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# The limit behavior of the Riemann solutions to the generalized Chaplygin gas equations with a source term ☆

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The paper is concerned with the limit behavior of the Riemann solutions to the inhomogeneous generalized Chaplygin gas equations. The formation of delta shock waves and the vacuum states are identified and analyzed as the pressure vanishes. Unlike the homogeneous case, the Riemann solutions are no longer self-similar. As the pressure vanishes, the Riemann solutions to the generalized Chaplygin gas equations with a friction term converge to the Riemann solutions to the pressureless gas dynamics model with a body force.



## 1. Introduction

One-dimensional isentropic flow in the gas dynamics with a source term can be written as

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P)_x = \beta\rho, \end{cases} \tag{1.1}$$

where $\beta$ is a constant, $\rho$, $u$ represent the density and the velocity respectively. $P = P(\rho, \varepsilon)$ is the scaled generalized Chaplygin gas pressure $P = \varepsilon p$ where $\varepsilon > 0$ and

$$p = -\frac{A}{\rho^\alpha},\ 0 < \alpha \leq 1, \tag{1.2}$$

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where $A > 0$ is a constant. It is easy to see that $\lim_{\varepsilon \to 0} P(\rho, \varepsilon) = 0$. The model of the generalized Chaplygin gas explains the acceleration of the universe through an exotic equation of state (1.2) causing it acts like dark matter at high density and like dark energy at low density [17]. Thus, the generalized Chaplygin gas allows for an unification of dark energy and dark matter [1,2,15,24].

Sun [25] considered the inhomogeneous generalized Chaplygin gas equations (1.1)–(1.2) and obtained the non-self-similar Riemann solutions by introducing a new state variable

$$v(x,t) = u(x,t) - \beta t. \tag{1.3}$$

The new velocity (1.3) was introduced by Faccanoni and Mangeney [8] to study the Riemann problem of the shallow water equations.

When $\alpha = 1$, (1.2) is the equation of state of the Chaplygin gas. For the Chaplygin gas equations with a friction term, Shen [21] obtained the Riemann solutions. The Riemann problem with delta initial data and the vanishing pressure limit problem were considered in [9] and [10] respectively.

For the homogeneous generalized Chaplygin gas equations, $\beta = 0$ in (1.1)–(1.2), Wang [26] studied the Riemann problem. Sheng, Wang and Yin [23] studied the vanishing pressure limits of the Riemann solutions.

For the Chaplygin gas equations without a source term, there are many results, the readers are referred to [3,11,12,16,13,28,27].

The limit system of (1.1)–(1.2) as $\varepsilon \to 0$ formally becomes the pressureless gas dynamics model with a source term

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = \beta \rho. \end{cases} \tag{1.4}$$

We can also obtain system (1.4) by taking the constant pressure where the force is assumed to be the gravity with $\beta$ being the gravity constant [6]. System (1.4) can describe the motion process of free particles sticking under collision in the low temperature and the information of large-scale structures in the universe [7,18]. Shen [20] considered both the Riemann problem and the Riemann problem with delta initial data of system (1.4).

Li [14] introduced the method of vanishing pressure limit to study the isothermal gases dynamics model. Chen and Liu [4] identified and analyzed the formation of delta shocks and vacuum states in Riemann solutions to the Euler equations for isentropic fluids. They made a further step later to generalize the results to the nonisentropic fluids [5]. For more results on vanishing pressure, we refer the readers to [22,19,23,29,30].

In this paper, we focus on the vanishing pressure limits of Riemann solutions to the inhomogeneous generalized Chaplygin gas equations (1.1)–(1.2). Unlike the homogeneous case, the Riemann solutions are no longer self-similar. Moreover, the generalized Chaplygin gas equations (1.1)–(1.2) differ from the Chaplygin gas equations. In the present case, the characteristic fields are genuinely nonlinear, while in the latter case, the characteristic fields are linearly degenerate.

Now, we give our main results.

**Theorem 1.1.** *When the parameter $\varepsilon \to 0$, Riemann solutions of system (1.1)–(1.2) converge to the Riemann solutions of system (1.4). There are three cases.*

*(1) When $u_- > u_+$, the two shock wave solutions firstly converge to a delta shock solution as $\varepsilon$ drops to a certain parameter value $\varepsilon_2$ which depends on the initial data, then as $\varepsilon$ goes to zero, the delta shock wave converges to a delta shock solution of (1.4).*

*(2) When $u_- < u_+$, as $\varepsilon \to 0$, the two rarefaction wave solutions converge to two contact discontinuities connecting the states $(u_\pm + \beta t, \rho_\pm)$ with a vacuum state between them.*

*(3) When* $u_- = u_+$, *as* $\varepsilon \to 0$, *the Riemann solutions converge to a contact discontinuity connecting the states* $(u_\pm + \beta t, \rho_\pm)$.

The organization of this article is as follows: In section 2 and section 3, we review the Riemann solutions to (1.4), (1.1) respectively. In section 4, we study the vanishing pressure limits of Riemann solutions to (1.1)–(1.2).

## 2. Riemann problem for (1.4)

In this section, we give a sketch of the results on the Riemann problem to system (1.4) in [20].

By a change of variable (1.3), system (1.4) can be rewritten as

$$\begin{cases} \rho_t + (\rho(v+\beta t))_x = 0, \\ (\rho v)_t + (\rho v(v+\beta t))_x = 0. \end{cases} \tag{2.1}$$

Consider the Riemann problem of (2.1) with the Riemann initial data

$$(v, \rho)(x, 0) = \begin{cases} (v_-, \rho_-), & x < 0, \\ (v_+, \rho_+), & x > 0. \end{cases} \tag{2.2}$$

From (1.3), we obtain $v_\pm = u_\pm$. We use $u_\pm$ to denote $v_\pm$ throughout this paper.

A double eigenvalue of (2.1) is $\lambda = v + \beta t$. The corresponding eigenvector is $\overrightarrow{r} = (1, 0)^\top$. It is easy to know that $\nabla\lambda \cdot \overrightarrow{r} = 0$. This means that system (2.1) is linearly degenerate. The elementary waves are contact discontinuities.

For a discontinuity $\sigma(t) = x'(t)$, the Rankine–Hugoniot conditions hold

$$\begin{cases} -\sigma(t)\,[\rho] + [\rho(v+\beta t)] = 0, \\ -\sigma(t)\,[\rho v] + [\rho v(v+\beta t)] = 0, \end{cases} \tag{2.3}$$

where $[\rho] = \rho - \rho_-$. From (2.3), we solve for the contact discontinuity $J(u_-, \rho_-)$:

$$\sigma(t) = v + \beta t = u_- + \beta t. \tag{2.4}$$

In the case $u_- < u_+$, the Riemann solution of (2.1)–(2.2) consists of two contact discontinuities with a vacuum state between them. The solution can be expressed by:

$$(u_-, \rho_-) + J_1 + Vac + J_2 + (u_+, \rho_+), \tag{2.5}$$

where "+" means "follow by".

In the case $u_- = u_+$, the Riemann solution can be expressed by:

$$(u_-, \rho_-) + J + (u_+, \rho_+). \tag{2.6}$$

While in the case $u_- > u_+$, the Riemann solution contains a delta shock wave. The Riemann solution can be expressed by:

$$(u_-, \rho_-) + \delta S + (u_+, \rho_+). \tag{2.7}$$

The delta shock wave satisfies the generalized Rankine–Hugoniot conditions [20]

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho(v+\beta t)], \\ \dfrac{d(w(t)v_\delta(t))}{dt} = u_\delta(t)[\rho v] - [\rho v(v+\beta t)], \end{cases} \tag{2.8}$$

where $w(t)$ and $u_\delta(t) = v_\delta + \beta t$ are weight and velocity of delta shock wave respectively, and $(x, w)(0) = (0, 0)$.

By direct calculations, we have

$$v_\delta = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \quad x(t) = v_\delta t + \frac{1}{2}\beta t^2, \quad w(t) = \sqrt{\rho_-\rho_+}(u_- - u_+)t. \tag{2.9}$$

In order to ensure the uniqueness of the Riemann solution, the delta shock wave should satisfy the generalized entropy condition

$$u_+ + \beta t < u_\delta(t) < u_- + \beta t. \tag{2.10}$$

In summary, we obtain the Riemann solutions to system (1.4) as follows

(1) For $u_- > u_+$, the Riemann solution can be expressed by

$$(u_- + \beta t, \rho_-) + \delta S + (u_+ + \beta t, \rho_+). \tag{2.11}$$

(2) For $u_- < u_+$, the Riemann solution can be expressed by

$$(u_- + \beta t, \rho_-) + J_1 + Vac + J_2 + (u_+ + \beta t, \rho_+). \tag{2.12}$$

(3) For $u_- = u_+$, the Riemann solution can be expressed by

$$(u_- + \beta t, \rho_-) + J + (u_+ + \beta t, \rho_+). \tag{2.13}$$

## 3. Riemann problem for (1.1)–(1.2)

In this section, we summarize results on the Riemann problem of (1.1)–(1.2), see [25] for the detail.

Without loss of generality, we take $A = 1$ throughout this paper.

Using (1.3), system (1.1)–(1.2) is reformulated as

$$\begin{cases} \rho_t + (\rho(v+\beta t))_x = 0, \\ (\rho v)_t + \left(\rho v(v+\beta t) - \dfrac{\varepsilon}{\rho^\alpha}\right)_x = 0, \end{cases} \tag{3.1}$$

which is a system of homogeneous conservative equations. Notice that $t$ appears in the equations.

System (3.1) has two eigenvalues

$$\lambda_1 = v + \beta t - \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}}, \quad \lambda_2 = v + \beta t + \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}} \tag{3.2}$$

with corresponding right eigenvectors

$$\overrightarrow{r}_1 = (1, -\sqrt{\alpha\varepsilon}\rho^{-\frac{3+\alpha}{2}})^T, \quad \overrightarrow{r}_2 = (1, \sqrt{\alpha\varepsilon}\rho^{-\frac{3+\alpha}{2}})^T. \tag{3.3}$$

Direct calculations give $\nabla\lambda_1 \cdot \overrightarrow{r}_1 \neq 0$ and $\nabla\lambda_2 \cdot \overrightarrow{r}_2 \neq 0$ for $0 < \alpha < 1$, which means that both characteristic fields are genuinely nonlinear. We now solve the Riemann problem for (3.1).

Given a state $(u_-, \rho_-)$ in the phase plane, the rarefaction wave curves are the sets of states that can be connected on the right by a 1-rarefaction or 2-rarefaction wave in the form

$$R_1^\varepsilon(u_-, \rho_-): \begin{cases} \dfrac{dx}{dt} = \lambda_1 = v + \beta t - \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}}, \\ v - \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_- - \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}, \ \rho < \rho_-, \\ \lambda_1(u_-, \rho_-) < \lambda_1(v, \rho), \end{cases} \tag{3.4}$$

and

$$R_2^\varepsilon(u_-, \rho_-): \begin{cases} \dfrac{dx}{dt} = \lambda_2 = v + \beta t + \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}}, \\ v + \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_- + \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}, \ \rho > \rho_-. \\ \lambda_2(u_-, \rho_-) < \lambda_2(v, \rho). \end{cases} \tag{3.5}$$

Let $\sigma(t) = x'(t)$ be the speed of a discontinuity $x = x(t)$, then the Rankine–Hugoniot conditions read

$$\begin{cases} -\sigma(t)[\rho] + [\rho(v + \beta t)] = 0, \\ -\sigma(t)[\rho v] + \left[\rho v(v + \beta t) - \dfrac{\varepsilon}{\rho^\alpha}\right] = 0. \end{cases} \tag{3.6}$$

From (3.6), given a state $(u_-, \rho_-)$ in the phase plane, the shock wave curves are the sets of states that can be connected on the right by a 1-shock or 2-shock wave in the form

$$S_1^\varepsilon(u_-, \rho_-): \begin{cases} \sigma_1^\varepsilon(t) = u_- + \beta t - \left(\dfrac{\rho}{\rho_-}\dfrac{[P]}{[\rho]}\right)^{\frac{1}{2}}, \\ v = u_- - \left(\dfrac{1}{\rho\rho_-}\dfrac{[P]}{[\rho]}\right)^{\frac{1}{2}}(\rho - \rho_-), \end{cases} \rho > \rho_-, \tag{3.7}$$

and

$$S_2^\varepsilon(u_-, \rho_-): \begin{cases} \sigma_2^\varepsilon(t) = u_- + \beta t + \left(\dfrac{\rho}{\rho_-}\dfrac{[P]}{[\rho]}\right)^{\frac{1}{2}}, \\ v = u_- + \left(\dfrac{1}{\rho\rho_-}\dfrac{[P]}{[\rho]}\right)^{\frac{1}{2}}(\rho - \rho_-), \end{cases} \rho < \rho_-. \tag{3.8}$$

In the phase plane, given a state $(u_-, \rho_-)$, we draw curves (3.4) and (3.5) for $\rho < \rho_-$ and $\rho > \rho_-$ denoted by $R_1^\varepsilon$ and $R_2^\varepsilon$, respectively, see Fig. 3.1. Meanwhile, we draw curves (3.7) and (3.8) for $\rho > \rho_-$ and $\rho < \rho_-$ denoted by $S_1^\varepsilon$ and $S_2^\varepsilon$, respectively. Curves $R_1^\varepsilon$ and $R_2^\varepsilon$ have asymptotic lines: the positive $v$-axis and $v = u_- + \frac{2\sqrt{\varepsilon\alpha}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}$, respectively. Curves $S_1^\varepsilon$ and $S_2^\varepsilon$ have asymptotic lines: $v = u_- - \sqrt{\varepsilon}\rho_-^{-\frac{1+\alpha}{2}}$ and negative $v$-axis, respectively. Through the point $(u_- - 2\sqrt{\varepsilon}\rho_-^{-\frac{1+\alpha}{2}}, \rho_-)$, we draw curve

$$u + \sqrt{\varepsilon}\rho^{-\frac{1+\alpha}{2}} = u_- - \sqrt{\varepsilon}\rho_-^{-\frac{1+\alpha}{2}}, \tag{3.9}$$

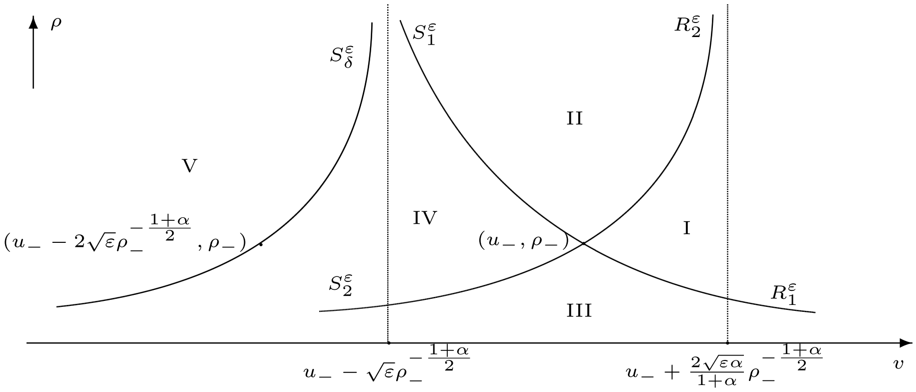



**Fig. 3.1.** The phase plane $(v, \rho)$.

denoted by $S_\delta^\varepsilon$. Curve $S_\delta^\varepsilon$ has two asymptotic lines: $v = u_- - \sqrt{\varepsilon}\rho_-^{-\frac{1+\alpha}{2}}$ and negative $v$-axis. Thus the phase plane is divided into five regions, see Fig. 3.1.

When $(u_+, \rho_+) \in (\mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV})(u_-, \rho_-)$, the Riemann solutions for (3.1) consist of rarefaction waves or shock waves. When $(u_+, \rho_+) \in \mathrm{V}$, we use a delta shock wave to connect states $(u_-, \rho_-)$ and $(u_+, \rho_+)$.

The detail can be found in [20]. The delta shock wave satisfies the generalized Rankine–Hugoniot conditions:

$$\begin{cases} \dfrac{dx(t,\varepsilon)}{dt} = u_\delta(t,\varepsilon) = v_\delta^\varepsilon + \beta t, \\ \dfrac{dw(t,\varepsilon)}{dt} = u_\delta(t,\varepsilon)[\rho] - [\rho(v+\beta t)], \\ \dfrac{d(w(t,\varepsilon)v_\delta^\varepsilon(t))}{dt} = u_\delta(t,\varepsilon)[\rho v] - [\rho v(v+\beta t)], \end{cases} \tag{3.10}$$

where $(x, w)(0, \varepsilon) = 0$, $w(t, \varepsilon)$ and $u_\delta(t, \varepsilon) = v_\delta^\varepsilon + \beta t$ are weight and velocity of delta shock wave respectively, $v_\delta^\varepsilon$ indicates the intermediate variable on this delta shock wave curve.

Solving (3.10), we have that

$$\begin{cases} w(t,\varepsilon) = \sqrt{\rho_-\rho_+\left((u_+ - u_-)^2 - \varepsilon\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right)\left(\frac{1}{\rho_-^\alpha} - \frac{1}{\rho_+^\alpha}\right)\right)} \quad t, \\ v_\delta^\varepsilon = \dfrac{\rho_+ u_+ - \rho_- u_- + \frac{\mathrm{d}\omega(t,\varepsilon)}{\mathrm{d}t}}{\rho_+ - \rho_-}, \\ x(t,\varepsilon) = v_\delta^\varepsilon t + \dfrac{1}{2}\beta t^2, \end{cases} \tag{3.11}$$

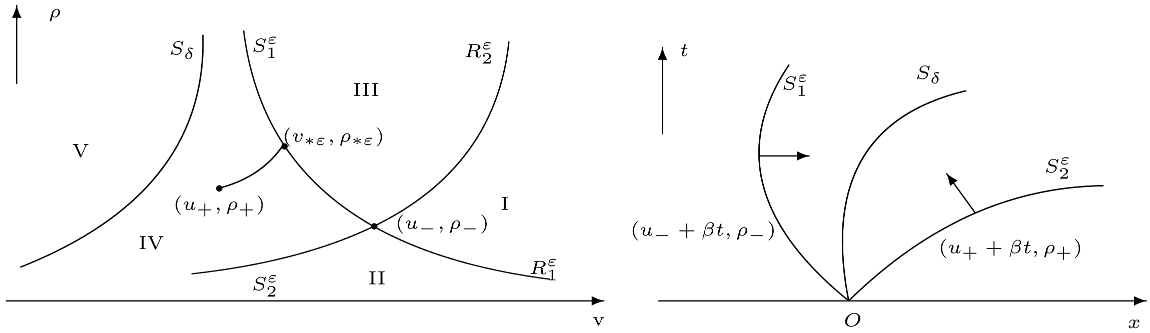
for $\rho_+ \neq \rho_-$ and

$$(t,\varepsilon) = (\rho_- u_- - \rho_+ u_+)t, \quad v_\delta^\varepsilon = \frac{1}{2}(u_+ + u_-), \quad x(t,\varepsilon) = \frac{1}{2}(u_+ + u_-) + \frac{1}{2}\beta t^2 \tag{3.12}$$

for $\rho_+ = \rho_-$. In addition, the delta shock wave should satisfy $\delta$-entropy condition

$$u_+ + \beta t + \sqrt{\alpha\varepsilon}\rho_+^{-\frac{1+\alpha}{2}} < u_\delta(t,\varepsilon) < u_- + \beta t - \sqrt{\alpha\varepsilon}\rho_-^{-\frac{1+\alpha}{2}}. \tag{3.13}$$

Therefore, for any given right state $(u_+, \rho_+)$, there exists a Riemann solution of (3.1). When $(u_+, \rho_+) \in (\mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{V})(u_-, \rho_-)$, the configurations of which are as follows:




(a) The phase plane

(b) The $(x, t)$ plane.

**Fig. 4.1.** Riemann solution when $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$.

1. $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$ : $(u_-, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon}, \rho_{*\varepsilon}) + R_2^\varepsilon + (u_+, \rho_+)$;
2. $(u_+, \rho_+) \in \mathrm{II}(u_-, \rho_-)$ : $(u_-, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon}, \rho_{*\varepsilon}) + S_2^\varepsilon + (u_+, \rho_+)$;
3. $(u_+, \rho_+) \in \mathrm{III}(u_-, \rho_-)$ : $(u_-, \rho_-) + S_1^\varepsilon + (v_{*\varepsilon}, \rho_{*\varepsilon}) + R_2^\varepsilon + (u_+, \rho_+)$;
4. $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$ : $(u_-, \rho_-) + S_1^\varepsilon + (v_{*\varepsilon}, \rho_{*\varepsilon}) + S_2^\varepsilon + (u_+, \rho_+)$;
5. $(u_+, \rho_+) \in \mathrm{V}(u_-, \rho_-)$ : $(u_-, \rho_-) + \delta S + (u_+, \rho_+)$,

where $(v_{*\varepsilon}, \rho_{*\varepsilon})$ is the intermediate state. If $(u_+, \rho_+) \in (R_1^\varepsilon \cup S_1^\varepsilon \cup R_2^\varepsilon \cup S_2^\varepsilon)$, we use one rarefaction wave or one shock wave to connect $(u_-, \rho_-)$ and $(u_+, \rho_+)$. By using (1.3), we obtain the Riemann solutions of (1.1) as follows

1. $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$ :
$(u_- + \beta t, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + R_2^\varepsilon + (u_+ + \beta t, \rho_+)$;
2. $(u_+, \rho_+) \in \mathrm{II}(u_-, \rho_-)$ :
$(u_- + \beta t, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + S_2^\varepsilon + (u_+ + \beta t, \rho_+)$;
3. $(u_+, \rho_+) \in \mathrm{III}(u_-, \rho_-)$
$(u_- + \beta t, \rho_-) + S_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + R_2^\varepsilon + (u_+ + \beta t, \rho_+)$;
4. $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$
$(u_- + \beta t, \rho_-) + S_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + S_2^\varepsilon + (u_+ + \beta t, \rho_+)$;
5. $(u_+, \rho_+) \in \mathrm{V}(u_-, \rho_-)$
$(u_- + \beta t, \rho_-) + \delta S + (u_+ + \beta t, \rho_+)$.

## 4. Limits of Riemann solutions to (1.1)–(1.2)

In this section, we investigate the limit behavior of the Riemann solutions to system (1.1)–(1.2), that is, the formation of delta shock and the vacuum states when pressure vanishes, respectively in the case $u_- > u_+$ and in the case $u_- < u_+$.

### *4.1. Formation of delta shock wave*

In this subsection, we study the limit behavior of Riemann solutions when $u_- > u_+$, see Fig. 4.1(a). We divide our discussion into two steps:

identify and analyze the formation of delta shock wave when $\varepsilon$ tends to a certain value, and display how the strength and propagation speed of the delta shock wave change when $\varepsilon$ tends to zero.

**Lemma 4.1.** *Assume $u_- > u_+$, then there exist two certain values $\varepsilon_1 > \varepsilon_2 > 0$, such that $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$; $(u_+, \rho_+) \in \mathrm{V}(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_2$.*

**Proof.** Suppose that $u_- > u_+$ and $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$, we have (see Fig. 4.1(a)),

$$u_+ < u_- - \left(\frac{[P]}{\rho_+\rho_-[\rho]}\right)^{\frac{1}{2}} (\rho_+ - \rho_-), \quad \rho_+ > \rho_-, \tag{4.1}$$

$$u_+ < u_- + \left(\frac{[P]}{\rho_+\rho_-[\rho]}\right)^{\frac{1}{2}} (\rho_+ - \rho_-),, \quad \rho_+ < \rho_-, \tag{4.2}$$

and

$$u_+ > u_- - \sqrt{\varepsilon}\left(\rho_+^{-\frac{1+\alpha}{2}} + \rho_-^{-\frac{1+\alpha}{2}}\right). \tag{4.3}$$

If $\rho_+ \neq \rho_-$, we deduce from (4.1)–(4.2) and (1.2) that

$$\sqrt{\varepsilon}\left|\left(\frac{\left(\frac{1}{\rho_-^\alpha} - \frac{1}{\rho_+^\alpha}\right)}{\rho_+\rho_-(\rho_+ - \rho_-)}\right)^{\frac{1}{2}} (\rho_+ - \rho_-)\right| < u_- - u_+, \tag{4.4}$$

i.e.,

$$\varepsilon < \frac{\rho_-^{\alpha+1}\rho_+^{\alpha+1}(u_- - u_+)^2}{(\rho_+^\alpha - \rho_-^\alpha)(\rho_+ - \rho_-)}. \tag{4.5}$$

Let

$$\varepsilon_1 = \frac{\rho_-^{\alpha+1}\rho_+^{\alpha+1}(u_- - u_+)^2}{(\rho_+^\alpha - \rho_-^\alpha)(\rho_+ - \rho_-)}, \tag{4.6}$$

then $(u_+, \rho_+) \in \mathrm{IV} \cup \mathrm{V}(u_-, \rho_-)$ when $\varepsilon < \varepsilon_1$. According to (4.3), we have

$$\varepsilon > \left(\frac{(u_- - u_+)}{\rho_-^{-\frac{1+\alpha}{2}} + \rho_+^{-\frac{1+\alpha}{2}}}\right)^2 = \left(\frac{\rho_-^{\frac{1+\alpha}{2}}\rho_+^{\frac{1+\alpha}{2}}(u_- - u_+)}{\rho_-^{\frac{1+\alpha}{2}} + \rho_+^{\frac{1+\alpha}{2}}}\right)^2. \tag{4.7}$$

Let

$$\varepsilon_2 = \left(\frac{\rho_-^{\frac{1+\alpha}{2}}\rho_+^{\frac{1+\alpha}{2}}(u_- - u_+)}{\rho_-^{\frac{1+\alpha}{2}} + \rho_+^{\frac{1+\alpha}{2}}}\right)^2, \tag{4.8}$$

then $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$ when $\varepsilon_2 < \varepsilon < \varepsilon_1$, and $(u_+, \rho_+) \in \mathrm{V}(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_2$.

If $\rho_+ = \rho_-$, this conclusion is also true. The proof is completed. □

Lemma 4.1 implies that there is no delta shock wave when $\varepsilon > \varepsilon_2$.

Now we show how the strength and propagation speed of the delta shock wave change when $\varepsilon$ tends to zero.

We first consider the situation $(u_+, \rho_+) \in \mathrm{IV}(u_-, \rho_-)$. In this situation, the Riemann solution to (3.1) and (2.2) presented in Section 3 is

$$S_1^\varepsilon:\begin{cases} v_{*\varepsilon}=u_- - \sqrt{\varepsilon\left(\frac{1}{\rho_-}-\frac{1}{\rho_{*\varepsilon}}\right)\left(\frac{1}{\rho_-^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}, \\ \sigma_1^\varepsilon(t)=u_-+\beta t-\sqrt{\frac{\varepsilon\rho_{*\varepsilon}\left(\frac{1}{\rho_-^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}{\rho_-(\rho_{*\varepsilon}-\rho_-)}}, \end{cases} \quad \rho_{*\varepsilon}>\rho_-, \tag{4.9}$$

$$S_2^\varepsilon:\begin{cases} v_{*\varepsilon}=u_+ + \sqrt{\varepsilon\left(\frac{1}{\rho_+}-\frac{1}{\rho_{*\varepsilon}}\right)\left(\frac{1}{\rho_+^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}, \\ \sigma_2^\varepsilon(t)=u_++\beta t+\sqrt{\frac{\varepsilon\rho_+\left(\frac{1}{\rho_{*\varepsilon}^\alpha}-\frac{1}{\rho_+^\alpha}\right)}{\rho_{*\varepsilon}(\rho_+-\rho_{*\varepsilon})}}, \end{cases} \quad \rho_{*\varepsilon}>\rho_+, \tag{4.10}$$

where $(v_{*\varepsilon},\rho_{*\varepsilon})$ is the intermediate state, see Fig. 4.1(a). By using the first equation of (4.9) and (4.11), we have

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_2}\left(\sqrt{\varepsilon\left(\frac{1}{\rho_-}-\frac{1}{\rho_{*\varepsilon}}\right)\left(\frac{1}{\rho_-^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}+\sqrt{\varepsilon\left(\frac{1}{\rho_+}-\frac{1}{\rho_{*\varepsilon}}\right)\left(\frac{1}{\rho_+^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}\right)\\&=u_--u_+,\ \rho_{*\varepsilon}>\rho_\pm.\end{aligned} \tag{4.11}$$

It follows that

$$\lim_{\varepsilon\to\varepsilon_2}\rho_{*\varepsilon}=\infty. \tag{4.12}$$

**Lemma 4.2.** *Let*

$$u_\delta(t,\varepsilon_2)=\frac{u_-\rho_-^{\frac{1+\alpha}{2}}+u_+\rho_+^{\frac{1+\alpha}{2}}}{\rho_-^{\frac{1+\alpha}{2}}+\rho_+^{\frac{1+\alpha}{2}}}+\beta t, \tag{4.13}$$

*then*

$$\lim_{\varepsilon\to\varepsilon_2}u_{*\varepsilon}=\lim_{\varepsilon\to\varepsilon_2}(v_{*\varepsilon}+\beta t)=\lim_{\varepsilon\to\varepsilon_2}\sigma_1^\varepsilon(t)=\lim_{\varepsilon\to\varepsilon_2}\sigma_2^\varepsilon(t)=u_\delta(t,\varepsilon_2), \tag{4.14}$$

$$\lim_{\varepsilon\to\varepsilon_2}\int_{x_1(t,\varepsilon)}^{x_2(t,\varepsilon)}\rho_{*\varepsilon}dx=\left(u_\delta(t,\varepsilon_2)[\rho]-[\rho(v+\beta t)]\right)t, \tag{4.15}$$

$$\lim_{\varepsilon\to\varepsilon_2}\int_{x_1(t,\varepsilon)}^{x_2(t,\varepsilon)}\rho_{*\varepsilon}v_{*\varepsilon}dx=\left(u_\delta(t,\varepsilon_2)[\rho v]-\left[\rho v(v+\beta)-\frac{\varepsilon}{\rho^\alpha}\right]\right)t, \tag{4.16}$$

*where* $[\rho]=\rho_+-\rho_-$,

$$x_1(t,\varepsilon)=\int_0^t\sigma_1^\varepsilon(\tau)d\tau=\left(u_--\sqrt{\frac{\varepsilon\rho_{*\varepsilon}\left(\frac{1}{\rho_-^\alpha}-\frac{1}{\rho_{*\varepsilon}^\alpha}\right)}{\rho_-(\rho_{*\varepsilon}-\rho_-)}}\right)t+\frac{1}{2}\beta t^2, \tag{4.17}$$

$$x_2(t,\varepsilon)=\int_0^t\sigma_2^\varepsilon(\tau)d\tau=\left(u_++\sqrt{\frac{\varepsilon\rho_+\left(\frac{1}{\rho_{*\varepsilon}^\alpha}-\frac{1}{\rho_+^\alpha}\right)}{\rho_{*\varepsilon}(\rho_+-\rho_{*\varepsilon})}}\right)t+\frac{1}{2}\beta t^2. \tag{4.18}$$

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**Proof.** Letting $\varepsilon \to \varepsilon_2$, it follows from (1.3), (4.9) and (4.12) that

$$\lim_{\varepsilon\to\varepsilon_2} u_{*\varepsilon} = \lim_{\varepsilon\to\varepsilon_2} (v_{*\varepsilon} + \beta t) = \lim_{\varepsilon\to\varepsilon_2} \left( u_- - \sqrt{\varepsilon \left( \frac{1}{\rho_-} - \frac{1}{\rho_{*\varepsilon}} \right) \left( \frac{1}{\rho_-^\alpha} - \frac{1}{\rho_{*\varepsilon}^\alpha} \right)} \right) \\ = u_\delta(t, \varepsilon_2). \tag{4.19}$$

From (4.9)–(4.10) and (4.12), we get

$$\lim_{\varepsilon\to\varepsilon_2} \sigma_1^\varepsilon(t) = \lim_{\varepsilon\to\varepsilon_2} \sigma_2^\varepsilon(t) = u_\delta(t, \varepsilon_2). \tag{4.20}$$

Thus, we conclude from (4.19)–(4.20) that the two shock waves $S_1^\varepsilon$ and $S_2^\varepsilon$ will coincide when $\varepsilon \to \varepsilon_2$.

In view of the first equation of Rankine–Hugoniot conditions (3.6) for both $S_1^\varepsilon$ and $S_2^\varepsilon$, we find

$$\begin{cases} \sigma_1^\varepsilon(t)(\rho_{*\varepsilon} - \rho_-) = \rho_{*\varepsilon}(v_{*\varepsilon} + \beta t) - \rho_-(u_- + \beta t), \\ \sigma_2^\varepsilon(t)(\rho_+ - \rho_{*\varepsilon}) = \rho_+(u_+ + \beta t) - \rho_{*\varepsilon}(v_{*\varepsilon} + \beta t). \end{cases} \tag{4.21}$$

From (4.20)–(4.21), we have

$$\lim_{\varepsilon\to\varepsilon_2} (\sigma_2^\varepsilon(t) - \sigma_1^\varepsilon(t))\rho_{*\varepsilon} = u_\delta(t, \varepsilon_2)[\rho] - [\rho(v + \beta t)]. \tag{4.22}$$

Using (4.22), we get

$$\lim_{\varepsilon\to\varepsilon_2} \int_{x_1(t,\varepsilon)}^{x_2(t,\varepsilon)} \rho_{*\varepsilon} dx = \lim_{\varepsilon\to\varepsilon_2} (\sigma_2^\varepsilon(t) - \sigma_1^\varepsilon(t))\rho_{*\varepsilon} t = (u_\delta(t, \varepsilon_2)[\rho] - [\rho(v + \beta t)])t \tag{4.23}$$

which proves (4.15).

We can obtain (4.16) similarly. The proof is completed. □

Lemma 4.2 shows that the two shock waves $S_1^\varepsilon$ and $S_2^\varepsilon$ coincide as $\varepsilon$ tends to $\varepsilon_2$, see Fig. 4.1(b). From (4.12) and (4.15), we find that $\rho_{*\varepsilon}$ possesses a singularity which is a weighed Dirac delta function with speed $u_\delta(t, \varepsilon_2)$. Now, if $\rho_+ \neq \rho_-$, we deduce from (3.11) and (4.8) that

$$\lim_{\varepsilon\to\varepsilon_2} w(t, \varepsilon) = w(t, \varepsilon_2) = (u_\delta(t, \varepsilon_2)[\rho] - [\rho(v + \beta t)])t, \tag{4.24}$$

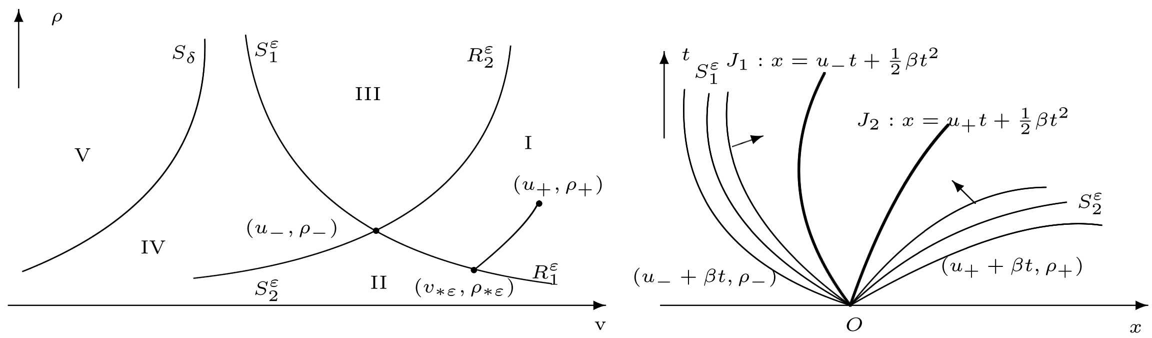
$$\lim_{\varepsilon\to\varepsilon_2} u_\delta(t, \varepsilon) = u_\delta(t, \varepsilon_2) = v_\delta^{\varepsilon_2} + \beta t. \tag{4.25}$$

It is easy to see that the quantities $\omega(t, \varepsilon)$, $u_\delta(t, \varepsilon)$ and the limits of $u_{*\varepsilon}$, $\sigma_1^\varepsilon$ and $\sigma_2^\varepsilon$ are consistent with (4.15) when $\varepsilon$ arrives at $\varepsilon_2$. For $\rho_+ = \rho_-$, we can obtain the same conclusion. Therefore, as $\varepsilon \to \varepsilon_2$, the limit of the two shock waves of (1.1)–(1.2) is the delta shock solution of (1.1)–(1.2) corresponding to the case of the Riemann solution of the case $(u_+, \rho_+) \in S_\delta$, see Fig. 4.1(a).

Next, we discuss the situation $0 < \varepsilon < \varepsilon_2$, in which $(u_+, \rho_+) \in \mathrm{V}(u_-, \rho_-)$. In this situation, the Riemann solution to (1.1)–(1.2) is a delta shock wave solution. If $\rho_+ \neq \rho_-$, we deduce from (3.11) that

$$\frac{\partial w(t, \varepsilon)}{\partial \varepsilon} < 0 \quad \text{and} \quad \frac{\partial u_\delta(t, \varepsilon)}{\partial \varepsilon} < 0. \tag{4.26}$$

Using (4.26), we know that both the strength and the propagation speed of the delta shock wave increase as $\varepsilon$ decreases. Furthermore, taking the limit $\varepsilon \to 0$ in (3.11) leads to




(a) The phase plane (b) The $(x, t)$ plane.

**Fig. 4.2.** Riemann solution when $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$.

$$\lim_{\varepsilon\to 0} w(t,\varepsilon) = \sqrt{\rho_+\rho_-}(u_- - u_+)t, \tag{4.27}$$

$$\lim_{\varepsilon\to 0} u_\delta(t,\varepsilon) = \frac{\sqrt{\rho_+}u_+ + \sqrt{\rho_-}u_-}{\sqrt{\rho_+} + \sqrt{\rho_-}} + \beta t. \tag{4.28}$$

By (4.27)–(4.28), we deduce that the delta shock solution to (1.1)–(1.2) converges to the Riemann solution of (1.4). The same conclusion holds for $\rho_+ = \rho_-$.

Therefore, as $\varepsilon \to 0$, the delta shock solution is nothing but the Riemann solution to (1.4). Therefore, we have proved the first part of Theorem 1.1.

### *4.2. Formation of vacuum states*

In this subsection, we will show the limit behavior of Riemann solutions of (1.1)–(1.2), in the case $u_- < u_+$ as the pressure decreases, see Fig. 4.2(a).

**Lemma 4.3.** *If $u_- < u_+$, then there exists $\varepsilon_3 > 0$ such that $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$ when $0 < \varepsilon < \varepsilon_3$.*

**Proof.** If $\rho_+ = \rho_-$, then the conclusion is obviously true. Next, we discuss the situation $\rho_+ \neq \rho_-$. Assume that $u_- < u_+$ and $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$, we obtain (see Fig. 4.2(a)):

$$u_+ > u_- + \frac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\left(\rho_+^{-\frac{1+\alpha}{2}} - \rho_-^{-\frac{1+\alpha}{2}}\right), \quad \rho_+ < \rho_-, \tag{4.29}$$

$$u_+ > u_- - \frac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\left(\rho_+^{-\frac{1+\alpha}{2}} - \rho_-^{-\frac{1+\alpha}{2}}\right), \quad \rho_+ > \rho_-. \tag{4.30}$$

According to (4.29) and (4.30), we have

$$\varepsilon < \frac{(1+\alpha)^2\rho_-^{1+\alpha}\rho_+^{1+\alpha}(u_+ - u_-)^2}{4\alpha\left(\rho_+^{\frac{1+\alpha}{2}} - \rho_-^{\frac{1+\alpha}{2}}\right)^2}. \tag{4.31}$$

Letting

$$\varepsilon_3 = \frac{(1+\alpha)^2\rho_-^{1+\alpha}\rho_+^{1+\alpha}(u_+ - u_-)^2}{4\alpha\left(\rho_+^{\frac{1+\alpha}{2}} - \rho_-^{\frac{1+\alpha}{2}}\right)^2}, \tag{4.32}$$

we find $(u_+, \rho_+) \in \mathrm{I}(u_-, \rho_-)$ when $\varepsilon < \varepsilon_3$. $\square$

As $u_- < u_+$, by Lemma 4.3, for any given $\varepsilon \in (0, \varepsilon_3)$, the Riemann solution of (1.1)–(1.2) is

$$(u_- + \beta t, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + R_2^\varepsilon + (u_+ + \beta t, \rho_+) \tag{4.33}$$

where

$$R_1^\varepsilon : \begin{cases} \dfrac{dx}{dt} = \lambda_1(v, \rho) = v + \beta t - \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}}, \\ v - \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_- - \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}, \quad \rho < \rho_-; \\ \lambda_1(u_-, \rho_-) \le \lambda_1(v, \rho), \end{cases} \tag{4.34}$$

$$R_2^\varepsilon : \begin{cases} \dfrac{dx}{dt} = \lambda_2(v, \rho) = v + \beta t + \sqrt{\alpha\varepsilon}\rho^{-\frac{1+\alpha}{2}}, \\ v + \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} = u_+ + \dfrac{2\sqrt{\alpha\varepsilon}}{1+\alpha}\rho_+^{-\frac{1+\alpha}{2}}, \quad \rho < \rho_+ \\ \lambda_2(v, \rho) \le \lambda_2(u_+, \rho_+), \end{cases} \tag{4.35}$$

(see Fig. 4.2(a)). By (4.34)–(4.35), we have

$$\rho_{*\varepsilon}^{-\frac{1+\alpha}{2}} = \frac{1+\alpha}{4\sqrt{\alpha\varepsilon}}\left(u_+ + \frac{\alpha\varepsilon}{1+\alpha}\rho_+^{-\frac{1+\alpha}{2}} - u_- + \frac{\alpha\varepsilon}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}}\right), \tag{4.36}$$

$$\lim_{\varepsilon\to 0} v = u_- \text{ on } R_1^\varepsilon, \qquad \lim_{\varepsilon\to 0} v = u_+ \text{ on } R_2^\varepsilon, \tag{4.37}$$

and

$$\begin{cases} \lambda_1 = \dfrac{1-\alpha}{2}v + \dfrac{1+\alpha}{2}u_- - \sqrt{\alpha\varepsilon}\rho_-^{-\frac{1+\alpha}{2}} + \beta t, \\ \lambda_2 = \dfrac{1-\alpha}{2}v + \dfrac{1+\alpha}{2}u_+ + \sqrt{\alpha\varepsilon}\rho_-^{-\frac{1+\alpha}{2}} + \beta t. \end{cases} \tag{4.38}$$

Combining (4.36)–(4.38), we have

$$\lim_{\varepsilon\to 0}\rho_{*\varepsilon} = 0, \quad \lim_{\varepsilon\to 0}\lambda_1 = u_- + \beta t, \quad \lim_{\varepsilon\to 0}\lambda_2 = u_+ + \beta t. \tag{4.39}$$

The above identities assert that as $\varepsilon \to 0$, $\rho_{*\varepsilon}$ vanishes and two rarefaction waves $R_1^\varepsilon$ and $R_2^\varepsilon$ become two contact discontinuities connecting the state $(u_\pm + \beta t, \rho_\pm)$ and the vacuum $\rho_{*\varepsilon} = 0$, which is one kind of Riemann solution of (1.4), see Fig. 4.2(b). Therefore, we have proved the second part of Theorem 1.1.

*4.3. Limit of Riemann solutions for $u_- = u_+$*

In this subcase, we discuss $(u_+, \rho_+) \in \mathrm{II}(u_-, \rho_-) \cup \mathrm{III}(u_-, \rho_-)$.

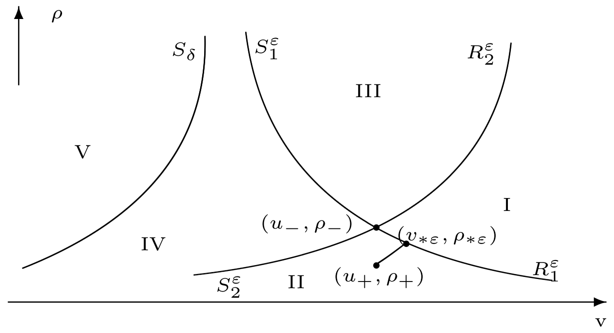
If $(u_+, \rho_+) \in \mathrm{II}(u_-, \rho_-)$, then the Riemann solution of (1.1)–(1.2) is

$$(u_- + \beta t, \rho_-) + R_1^\varepsilon + (v_{*\varepsilon} + \beta t, \rho_{*\varepsilon}) + S_2^\varepsilon + (u_+ + \beta t, \rho_+), \tag{4.40}$$

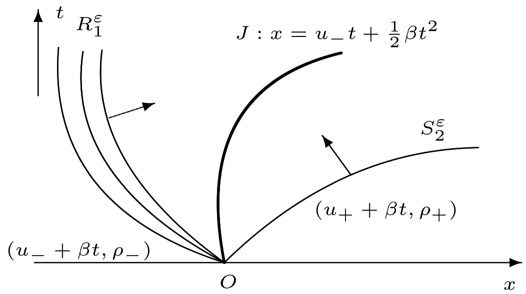
where $R_1^\varepsilon$, $S_2^\varepsilon$ are given by (4.34) and (4.10) respectively, $(v_{*\varepsilon} + \beta t, \rho_{*\varepsilon})$ is the intermediate state. From (4.34) and (4.10) and $\rho_+ < \rho_{*\varepsilon} < \rho_-$, we have

$$\lim_{\varepsilon\to 0} v_{*\varepsilon} + \beta t = \lim_{\varepsilon\to 0}\lambda_1(v_{*\varepsilon}, \rho_{*\varepsilon}) = \sigma_2^\varepsilon(t) = u_- + \beta t. \tag{4.41}$$

For $(u_+, \rho_+) \in \mathrm{III}(u_-, \rho_-)$, we can obtain same conclusion.




(a) The phase plane




(b) The $(x, t)$ plane.

**Fig. 4.3.** Riemann solution when $(u_+, \rho_+) \in \mathrm{II}(u_-, \rho_-)$.

Thus, if $u_+ = u_-$, when $\varepsilon \to 0$, the limit of the Riemann solution is one kind of the Riemann solution to (1.4), see Fig. 4.3(b). Therefore, we have proved the third part of Theorem 1.1.

## 5. Discussion

We have studied the limit behavior of the Riemann solutions to the generalized Chaplygin gas equations with a source term. We find the formation of delta shock wave and vacuum states in the Riemann solutions to system (1.1)–(1.2) when pressure vanishes as $\varepsilon \to 0$. Specifically, for $u_- > u_+$, when $\varepsilon \to \varepsilon_2$, for some $\varepsilon_2 > 0$, the two shock waves $S_1^\varepsilon$ and $S_2^\varepsilon$ coincide and the Riemann solution converges to a delta shock wave solution of the same system (1.1)–(1.2). When $\varepsilon$ continues to drop, both the strength and propagation speed of this delta shock become stronger. In the end, as $\varepsilon \to 0$, the delta shock solution is nothing but the Riemann solution to the pressureless gas dynamics with a body force. For $u_- < u_+$, when $\varepsilon$ tends to zero, we find that the two rarefaction waves converge to two contact discontinuities connecting the states $(u_\pm + \beta t, \rho_\pm)$ and the vacuum. For $u_- = u_+$, the Riemann solutions to system (1.1) and (1.2) converge to one contact discontinuity with the propagation speed $u_- + \beta t$. The results for the generalized Chaplygin gas equations with a source term in the current paper are qualitatively similar to those for the Chaplygin gas equations with a source term in [10].

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