(\frac{p-1}{p}(\chi\alpha - \xi\gamma) - \mu\right)\int_\Omega u^{p+1} \leq \frac{4c_D(p-1)}{(m+p-1)^2}\int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 + C_2 \tag{3.32}$$

for all $t \in (0, T_{max})$. Finally, we substitute (3.32) into (3.29) to discover

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \leq C_3 \text{ for all } t \in (0, T_{max}),$$

where $C_3 := C_1 + C_2$, which confirms (3.28). □

Finally, we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. With the aid of Lemma A.1 in [34] and Lemma 3.5, we obtain that u is bounded in $(0, T_{max})$. Thus, we can find a positive constant C independent of t such that

$$\|u\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{max}),$$

which together with Lemma 2.1 shows that $T_{max} = \infty$. Therefore, (u, v, w) is a global bounded classical solution to system (1.3) and the proof of Theorem 1.1 is completed. □

4 Convergence rate for $D(u) \equiv 1$

In this section, we treat the asymptotic behavior of the solution to system (1.3) with $D(u) \equiv 1$. In order to prepare our arguments concerning the large time behavior of the solution, we need to introduce the following property.

Lemma 4.1. (see [27, 47]) Suppose that $D(u) \equiv 1$, $n \geq 3$ and $\mu > \max\left\{0, \frac{n-2}{n}(\chi\alpha - \xi\gamma)\right\}$. Then for any nonnegative initial data $u_0 \in C(\overline{\Omega})$ the system (1.3) possesses a unique classical bounded solution (u, v, w) satisfying

$$\|u\|_{L^\infty(\Omega)} \leq c, \text{ for all } t \in (0, \infty),$$

where $c = c(\|u_0\|_{L^\infty(\Omega)})$. Moreover, there is a constant $\kappa \in (0,1)$ and $K > 0$ such that

$$\|u\|_{C^{\kappa, \frac{\kappa}{2}}(\Omega \times [t, t+1])} \leq K,$$
$$\|v\|_{C^{\kappa, \frac{\kappa}{2}}(\Omega \times [t, t+1])} \leq K,$$
$$\|w\|_{C^{\kappa, \frac{\kappa}{2}}(\Omega \times [t, t+1])} \leq K$$

hold for all $t > 1$.

Next, we can derive the following L^2-estimate of the solution to (1.3) by making full use of the Lyapunov function.

Lemma 4.2. Assume that $n \geq 3$, $D(u) \equiv 1$ and $\mu > \max\left\{\frac{n-2}{n}(\chi\alpha - \xi\gamma), \frac{r\chi^2\alpha^2}{8\mu\beta} + \frac{r\gamma^2\xi^2}{8\mu\delta}\right\}$. Then for any initial data $u_0 \in C^0(\overline{\Omega})$, the corresponding solution of (1.3) fulfills

$$\int_\Omega \left(u - \frac{r}{\mu}\right)^2 \to 0,$$
$$\int_\Omega \left(v - \frac{\alpha r}{\beta\mu}\right)^2 \to 0,$$
$$\int_\Omega \left(w - \frac{\gamma r}{\delta\mu}\right)^2 \to 0$$

as $t \to \infty$.

Proof. We construct a function

$$F(t) := \int_\Omega \left(u - \frac{r}{\mu} - \frac{r}{\mu}\ln\frac{\mu u}{r}\right) \tag{4.1}$$

and it is easy to verify that $s - \frac{r}{\mu} - \frac{r}{\mu}\ln\frac{\mu s}{r} \geq 0$ for all $s > 0$. Thus we have $F(t) \geq 0$. We collect (4.1) and the first equation of (1.3) to see that

$$\begin{aligned}\frac{d}{dt}F(t) &= \int_\Omega \frac{u - \frac{r}{\mu}}{u} u_t \\ &= \int_\Omega \frac{u - \frac{r}{\mu}}{u}(\Delta u - \nabla\cdot(\chi u \nabla v) + \nabla\cdot(\xi u\nabla w) + ru - \mu u^2) \\ &= -\frac{r}{\mu}\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi r}{\mu}\int_\Omega \frac{\nabla u\cdot\nabla v}{u} - \frac{\xi r}{\mu}\int_\Omega \frac{\nabla u\cdot\nabla w}{u} - \mu\int_\Omega\left(u - \frac{r}{\mu}\right)^2 \\ &\leq \frac{r\chi^2}{2\mu}\int_\Omega |\nabla v|^2 + \frac{r\xi^2}{2\mu}\int_\Omega |\nabla w|^2 - \mu\int_\Omega \left(u - \frac{r}{\mu}\right)^2.\end{aligned} \tag{4.2}$$

To estimate $\int_\Omega |\nabla v|^2$, we test the second equation of (1.3) by $v - \frac{r\alpha}{\beta\mu}$ and integrate by parts to compute

$$\begin{aligned}0 &= \int_\Omega \Delta v\left(v - \frac{r\alpha}{\beta\mu}\right) + \int_\Omega \alpha u\left(v - \frac{r\alpha}{\beta\mu}\right) - \int_\Omega \beta v\left(v - \frac{r\alpha}{\beta\mu}\right) \\ &= -\int_\Omega |\nabla v|^2 + \alpha\int_\Omega\left(u - \frac{r}{\mu}\right)\left(v - \frac{r\alpha}{\beta\mu}\right) - \beta\int_\Omega\left(v - \frac{r\alpha}{\beta\mu}\right)^2.\end{aligned} \tag{4.3}$$

Similarly, we have

$$0 = -\int_\Omega |\nabla w|^2 + \gamma\int_\Omega\left(u - \frac{r}{\mu}\right)\left(w - \frac{r\gamma}{\delta\mu}\right) - \delta\int_\Omega\left(w - \frac{r\gamma}{\delta\mu}\right)^2. \tag{4.4}$$

Collecting (4.2)-(4.4) and applying the Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}F(t) \leq &-\mu\int_\Omega\left(u - \frac{r}{\mu}\right)^2 + \frac{r\alpha\chi^2}{2\mu}\int_\Omega\left(u - \frac{r}{\mu}\right)\left(v - \frac{r\alpha}{\beta\mu}\right) - \frac{r\beta\chi^2}{2\mu}\int_\Omega\left(v - \frac{r\alpha}{\beta\mu}\right)^2 \\ &+ \frac{r\gamma\xi^2}{2\mu}\int_\Omega\left(u - \frac{r}{\mu}\right)\left(w - \frac{r\gamma}{\delta\mu}\right) - \frac{r\delta\xi^2}{2\mu}\int_\Omega\left(w - \frac{r\gamma}{\delta\mu}\right)^2 \\ \leq &-\left(\mu - \frac{r\chi^2\alpha^2}{8\mu\beta} - \frac{r\gamma^2\xi^2}{8\mu\delta}\right)\int_\Omega\left(u - \frac{r}{\mu}\right)^2.\end{aligned}$$

Due to $\mu > \max\left\{\frac{n-2}{n}(\chi\alpha - \xi\gamma), \frac{r\chi^2\alpha^2}{8\mu\beta} + \frac{r\gamma^2\xi^2}{8\mu\delta}\right\}$, we have

$$\frac{d}{dt}F(t) \leq -\epsilon \int_\Omega \left(u - \frac{r}{\mu}\right)^2, \tag{4.5}$$

where $\epsilon := \mu - \frac{r\chi^2\alpha^2}{8\mu\beta} - \frac{r\gamma^2\xi^2}{8\mu\delta} > 0$. Integrating (4.5) from $t_0 > 0$ to t, we infer that

$$F(t) - F(t_0) \leq -\epsilon \int_{t_0}^t \int_\Omega \left(u - \frac{r}{\mu}\right)^2 \text{ for all } t > t_0 > 0.$$

Thanks to $F(t) \geq 0$ and the boundedness of u, we get

$$\int_{t_0}^\infty \int_\Omega \left(u - \frac{r}{\mu}\right)^2 \leq \frac{F(t_0)}{\epsilon},$$

this implies

$$\int_\Omega \left(u - \frac{r}{\mu}\right)^2 \to 0 \text{ as } t \to \infty.$$

According to (4.3) and using the Young's inequality, we arrive at

$$\begin{aligned}
\int_\Omega |\nabla v|^2 &= \alpha \int_\Omega \left(u - \frac{r}{\mu}\right)\left(v - \frac{r\alpha}{\beta\mu}\right) - \beta \int_\Omega \left(v - \frac{r\alpha}{\beta\mu}\right)^2 \\
&\leq \frac{\beta}{2} \int_\Omega \left(v - \frac{r\alpha}{\beta\mu}\right)^2 + C(\alpha,\beta) \int_\Omega \left(u - \frac{r}{\mu}\right)^2 - \beta \int_\Omega \left(v - \frac{r\alpha}{\beta\mu}\right)^2 \\
&\leq C(\alpha,\beta) \int_\Omega \left(u - \frac{r}{\mu}\right)^2 - \frac{\beta}{2} \int_\Omega \left(v - \frac{r\alpha}{\beta\mu}\right)^2,
\end{aligned}$$

and hence

$$\frac{\beta}{2} \int_\Omega \left(v - \frac{r\alpha}{\beta\mu}\right)^2 \leq C(\alpha,\beta) \int_\Omega \left(u - \frac{r}{\mu}\right)^2 \to 0 \text{ as } t \to \infty.$$

Similarly, by virtue of (4.4), we have

$$\frac{\delta}{2} \int_\Omega \left(w - \frac{r\gamma}{\delta\mu}\right)^2 \leq C(\gamma,\delta) \int_\Omega \left(u - \frac{r}{\mu}\right)^2 \to 0 \text{ as } t \to \infty.$$

Thereupon, the proof of this lemma is completed. $\square$

Next we prove the uniform convergence.

Lemma 4.3. Let $D(u) \equiv 1$ and $\mu > \max\left\{\frac{n-2}{n}(\chi\alpha - \xi\gamma), \frac{r\chi^2\alpha^2}{8\mu\beta} + \frac{r\gamma^2\xi^2}{8\mu\delta}\right\}$. Then for any initial data $u_0 \in C^0(\overline{\Omega})$, we have

$$\left\|u - \frac{r}{\mu}\right\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty, \tag{4.6}$$

$$\left\|v - \frac{r\alpha}{\beta\mu}\right\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty \tag{4.7}$$

as well as

$$\left\|w - \frac{r\gamma}{\delta\mu}\right\|_{L^\infty(\Omega)} \to 0, \text{ as } t \to \infty. \tag{4.8}$$

Proof. If (4.6) was false, then we could find $l > 0$, $\{t_k\}_{k\in N} \subset (1, \infty)$ and $\{x_k\}_{k\in N} \subset \Omega$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\left|u(x_k, t_k) - \frac{r}{\mu}\right| \geq l \text{ for all } k \in N.$$

According to Lemma 4.1, we know that u and hence also $u - \frac{r}{\mu}$ is uniformly continuous in $\Omega \times (1, \infty)$. Thus there exist two positive constants m and ρ such that

$$\left|u(x, t) - \frac{r}{\mu}\right| \geq \frac{l}{2} \text{ for all } x \in B_\rho(x_k) \cap \Omega \text{ and } t \in (t_k, t_k + m)$$

for arbitrary $k \in N$. The smoothness of $\partial\Omega$ implies the number $c := \inf_{k\in N} |B_\rho(x_k) \cap \Omega|$ must be positive. Then we obtain

$$\int_{t_k}^{t_k+m} \int_\Omega \left(u(x,t) - \frac{r}{\mu}\right)^2 dxdt \geq \int_{t_k}^{t_k+m} \int_{B_\rho(x_k)\cap\Omega} \left(u(x,t) - \frac{r}{\mu}\right)^2 dxdt \geq \frac{l^2mc}{4} \text{ for all } k \in N.$$

However, from Lemma 4.2, we have

$$\frac{l^2mc}{4} \leq \int_{t_k}^{t_k+m} \int_\Omega \left(u(x,t) - \frac{r}{\mu}\right)^2 dxdt \leq \int_{t_k}^{\infty}\int_\Omega \left(u(x,t) - \frac{r}{\mu}\right)^2 dxdt \to 0$$

as $k \to \infty$. This is absurd and hence establishes (4.6). The desired statement (4.7) and (4.8) can be derived similarly. □

Proof of Theorem 1.2. Since

$$\lim_{s\to\frac{r}{\mu}} \frac{s - \frac{r}{\mu} - \frac{r}{\mu}\ln\frac{\mu s}{r}}{(s - \frac{r}{\mu})^2} = \frac{\mu}{2r}. \tag{4.9}$$

Gathering the estimates (4.6) and (4.9), we gain a positive constant t_1 such that

$$\frac{\mu}{4r}\left(u - \frac{r}{\mu}\right)^2 \leq u - \frac{r}{\mu} - \frac{r}{\mu}\ln\frac{\mu u}{r} \leq \frac{\mu}{r}\left(u - \frac{r}{\mu}\right)^2$$

for all $t > t_1$. Consequently, we see that

$$\frac{\mu}{4r}\int_\Omega \left(u - \frac{r}{\mu}\right)^2 \leq F(t) \leq \frac{\mu}{r}\int_\Omega \left(u - \frac{r}{\mu}\right)^2 \tag{4.10}$$

holds for all $t > t_1$. Therefore we can conclude from (4.5) and (4.10) that

$$\frac{d}{dt}F(t) \leq -\frac{r\epsilon}{\mu}F(t) \text{ for all } t > t_1,$$

and this implies

$$F(t) \leq F(t_1)e^{-\frac{r\epsilon}{\mu}(t-t_1)} \text{ for all } t > t_1.$$

In light of the Gagliardo-Nirenberg inequality, we can find positive constant C_1 such that

$$\left\|u - \frac{r}{\mu}\right\|_{L^\infty(\Omega)} \leq C_1 \left\|u - \frac{r}{\mu}\right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \left\|u - \frac{r}{\mu}\right\|_{L^2(\Omega)}^{\frac{2}{n+2}}. \tag{4.11}$$

In the same way as in [35, Lemma 3.14], we can obtain a constant $C_2 > 0$ such that

$$\left\| u - \frac{r}{\mu} \right\|_{W^{1,\infty}(\Omega)} \leq C_2. \tag{4.12}$$

Recalling (4.10), it follows that

$$\int_\Omega \left(u - \frac{r}{\mu} \right)^2 \leq \frac{4r}{\mu} F(t) \leq \frac{4r}{\mu} F(t_1) e^{-\frac{r\epsilon}{\mu}(t-t_1)} \text{ for all } t > t_1. \tag{4.13}$$

Finally, using (4.11), (4.12) and (4.13), we observe that

$$\left\| u - \frac{r}{\mu} \right\|_{L^\infty(\Omega)} \leq C_3 e^{-\frac{r\epsilon}{\mu(n+2)}(t-t_1)} \text{ for all } t > t_1 \tag{4.14}$$

holds for some certain $C_3 > 0$. Similar arguments give the desired estimates of v and w

$$\left\| v - \frac{\alpha r}{\beta\mu} \right\|_{L^\infty(\Omega)} \leq C_3 e^{-\frac{r\epsilon}{\mu(n+2)}(t-t_1)} \text{ for all } t > t_1 \tag{4.15}$$

and

$$\left\| w - \frac{\gamma r}{\delta\mu} \right\|_{L^\infty(\Omega)} \leq C_3 e^{-\frac{r\epsilon}{\mu(n+2)}(t-t_1)} \text{ for all } t > t_1. \tag{4.16}$$

Collecting (4.14)-(4.16), we obtain (1.7). $\square$

5 The case $D(u) \equiv 0$

In this section, we consider the solution to system (1.3) when $D(u) \equiv 0$. In order to construct a regular solution, we need to introduce the following approximating equations

$$\begin{cases} u_{\epsilon t} = \epsilon\Delta u_\epsilon - \nabla\cdot(\chi u_\epsilon \nabla v_\epsilon) + \nabla\cdot(\xi u_\epsilon \nabla w_\epsilon) + r u_\epsilon - \mu u_\epsilon^2, \\ 0 = \Delta v_\epsilon + \alpha u_\epsilon - \beta v_\epsilon, \\ 0 = \Delta w_\epsilon + \gamma u_\epsilon - \delta w_\epsilon, \\ u_\epsilon(x,0) = u_{\epsilon 0}, \\ \frac{\partial u_\epsilon}{\partial \nu} = \frac{\partial v_\epsilon}{\partial \nu} = \frac{\partial w_\epsilon}{\partial \nu} = 0, \end{cases} \tag{5.1}$$

where $\epsilon \in (0,1)$, $\chi, \xi, \alpha, \beta, \gamma, \delta, r, \mu$ are positive parameters. For the above system, local existence of the classical solutions can be proved.

Lemma 5.1. Let $u_{\epsilon 0} \in W^{1,\infty}(\Omega)$, $\epsilon \in (0,1)$, Then there is a $T_\epsilon \in (0,\infty]$ such that (5.1) has a nonnegative classical solution existing for any time $t < T_\epsilon$. Moreover, if $T_\epsilon < \infty$, then

$$\|u_\epsilon\|_{L^\infty(\Omega)} \to \infty \text{ as } t \to T_\epsilon.$$

Proof. The proof of this lemma is standard, for details we refer the readers to [25].

5.1 A priori estimate for ∇u_ϵ

We will derive a priori estimate for ∇u_ϵ when $\|u_\epsilon\|_{L^\infty(\Omega)}$ is uniformly bounded.

Lemma 5.2. Let $(u_\epsilon, v_\epsilon, w_\epsilon)$ be the classical solution of system (5.1). Ω is a bounded convex domain with smooth boundary. Assume $\|u_\epsilon\|_{L^\infty(\Omega)} \le C_0 < \infty$ on $(0,T)$ where C_0 is a positive constant independent of ϵ. Then for some $q > n$, there exists a $T_1 > 0$ such that $\nabla u_\epsilon \in L^\infty((0,T_1), L^q(\Omega))$.

Proof. Applying the second and third equations of (1.3), we discover

$$\nabla \cdot (u_\epsilon \nabla v_\epsilon) = \nabla u_\epsilon \cdot \nabla v_\epsilon + u_\epsilon \Delta v_\epsilon = \nabla u_\epsilon \cdot \nabla v_\epsilon + u_\epsilon(\beta v_\epsilon - \alpha u_\epsilon)$$

and

$$\nabla \cdot (u_\epsilon \nabla w_\epsilon) = \nabla u_\epsilon \cdot \nabla w_\epsilon + u_\epsilon \Delta w_\epsilon = \nabla u_\epsilon \cdot \nabla w_\epsilon + u_\epsilon(\delta w_\epsilon - \gamma u_\epsilon).$$

Thus we rewrite the first equation of (5.1) as follows:

$$u_{\epsilon t} = \epsilon \Delta u_\epsilon - \chi \nabla u_\epsilon \cdot \nabla v_\epsilon + \xi \nabla u_\epsilon \cdot \nabla w_\epsilon + (r - \chi\beta v_\epsilon + \xi\delta w_\epsilon)u_\epsilon + (\chi\alpha - \xi\gamma - \mu)u_\epsilon^2,$$

and hence we have

$$\begin{aligned}\nabla u_{\epsilon t} =& \epsilon \nabla\Delta u_\epsilon - \chi\nabla(\nabla u_\epsilon)\cdot\nabla v_\epsilon - \chi\nabla u_\epsilon \cdot \nabla(\nabla v_\epsilon) + \xi\nabla(\nabla u_\epsilon)\cdot\nabla w_\epsilon + \xi\nabla u_\epsilon\cdot\nabla(\nabla w_\epsilon)\\ &+ \xi\delta u_\epsilon\nabla w_\epsilon - \chi\beta u_\epsilon \nabla v_\epsilon + (r - \chi\beta v_\epsilon + \xi\delta w_\epsilon)\nabla u_\epsilon + 2(\chi\alpha - \xi\gamma - \mu)u_\epsilon\nabla u_\epsilon.\end{aligned} \tag{5.2}$$

Multiplying (5.2) by $|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon$ with $q > n$ and integrating over Ω, we see that

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\nabla u_\epsilon\|^q_{L^q(\Omega)} =& \epsilon\int_\Omega |\nabla u_\epsilon|^{q-2}\nabla u_\epsilon \cdot \nabla\Delta u_\epsilon - \chi\int_\Omega \nabla(\nabla u_\epsilon)\cdot\nabla v_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon\\ &- \chi\int_\Omega \nabla u_\epsilon\cdot\nabla(\nabla v_\epsilon)|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon + \xi\int_\Omega \nabla(\nabla u_\epsilon)\cdot\nabla w_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon\\ &+ \xi\int_\Omega \nabla u_\epsilon\cdot\nabla(\nabla w_\epsilon)|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon + \xi\delta\int_\Omega u_\epsilon\nabla w_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon\\ &- \chi\beta\int_\Omega u_\epsilon\nabla v_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon + \int_\Omega (r - \chi\beta v_\epsilon + \xi\delta w_\epsilon)\nabla u_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon\\ &+ \int_\Omega 2(\chi\alpha - \xi\gamma - \mu)u_\epsilon\nabla u_\epsilon|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon\\ =:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9.\end{aligned}$$

In treating I_1, we make use of the pointwise identity

$$\nabla u_\epsilon \cdot \nabla\Delta u_\epsilon = \frac{1}{2}\Delta(|\nabla u_\epsilon|^2) - |D^2 u_\epsilon|^2$$

to obtain

$$I_1 = -\epsilon\int_\Omega |\nabla u_\epsilon|^{q-2}|D^2u_\epsilon|^2 + \frac{\epsilon}{2}\int_{\partial\Omega}|\nabla u_\epsilon|^{q-2}\frac{\partial|\nabla u_\epsilon|^2}{\partial\nu} - \frac{(q-2)\epsilon}{4}\int_\Omega |\nabla u_\epsilon|^{q-4}(\nabla|\nabla u_\epsilon|^2)^2.$$

Since $\frac{\partial|\nabla u_\epsilon|^2}{\partial\nu} < 0$ on $\partial\Omega$ for convex domains, it is direct that $I_1 \le 0$. Clearly,

$$I_2 = -\frac{\chi}{q}\int_\Omega \nabla|\nabla u_\epsilon|^q\cdot\nabla v_\epsilon = \frac{\chi}{q}\int_\Omega |\nabla u_\epsilon|^q\Delta v_\epsilon = \frac{\chi}{q}\int_\Omega |\nabla u_\epsilon|^q(\beta v_\epsilon - \alpha u_\epsilon).$$

Thus we have

$$I_2 \leq \frac{\chi}{q} \int_\Omega |\nabla u_\epsilon|^q (\beta \|v_\epsilon\|_{L^\infty(\Omega)} + \alpha \|u_\epsilon\|_{L^\infty(\Omega)}).$$

Similarly, we deduce

$$I_4 \leq \frac{\xi}{q} \int_\Omega |\nabla u_\epsilon|^q (\delta \|w_\epsilon\|_{L^\infty(\Omega)} + \gamma \|u_\epsilon\|_{L^\infty(\Omega)}),$$

$$I_6 \leq \xi\delta \int_\Omega |\nabla u_\epsilon|^{q-1} (\|u_\epsilon\|_{L^\infty(\Omega)} \|\nabla w_\epsilon\|_{L^\infty(\Omega)}),$$

$$I_7 \leq \chi\beta \int_\Omega |\nabla u_\epsilon|^{q-1} (\|u_\epsilon\|_{L^\infty(\Omega)} \|\nabla v_\epsilon\|_{L^\infty(\Omega)}),$$

$$I_8 \leq \int_\Omega |\nabla u_\epsilon|^q (r - \chi\beta \|v_\epsilon\|_{L^\infty(\Omega)} + \xi\delta \|w_\epsilon\|_{L^\infty(\Omega)})$$

as well as

$$I_9 \leq \int_\Omega (2(\chi\alpha - \xi\gamma - \mu) \|u_\epsilon\|_{L^\infty(\Omega)}) |\nabla u_\epsilon|^q.$$

Additionally, we obtain

$$I_3 \leq \chi \|\nabla^2 v_\epsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_\epsilon|^q$$

and

$$I_5 \leq \xi \|\nabla^2 w_\epsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_\epsilon|^q.$$

Via classical elliptic L^p estimates, we have

$$\|v_\epsilon\|_{W^{2,q}(\Omega)} \leq C_1 \|u_\epsilon\|_{L^q(\Omega)},$$

$$\|w_\epsilon\|_{W^{2,q}(\Omega)} \leq C_2 \|u_\epsilon\|_{L^q(\Omega)},$$

$$\|\nabla^2 v_\epsilon\|_{L^\infty(\Omega)} \leq C \|\nabla^2 v_\epsilon\|_{W^{1,q}(\Omega)} \leq C_3 \|u_\epsilon\|_{W^{1,q}(\Omega)}$$

and

$$\|\nabla^2 v_\epsilon\|_{L^\infty(\Omega)} \leq C \|\nabla^2 v_\epsilon\|_{W^{1,q}(\Omega)} \leq C_4 \|u_\epsilon\|_{W^{1,q}(\Omega)}$$

hold for some appropriate positive constant C_1, C_2, C_3 and C_4. Collecting all estimates, we can find positive constants C_5 and C_6 such that

$$\frac{d}{dt} \|\nabla u_\epsilon\|^q_{L^q(\Omega)} \leq C_5 \|\nabla u_\epsilon\|^q_{L^q(\Omega)} + C_6 \|\nabla u_\epsilon\|^{q+1}_{L^q(\Omega)}. \tag{5.3}$$

This ordinary differential inequality gives a $T_1 > 0$ such that $\nabla u_\epsilon \in L^\infty((0, T_1), L^q(\Omega))$. $\square$

Next, we devote to showing the existence of $T(D)$ independent of ϵ such that the solution u_ϵ of (5.1) exists on $(0, T(D))$. We begin with the following lemma which we can see details in [25, Lemma 18].

Lemma 5.3. Suppose that $f : [0, \infty) \to \mathbb{R}$ is nondecreasing and locally Lipschitz continuous. Let y be the solution of $y^{'}(t) = f(y(t))$ with initial data $y(0) = y_0$ on some interval $(0, T)$. If a continuous function $z : [0, T) \to \mathbb{R}$ fulfills

$$z(t) \le z(0) + \int_0^t f(z(\tau))d\tau \ \text{ for all } \ t \in (0, T), \ \ z(0) \le y_0.$$

Then we have $z(t) \le y(t)$ for all $t \in (0, T)$.

Lemma 5.4. Let $q > n$. For any $D > 0$, if $\|u_{\epsilon 0}\|_{W^{1,q}(\Omega)} \le D$ for arbitrary $\epsilon \in (0, 1)$, then there are some numbers $T(D) > 0$ and $M(D) > 0$ such that the regular solution of (5.1) exists on $\Omega \times (0, T(D))$ satisfying $\|\nabla u_\epsilon\|_{L^q(\Omega)} \le (D^q + 1)^{\frac{1}{q}}$ and $\|u_\epsilon\|_{L^\infty(\Omega\times(0,T(D)))} \le M(D)$.

Proof. For any $\varphi \in W^{1,q}(\Omega)$ where $\infty > q > n$, the classical Sobolev inequality shows that

$$\|\varphi\|_{L^\infty(\Omega)} \le C\|\varphi\|_{L^q(\Omega)} + C\|\nabla\varphi\|_{L^q(\Omega)}$$

where $C = C(\Omega, q)$. Applying the interpolation inequality, we deduce

$$\|\varphi\|_{L^q(\Omega)} \le \|\varphi\|^a_{L^1(\Omega)}\|\varphi\|^{1-a}_{L^\infty(\Omega)}$$

where $a = \frac{1}{q}$. Next, we use Young's inequality to discover

$$\|\varphi\|_{L^q(\Omega)} \le C(q)\|\varphi\|_{L^1(\Omega)} + \frac{1}{2C}\|\varphi\|_{L^\infty(\Omega)}$$

and hence we have

$$\|\varphi\|_{L^\infty(\Omega)} \le 2CC(q)\|\varphi\|_{L^1(\Omega)} + 2C\|\nabla\varphi\|_{L^q(\Omega)}.$$

We first fix two constants $c_1 = \max\{2CC(q), 2C\} > 0$ and $c_2 > 0$ such that

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1(\|\nabla\varphi\|_{L^q(\Omega)} + \|\varphi\|_{L^1(\Omega)}) \tag{5.4}$$

and

$$\|\varphi\|_{L^1(\Omega)} \le c_2\|\varphi\|_{W^{1,q}(\Omega)} \tag{5.5}$$

for all $\varphi \in W^{1,q}(\Omega)$. Applying (5.3), one can find a positive constant C_1 fulfilling

$$\begin{cases} \frac{d}{dt}\|\nabla u_\epsilon\|^q_{L^q(\Omega)} \le C_1(1 + \|\nabla u_\epsilon\|_{L^q(\Omega)})\|\nabla u_\epsilon\|^q_{L^q(\Omega)}, \\ \|\nabla u_{\epsilon 0}\|^q_{L^q(\Omega)} \le D^q. \end{cases}$$

We denote y the solution to

$$\begin{cases} y'(t) = C_1(1 + y^{\frac{1}{q}})y, \\ y(0) = D^q \end{cases}$$

and denote a number $T(D)$ such that $y(t) \leq D^q + 1$ on $t \in (0, T(D))$. Lemma 5.3 leads us to the conclusion that

$$\|\nabla u_\epsilon\|_{L^q(\Omega)} \leq (D^q + 1)^{\frac{1}{q}} \tag{5.6}$$

on $(0, T(D))$. Recalling the first equation of (5.1) and employing the standard ODE comparison argument, we conclude

$$\|u_\epsilon\|_{L^1(\Omega)} \leq \max\left\{c_2 D, \frac{r|\Omega|}{\mu}\right\} =: \frac{c_3}{c_1}. \tag{5.7}$$

A combination of (5.4)-(5.7) and the fact $\|u_{\epsilon 0}\|_{W^{1,q}(\Omega)} \leq D$ yields that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq (D^q + 1)^{\frac{1}{q}} + c_3 := M(D) \quad \text{on} \quad t \in (0, T(D)),$$

where $T(D)$ is independent of ϵ. □

5.2 Existence of the regular solution.

In order to achieve a strong precompactness property of $\{u_\epsilon\}_{\epsilon \in (0,1)}$ by using the Anbin-Lions Theorem, we should do some estimates on $u_{\epsilon t}$ first. We multiply the first equation in (5.1) by φ, where $\varphi \in C_0^\infty(\Omega)$, then integrating over Ω, we obtain

$$\begin{aligned}
\left|\int_\Omega u_{\epsilon t}\varphi\right| =& \left|\int_\Omega \epsilon\Delta u_\epsilon \varphi - \nabla\cdot(\chi u_\epsilon \nabla v_\epsilon)\varphi + \nabla\cdot(\xi u_\epsilon \nabla w_\epsilon)\varphi + r u_\epsilon\varphi - \mu u_\epsilon^2\varphi\right| \\
\leq& \epsilon\left|\int_\Omega \nabla u_\epsilon \nabla\varphi\right| + \chi\left|\int_\Omega u_\epsilon \nabla v_\epsilon \nabla\varphi\right| + \xi\left|\int_\Omega u_\epsilon \nabla w_\epsilon \nabla\varphi\right| + r\left|\int_\Omega u_\epsilon\varphi\right| + \mu\left|\int_\Omega u_\epsilon^2\varphi\right| \\
\leq& \epsilon\|\nabla u_\epsilon\|_{L^q(\Omega)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} + \chi\|u_\epsilon\|_{L^\infty(\Omega)}\|\nabla v_\epsilon\|_{L^q(\Omega)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} \\
&+ \xi\|u_\epsilon\|_{L^\infty(\Omega)}\|\nabla w_\epsilon\|_{L^q(\Omega)}\|\nabla\varphi\|_{L^{\frac{q}{q-1}}(\Omega)} + r\|u_\epsilon\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} + \mu\|u_\epsilon\|^2_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)}.
\end{aligned}$$

Using the Hölder inequality, we have

$$r\|u_\epsilon\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} \leq Cr\|u_\epsilon\|_{L^\infty(\Omega)}\|\varphi\|_{L^{\frac{q}{q-1}}(\Omega)}$$

and

$$\mu\|u_\epsilon\|^2_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} \leq C\mu\|u_\epsilon\|^2_{L^\infty(\Omega)}\|\varphi\|_{L^{\frac{q}{q-1}}(\Omega)}.$$

According to Lemma 5.4, let $\|u_{\epsilon 0}\|_{W^{1,q}(\Omega)} \leq D$ for some $D > 0$, we can find a $\epsilon - independent$ $T(D) > 0$ such that the solution u_ϵ to (5.1) with initial data $u_{\epsilon 0}$ exists on interval $(0, T(D))$

satisfying $\|\nabla u_\epsilon\|_{L^q(\Omega)} \le (D^q+1)^{\frac{1}{q}}$ and $\|u_\epsilon\|_{L^\infty(\Omega\times(0,T(D)))} \le M(D)$. Thus we obtain that for $t \in (0,T(D))$, there is a positive constant $C(D)$ such that

$$\left|\int_\Omega u_{\epsilon t}\varphi\right| \le C(D)\|\varphi\|_{W^{1,\frac{q}{q-1}}(\Omega)}.$$

Moreover, taking the supremum over φ with $\|\varphi\|_{W^{1,\frac{q}{q-1}}(\Omega)} = 1$, we infer that

$$\|u_{\epsilon t}\|_{(W^{1,\frac{q}{q-1}}(\Omega))^*} \le C(D).$$

Consequently, we obtain

$$\|u_{\epsilon t}\|_{L^q((0,T(D));(W^{1,\frac{q}{q-1}}(\Omega))^*)} \le C(T(D),D).$$

By Aubin-Lions Theorem (see [37]) and the fact $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \hookrightarrow (W^{1,\frac{q}{q-1}}(\Omega))^*$, we thus infer that there exists a subsequence $\{\epsilon_j\} \searrow 0$ such that

$$u_{\epsilon_j} \to u \ \text{ weakly}^* \ \text{ in } \ L^\infty((0,T(D));W^{1,q}(\Omega)), \tag{5.8}$$

$$u_{\epsilon_j} \to u \ \text{ weakly } \ \text{ in } \ L^q((0,T(D));W^{1,q}(\Omega)) \tag{5.9}$$

and

$$u_{\epsilon_j} \to u \ \text{ strongly } \ \text{ in } \ L^q((0,T(D));L^q(\Omega)). \tag{5.10}$$

Elliptic regularity theory applied to the second and third equations in (5.1) leads to

$$v_{\epsilon_j} \to v \ \text{ strongly } \ \text{ in } \ L^q((0,T(D));W^{2,q}(\Omega)) \tag{5.11}$$

and

$$w_{\epsilon_j} \to w \ \text{ strongly } \ \text{ in } \ L^q((0,T(D));W^{2,q}(\Omega)). \tag{5.12}$$

We fix any $\varphi \in C_0^\infty((0,T);\Omega) \subset C^{1,1}((0,T);(W^{1,q}(\Omega))^*)$ for all $T \in (0,\infty)$. Since u_{ϵ_j} solve (5.1) weakly at least up to time $T(D)$, then we can conclude

$$\begin{aligned} -\int_0^{T(D)}\int_\Omega u_{\epsilon_j}\varphi_t - \int_0^{T(D)}\int_\Omega u_{\epsilon_j 0}\varphi(0) = &-\epsilon_j\int_0^{T(D)}\int_\Omega \nabla u_{\epsilon_j}\cdot\nabla\varphi + \chi\int_0^{T(D)}\int_\Omega u_{\epsilon_j}\nabla v_{\epsilon_j}\cdot\nabla\varphi \\ &-\xi\int_0^{T(D)}\int_\Omega u_{\epsilon_j}\nabla w_{\epsilon_j}\cdot\nabla\varphi \\ &+ r\int_0^{T(D)}\int_\Omega u_{\epsilon_j}\varphi - \mu\int_0^{T(D)}\int_\Omega u_{\epsilon_j}^2\varphi. \end{aligned}$$

From (5.8) and (5.10) we can deduce that

$$\int_\Omega |\nabla u(\cdot,t)|^q \le \liminf_{k\to\infty}\int_\Omega |\nabla u_{\epsilon j(k)}(\cdot,t)|^q \ \text{ for all } \ t \in (0,T(D)).$$

Due to the bound of $\|\nabla u_{\epsilon_j}\|_{L^q(\Omega)}$ on $(0,T(D))$ is independent of ϵ, then we get

$$\epsilon_j \int_0^{T(D)} \int_\Omega \nabla u_{\epsilon_j} \cdot \nabla\varphi \to 0 \text{ as } \epsilon_j \to 0.$$

Next, we see

$$\begin{aligned}
&\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v_{\epsilon_j} \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u \nabla v \cdot \nabla\varphi \right| \\
\leq &\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v_{\epsilon_j} \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v \cdot \nabla\varphi \right| + \left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u \nabla v \cdot \nabla\varphi \right|
\end{aligned}$$

Then from (5.10) and (5.11), we arrive at

$$\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v_{\epsilon_j} \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v \cdot \nabla\varphi \right| \to 0 \text{ as } \epsilon_j \to 0$$

and

$$\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u \nabla v \cdot \nabla\varphi \right| \to 0 \text{ as } \epsilon_j \to 0.$$

Therefore, we discover

$$\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla v_{\epsilon_j} \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u \nabla v \cdot \nabla\varphi \right| \to 0 \text{ as } \epsilon_j \to 0.$$

Similarly, we have

$$\left| \int_0^{T(D)} \int_\Omega u_{\epsilon_j} \nabla w_{\epsilon_j} \cdot \nabla\varphi - \int_0^{T(D)} \int_\Omega u \nabla w \cdot \nabla\varphi \right| \to 0 \text{ as } \epsilon_j \to 0.$$

Moreover, (5.10) also yields

$$\begin{aligned}
\int_0^{T(D)} \int_\Omega u_{\epsilon_j} \varphi &\to \int_0^{T(D)} \int_\Omega u\varphi \text{ as } \epsilon_j \to 0, \\
\int_0^{T(D)} \int_\Omega u_{\epsilon_j}^2 \varphi &\to \int_0^{T(D)} \int_\Omega u^2\varphi \text{ as } \epsilon_j \to 0, \\
\int_0^{T(D)} \int_\Omega u_{\epsilon_j} \varphi_t &\to \int_0^{T(D)} \int_\Omega u\varphi_t \text{ as } \epsilon_j \to 0
\end{aligned}$$

as well as

$$\int_0^{T(D)} \int_\Omega u_{\epsilon_j 0} \varphi(0) \to \int_0^{T(D)} \int_\Omega u_0 \varphi(0) \text{ as } \epsilon_j \to 0.$$

For $\epsilon_j \to 0$, it follows that

$$\begin{aligned}
-\int_0^{T(D)} \int_\Omega u\varphi_t - \int_0^{T(D)} \int_\Omega u_0\varphi(0) =&\chi \int_0^{T(D)} \int_\Omega u\nabla v \cdot \nabla\varphi - \xi \int_0^{T(D)} \int_\Omega u\nabla w \cdot \nabla\varphi \\
&+ r \int_0^{T(D)} \int_\Omega u\varphi - \mu \int_0^{T(D)} \int_\Omega u^2 \varphi.
\end{aligned} \tag{5.13}$$

Next, we fix any $\zeta \in C_0^\infty((0,T);\Omega) \subset C((0,T);(L^q(\Omega))^*)$ to obtain

$$\int_0^{T(D)}\int_\Omega \Delta v_{\epsilon_j}\zeta - \int_0^{T(D)}\int_\Omega \beta v_{\epsilon_j}\zeta + \int_0^{T(D)}\int_\Omega \alpha u_{\epsilon_j}\zeta = 0.$$

As (5.11) implies $\Delta v_{\epsilon_j} \to \Delta v$ in $L^q((0,T(D));L^q(\Omega))$, we thus conclude

$$\int_0^{T(D)}\int_\Omega \Delta v_{\epsilon_j}\zeta \to \int_0^{T(D)}\int_\Omega \Delta v\zeta \ \text{ as } \ \epsilon_j \to 0,$$

$$\int_0^{T(D)}\int_\Omega \beta v_{\epsilon_j}\zeta \to \int_0^{T(D)}\int_\Omega \beta v\zeta \ \text{ as } \ \epsilon_j \to 0$$

and

$$\int_0^{T(D)}\int_\Omega \alpha u_{\epsilon_j}\zeta \to \int_0^{T(D)}\int_\Omega \alpha u\zeta \ \text{ as } \ \epsilon_j \to 0.$$

Then we arrive at

$$\int_0^{T(D)}\int_\Omega \Delta v\zeta - \int_0^{T(D)}\int_\Omega \beta v\zeta + \int_0^{T(D)}\int_\Omega \alpha u\zeta = 0. \tag{5.14}$$

Furthermore, we fix any $\eta \in C_0^\infty((0,T);\Omega) \subset C((0,T);(L^q(\Omega))^*)$ to obtain

$$\int_0^{T(D)}\int_\Omega \Delta w_{\epsilon_j}\eta - \int_0^{T(D)}\int_\Omega \delta w_{\epsilon_j}\eta + \int_0^{T(D)}\int_\Omega \gamma u_{\epsilon_j}\eta = 0.$$

Let $\epsilon_j \to 0$, we deduce

$$\int_0^{T(D)}\int_\Omega \Delta w\eta - \int_0^{T(D)}\int_\Omega \delta w\eta + \int_0^{T(D)}\int_\Omega \gamma u\eta = 0. \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), we can claim that (u,v,w) is a regular solution to system (1.3). Collecting all T fulfilling the above equations and taking their supremum, we can find a maximal existence time, say T_0. Now it remains to show that the constructed solution is nonnegative. Due to the maximum principle, $u \geq 0$ means that v and w are nonnegative. Therefore, it is sufficient to prove that u is nonnegative. Let $u_- = \max\{-u, 0\}$. Testing the first equation of (1.3) with u_-, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u_-^2 =& -\frac{\chi}{2}\int_\Omega u_-^2\Delta v + \frac{\xi}{2}\int_\Omega u_-^2\Delta w + r\int_\Omega u_-^2 - \mu\int_\Omega u_-^3\\ \leq& C(1 + \|u\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{L^\infty})\int_\Omega u_-^2.\end{aligned}$$

Then we deduce $u_- = 0$ for all $t < T_0$ due to the initial data $u_{0-} = 0$.

5.3 Boundedness of the solution when $\mu > \mu_*$

Based on the existence of the regular solution to (1.3), we set out to explore the boundedness of the solution when $\mu > \mu_*$.

Lemma 5.5. Let $n \geq 2$ and (u, v, w) be the regular solution of (1.3) in a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary. If $\mu > \mu_*$, where

$$\mu_* = \begin{cases} 0 & \text{if } \chi\alpha - \xi\gamma \leq 0, \\ \chi\alpha - \xi\gamma & \text{if } \chi\alpha - \xi\gamma > 0, \end{cases}$$

then u is uniformly bounded.

Proof. Multiplying both side of the first equation in (1.3) by u^n and integrating over Ω, we discover

$$\begin{aligned} \int_\Omega u^n u_t =& -\chi \int_\Omega u^n \nabla \cdot (u \nabla v) + \xi \int_\Omega u^n \nabla \cdot (u \nabla w) + r \int_\Omega u^{n+1} - \mu \int_\Omega u^{n+2} \\ =& -\frac{n\chi}{n+1} \int_\Omega u^{n+1} \Delta v + \frac{n\xi}{n+1} \int_\Omega u^{n+1} \Delta w + r \int_\Omega u^{n+1} - \mu \int_\Omega u^{n+2} \\ \leq& -\left(\mu - \frac{n}{n+1}(\chi\alpha - \xi\gamma)\right) \int_\Omega u^{n+2} + r \int_\Omega u^{n+1} + \frac{n\xi\delta}{n+1} \int_\Omega u^{n+1} w. \end{aligned}$$

Due to $\mu > \mu_*$, we derive $\mu - \frac{n}{n+1}(\chi\alpha - \xi\gamma) > 0$. Using Young's inequality along with Lemma 2.3, we arrive at

$$\frac{1}{n+1} \frac{d}{dt} \int_\Omega u^{n+1} = \int_\Omega u^n u_t \leq -\frac{\left(\mu - \frac{n}{n+1}(\chi\alpha - \xi\gamma)\right)}{2} \int_\Omega u^{n+2} + r \int_\Omega u^{n+1} + C_1,$$

where $C_1 = C_1(\chi, \alpha, \xi, \beta, \gamma, \delta, n)$ is a positive constant. Thus a standard ODE comparison argument implies boundedness of $\|u\|_{L^{n+1}(\Omega)}$. On the other hand, based on the boundedness of $\|u\|_{L^{n+1}(\Omega)}$, elliptic regularity theory applied to the third equation in (1.3) leads to a constant $C_2 > 0$ fulfilling

$$\|w\|_{W^{1,\infty}(\Omega)} \leq C_2.$$

Next, testing the first equation of (1.3) by u^{p-1} and integrating over Ω, we arrive at

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_\Omega u^p =& -\chi \int_\Omega u^{p-1} \nabla \cdot (u \nabla v) + \xi \int_\Omega u^{p-1} \nabla \cdot (u \nabla w) + r \int_\Omega u^p - \mu \int_\Omega u^{p+1} \\ \leq& -\left(\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)\right) \int_\Omega u^{p+1} + r \int_\Omega u^p + \frac{(p-1)\xi\delta}{p} \int_\Omega u^p w \\ \leq& -\left(\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)\right) \int_\Omega u^{p+1} + (r + \xi\delta \|w\|_{L^\infty(\Omega)}) \int_\Omega u^p. \end{aligned}$$

Based on the boundedness of $\|w\|_{L^\infty(\Omega)}$, we denote $C_3 = r + \xi\delta\|w\|_{L^\infty(\Omega)}$. Applying Young's inequality, we deduce

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \leq -\frac{\left(\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)\right)}{2} \int_\Omega u^{p+1} + \left(\frac{\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)}{2C_3} \frac{p+1}{p}\right)^{-p} \frac{1}{p+1} |\Omega|.$$

Consequently,

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \leq \left(\frac{\mu - \frac{p-1}{p}(\chi\alpha - \xi\gamma)}{2C_3} \frac{p+1}{p}\right)^{-p} \frac{1}{p+1} |\Omega|.$$

Thus we have $\|u\|_{L^p(\Omega)} \le \left(\left(\frac{\mu - \frac{p-1}{p}(\chi\alpha-\xi\gamma)}{2C_3} \frac{p+1}{p} \right)^{-p} \frac{1}{p+1} |\Omega| t + \|u_0\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$. Let $p \to \infty$, we have for arbitrary $t < \infty$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \frac{2C_3}{\mu - (\chi\alpha - \xi\gamma)}.$$

Therefore, we obtain a uniform bound of u. $\square$

5.4 Blow up

Next we establish the finite-time blow-up of the regular solution to (1.3) in the case $\chi\alpha - \xi\gamma > 0$ and $0 < \mu < \chi\alpha - \xi\gamma$.

Lemma 5.6. Let (u, v, w) be the regular solution of (1.3) in $\Omega \subset \mathbb{R}^n$ for $n \ge 2$. Assume that $\chi\alpha - \xi\gamma > 0$ and $0 < \mu < \chi\alpha - \xi\gamma$. If there exists q with $1 < q < \infty$ such that the initial data $\|u_0\|_{L^q(\Omega)}$ is sufficiently large, then u becomes unbounded at a finite time.

Proof. Since $0 < \mu < \chi\alpha - \xi\gamma$. Then we can find q sufficiently large such that $\frac{q-1}{q}(\chi\alpha - \xi\gamma) - \mu > 0$. Testing the first equation of (1.3) by u^{q-1} and integrating over Ω, we deduce that

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega u^q =& -\chi \int_\Omega u^{q-1}\nabla\cdot(u\nabla v) + \xi \int_\Omega u^{q-1}\nabla\cdot(u\nabla w) + r\int_\Omega u^q - \mu\int_\Omega u^{q+1} \\ \ge& \left(\frac{q-1}{q}(\chi\alpha-\xi\gamma) - \mu\right)\int_\Omega u^{q+1} + r\int_\Omega u^q - \frac{q-1}{q}\int_\Omega u^q v. \end{aligned} \tag{5.16}$$

Noting the Sobolev embedding and elliptic regular theory of v, we can find a positive constant C_1 satisfying

$$\|v\|_{L^{q+1}(\Omega)} \le C\|v\|_{W^{1,\tilde{q}}(\Omega)} \le C_1\|u\|_{L^{\tilde{q}}(\Omega)}, \tag{5.17}$$

where $\frac{1}{\tilde{q}} = \frac{1}{q+1} + \frac{1}{n}$. On the other hand, since $1 < \tilde{q} < q+1$, the interpolation inequality leads to

$$\|v\|_{L^{q+1}(\Omega)} \le C_1\|u\|_{L^{\tilde{q}}(\Omega)} \le C_1\|u\|^{1-a}_{L^1(\Omega)}\|u\|^a_{L^{q+1}(\Omega)}, \tag{5.18}$$

where $a = \frac{q(n-1)-1}{qn} \in (0,1)$. Employing the Young's inequality, we estimate (5.16) as follows:

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q \ge \frac{\left(\frac{q-1}{q}(\chi\alpha-\xi\gamma)-\mu\right)}{2}\int_\Omega u^{q+1} + r\int_\Omega u^q - C_2\int_\Omega v^{q+1}, \tag{5.19}$$

where C_2 is a positive constant depending on q, χ, ξ, α, β and μ. Combining (5.18) with (5.19) and using the Young's inequality once more, one can find a constant $C_3 > 0$ such that

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega u^q \ge& \frac{\left(\frac{q-1}{q}(\chi\alpha-\xi\gamma)-\mu\right)}{2}\int_\Omega u^{q+1} + r\int_\Omega u^q - C_1C_2\left(\int_\Omega u\right)^{(1-a)(q+1)}\left(\int_\Omega u^{q+1}\right)^a \\ \ge& \frac{\left(\frac{q-1}{q}(\chi\alpha-\xi\gamma)-\mu\right)}{4}\int_\Omega u^{q+1} + r\int_\Omega u^q - C_3\left(\int_\Omega u\right)^{q+1}. \end{aligned} \tag{5.20}$$

Recalling the boundedness of $\|u\|_{L^1(\Omega)}$ and the fact $\int_\Omega u^{q+1} \geq C_\Omega \left(\int_\Omega u^q\right)^{\frac{q+1}{q}}$, we obtain from (5.20) that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q \geq \frac{\left(\frac{q-1}{q}(\chi\alpha-\xi\gamma)-\mu\right)C_\Omega}{4}\left(\int_\Omega u^q\right)^{\frac{q+1}{q}} + r\int_\Omega u^q - C_4,$$

where $C_4 := C_3\|u\|^{q+1}_{L^1(\Omega)}$. If $\|u_0\|_{L^q(\Omega)}$ is sufficiently large, namely $r\|u_0\|^q_{L^q(\Omega)} \geq C_4$, this implies $r\int_\Omega u^q - C_4 > 0$ for all $t > 0$. Then we arrive at

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u^q \geq \frac{C(\Omega,q,\chi,\alpha,\xi,\gamma,\mu)}{4}\left(\int_\Omega u^q\right)^{\frac{q+1}{q}}$$

and hence

$$\int_\Omega u^q \geq \left(\frac{1}{\left(\frac{1}{\int_\Omega u_0^q}\right)^{\frac{1}{q}} - \frac{C(\Omega,q,\chi,\alpha,\xi,\gamma,\mu)t}{4}}\right)^q.$$

The proof is complete. □

Now we can easily prove Theorem 1.3.

Proof of Theorem 1.3. According to the existence of the regular solution, Lemma 5.5 and Lemma 5.6, we complete the proof. □

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