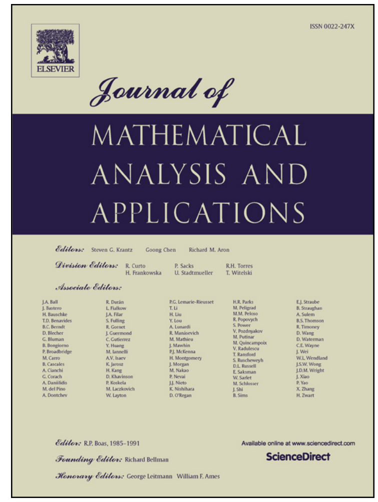

A fixed point theorem in n-Banach spaces and Ulam stability

Janusz Brzdęk, Krzysztof Ciepliński

PII: S0022-247X(18)30851-5
DOI: https://doi.org/10.1016/j.jmaa.2018.10.028
Reference: YJMAA 22616

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 5 May 2018

A fixed point theorem in n-Banach spaces and Ulam stability

Janusz Brzdęk

AGH University of Science and Technology, Faculty of Applied Mathematics, Mickiewicza 30, 30-059 Krakow, Poland

Krzysztof Ciepliński*

AGH University of Science and Technology, Faculty of Applied Mathematics, Mickiewicza 30, 30-059 Krakow, Poland

Abstract

Motivated by some issues in Ulam stability, we prove a fixed point theorem for operators acting on some classes of functions, with values in n-Banach spaces. We also present applications of it to Ulam stability of eigenvectors and some functional and difference equations.

Keywords: Fixed point theorem, n-normed space, Ulam stability, difference equation, functional equation.

1. Introduction

The following natural question arises in many areas of scientific investigations: what errors we commit replacing the exact solutions to some equations by functions that satisfy those equations only approximately (or vice versa). Some efficient tools to evaluate those errors can be found in the theory of Ulam's stability.

Roughly speaking, nowadays we say that an equation is Ulam stable in some class of functions if any function from that class, satisfying the equation approximately (in some sense), is near (in some way) to an exact solution of the equation (see Definition 2). The problem of such a stability was formulated for the first time by Ulam in 1940 for homomorphisms of metric groups; a solution to it was published a year later by Hyers for Banach spaces (for details see [23]).

In the last few decades, several stability issues of similar kind, for various (functional, differential, difference, integral) equations, have been investigated by many mathematicians (see [2, 6, 23, 24] for the comprehensive accounts of the

*Corresponding author.

Email addresses: `brzdek@agh.edu.pl, brzjan05@gmail.com` (Janusz Brzdęk), `cieplin@agh.edu.pl` (Krzysztof Ciepliński)

subject), but mainly in classical spaces. However, the notion of an approximate solution and the idea of nearness of two functions can be understood in various, nonstandard ways, depending on the needs and tools available in a particular situation. One of such non-classical measures of a distance can be introduced by the notion of the n-norm.

Recall that the concept of linear 2-normed space was introduced by S. Gähler in [18], and it seems that the first work on the Hyers-Ulam stability of functional equations in the complete 2-normed spaces (i.e., 2-Banach spaces) is [19]. After it some papers on the stability of other equations in such spaces have been published. The notion of 2-normed space was generalized by A. Misiak in [29], who introduced n-normed spaces. Results on the Hyers-Ulam stability of some functional equations in n-Banach spaces were obtained in [12, 17, 28, 34, 35].

It has been shown that there is a close connection between some fixed point theorems and the Ulam stability theory (see, e.g., [3, 14]). The aim of this paper is to prove a fixed point theorem in n-Banach spaces (see Theorem 3) and show that it has significant applications to the Ulam stability of eigenvectors (see Corollary 11) and some functional and difference equations (see Corollary 10 and the part following it). Let us mention yet that our Theorems 3 and 4 correspond to several outcomes from [4, 5, 10].

Throughout this paper $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{R}_+ := [0, \infty)$.

2. Preliminaries

In 1989, A. Misiak (see [29]) defined the n-normed spaces and studied their properties. The concept of an n-normed space is a generalization of the notions of a classical normed space and of a 2-normed space introduced by S. Gähler (see [18]). Let us also mention here that H. Gunawan and M. Mashadi (see [20]) showed that from every n-normed space one can derive an $(n-1)$-normed space and thus a normed space. More information on these spaces and on some problems investigated in them, among others in fixed point theory, can be found for instance in [11, 12, 13, 16, 17, 21, 27, 28, 34, 35].

Now, we recall some basic definitions and facts concerning n-normed spaces (for more details, we refer the reader to [12, 20, 29, 34, 35]).

Let $n \in \mathbb{N}$, X be a real linear space, which is at least n-dimensional, and $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ be a function satisfying the following conditions:

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$,

(N4) $\|x + y, x_2, \dots, x_n\| \le \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$

for any $\alpha \in \mathbb{R}$ and $x, y, x_1, \dots, x_n \in X$. Then the function $\|\cdot, \dots, \cdot\|$ is called an n-norm on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is said to be an n-normed space.

Let us mention two standard examples of n-norms:

1. The Euclidean n-norm $\|x_1, \ldots, x_n\|_E$ on $\mathbb{R}^n$ is given by

$$\|x_1, \ldots, x_n\|_E = |\det(x_{ij})| = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right), \tag{1}$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for $i \in \{1, \ldots, n\}$.

2. Let $(X, < \cdot, \cdot >)$ be an at least n-dimensional real inner product space. The standard n-norm on X is given by

$$\|x_1, \ldots, x_n\|_S = \begin{vmatrix} < x_1, x_1 > & \cdots & < x_1, x_n > \\ \vdots & \ddots & \vdots \\ < x_n, x_1 > & \cdots & < x_n, x_n > \end{vmatrix}^{1/2},$$

where $x_i \in X$ for $i \in \{1, \ldots, n\}$. If $X = \mathbb{R}^n$, then this n-norm is the same as the previous one.

If $(X, \|\cdot, \ldots, \cdot\|)$ is an n-normed space, then the function $\|\cdot, \ldots, \cdot\|$ is non-negative and

$$\Big\|\sum_{i=1}^{k} y_i, x_2, \ldots, x_n\Big\| \leq \sum_{i=1}^{k} \|y_i, x_2, \ldots, x_n\|$$

for any $k \in \mathbb{N}, x_2, \ldots, x_n \in X$ and $y_i \in X$ for $i \in \{1, \ldots, k\}$.

A sequence $(y_k)_{k\in\mathbb{N}}$ of elements of an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is called a Cauchy sequence if

$$\lim_{k,l\to\infty} \|y_k - y_l, x_2, \ldots, x_n\| = 0, \qquad x_2, \ldots, x_n \in X,$$

whereas $(y_k)_{k\in\mathbb{N}}$ is said to be convergent if there exists a $y \in X$ (called the limit of this sequence and denoted by $\lim_{k\to\infty} y_k$) with

$$\lim_{k\to\infty} \|y_k - y, x_2, \ldots, x_n\| = 0, \qquad x_2, \ldots, x_n \in X.$$

An n-normed space in which every Cauchy sequence is convergent is called an n-Banach space.

Let us also mention that in n-normed spaces every convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product are valid.

Moreover, we have the following properties stated in a form of a lemma in [35] (see also [12]).

Lemma 1. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. Then the following four conditions hold:*

(i) *if $x_1, \ldots, x_n \in X, \alpha \in \mathbb{R}$, $i, j \in \{1, \ldots, n\}$ and $i < j$, then*

$$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j + \alpha x_i, \ldots, x_n\|;$$

(ii) *if* $x, y, y_2, \ldots, y_n \in X$, *then*

$$\big| \|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\| \big| \leq \|x - y, y_2, \ldots, y_n\|;$$

(iii) *if* $x \in X$ *and*

$$\|x, y_2, \ldots, y_n\| = 0, \qquad y_2, \ldots, y_n \in X,$$

then $x = 0$;

(iv) *if* $(x_k)_{k \in \mathbb{N}}$ *is a convergent sequence of elements of* X, *then*

$$\lim_{k\to\infty} \|x_k, y_2, \ldots, y_n\| = \big\| \lim_{k\to\infty} x_k, y_2, \ldots, y_n \big\|, \qquad y_2, \ldots, y_n \in X.$$

The name of Ulam has been somehow connected with various definitions of stability (see, e.g., [1, 23, 31]), but roughly speaking, the following one describes our considerations in this paper (A^B denotes the family of all functions mapping a set B into a set A).

Definition 2. *Let* $(Y, \|\cdot, \ldots, \cdot\|)$ *be an* $(n+1)$*-normed space,* $S \neq \emptyset$ *be a set,* $\mathcal{D}_0 \subset \mathcal{D} \subset Y^S$ *and* $\mathcal{E} \subset {\mathbb{R}_+}^{S\times Y^n}$ *be nonempty,* $\mathcal{S} : \mathcal{E} \to {\mathbb{R}_+}^{S \times Y^n}$ *and* $\mathcal{T} : \mathcal{D} \to Y^S$. *We say that the equation*

$$\mathcal{T}(\psi) = \psi$$

is $\mathcal{S}$*–stable in* $\mathcal{D}_0$ *provided, for any* $\psi \in \mathcal{D}_0$ *and* $\delta \in \mathcal{E}$ *with*

$$\big\|\mathcal{T}(\psi)(t) - \psi(t), y_1, \ldots, y_n\big\| \leq \delta(t, y_1, \ldots, y_n), \qquad t \in S, y_1, \ldots, y_n \in Y,$$

there is a solution $\phi \in \mathcal{D}$ *of the equation such that*

$$\big\|\phi(t) - \psi(t), y_1, \ldots, y_n\big\| \leq (\mathcal{S}\delta)(t, y_1, \ldots, y_n), \qquad t \in S, y_1, \ldots, y_n \in Y.$$

3. Fixed point theorem

In the rest of the paper we assume that $m \in \mathbb{N}$ and $(Y, \|\cdot, \ldots, \cdot\|)$ is an $(m+1)$–Banach space. To simplify the notation we write

$$\|z, y\| := \|z, y_1, \ldots, y_m\|, \qquad z \in Y, y = (y_1, \ldots, y_m) \in Y^m.$$

Moreover, E always denotes a nonempty set and $\Delta : Y^E \times Y^E \to {\mathbb{R}_+}^{E \times Y^m}$ is defined by

$$\Delta(\xi, \mu)(x, y) := \|\xi(x) - \mu(x), y\|, \qquad \xi, \mu \in Y^E, x \in E, y \in Y^m.$$

Let $\emptyset \neq \mathcal{D} \subset {\mathbb{R}_+}^{E \times Y^m}$, $\emptyset \neq \mathcal{C} \subset Y^E$ and $\Lambda \colon \mathcal{D} \to {\mathbb{R}_+}^{E\times Y^m}$. We say that $\mathcal{T} \colon \mathcal{C} \to Y^E$ is $\Lambda-$contractive provided

$$\Delta(\mathcal{T}\xi, \mathcal{T}\mu)(x, y) \leq \Lambda\delta(x, y), \qquad x \in E, y \in Y^m,$$

for any $\xi, \mu \in \mathcal{C}$ and $\delta \in \mathcal{D}$ with

$$\Delta(\xi, \mu)(x, y) \leq \delta(x, y), \qquad x \in E, y \in Y^m.$$

Next, a subset $\mathcal{F}$ of Y^E is called pointwise closed if every $\chi \in Y^E$ such that

$$\chi(x) = \lim_{n\to\infty} \chi_n(x), \qquad x \in E,$$

with a sequence $(\chi_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$, belongs to $\mathcal{F}$.

Given $A \neq \emptyset$ and $f \in A^A$, we define $f^n \in A^A$ (for $n \in \mathbb{N}_0$) by:

$$f^0(x) = x, \qquad f^{n+1}(x) = f(f^n(x)), \qquad x \in A, n \in \mathbb{N}_0.$$

Finally, if $f, g \in \mathbb{R}^A$, then we write $f \leq g$ provided $f(x) \leq g(x)$ for every $x \in A$.

Now, we are in a position to present the above mentioned fixed point theorem; its proof is provided in the last section of the paper.

Theorem 3. *Let $\emptyset \neq \mathcal{C} \subset Y^E$ be pointwise closed, $\Lambda_n \colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$ for $n \in \mathbb{N}$, and $\mathcal{T} \colon \mathcal{C} \to \mathcal{C}$. Assume also that $\mathcal{T}^n$ is Λ_n – contractive for $n \in \mathbb{N}$, and there exist functions $\varepsilon \in {\mathbb{R}_+}^{E\times Y^m}$ and $\varphi \in \mathcal{C}$ such that*

$$\big\|\mathcal{T}\varphi(x) - \varphi(x), y\big\| \leq \varepsilon(x, y), \qquad x \in E, y \in Y^m, \tag{2}$$

$$\liminf_{n\to\infty} \Lambda_1\Big(\sum_{i=n}^{\infty} \Lambda_i\varepsilon\Big)(x, y) = 0, \qquad x \in E, y \in Y^m, \tag{3}$$

$$\varepsilon^*(x, y) := \sum_{i=0}^{\infty} \Lambda_i\varepsilon(x, y) < \infty, \qquad x \in E, y \in Y^m, \tag{4}$$

where $\Lambda_0\varepsilon(x, y) := \varepsilon(x, y)$ for $x \in E$ and $y \in Y^m$. Then, for each $x \in E$, there exists the limit

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x) \tag{5}$$

and the function $\psi \in \mathcal{C}$, defined in this way, is a unique fixed point of $\mathcal{T}$ with

$$\|\mathcal{T}^n\varphi(x) - \psi(x), y\| \leq \sum_{i=n}^{\infty} \Lambda_i\varepsilon(x, y), \qquad n \in \mathbb{N}_0, x \in E, y \in Y^m. \tag{6}$$

Moreover, the following two statements are valid:

(a) *for every sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers with $\lim_{n\to\infty} k_n = \infty$, ψ is the unique function in $\mathcal{C}$ such that*

$$\|\mathcal{T}^{k_n}\varphi(x) - \psi(x), y\| \leq \sum_{i=k_n}^{\infty} \Lambda_i\varepsilon(x, y), \qquad n \in \mathbb{N}, x \in E, y \in Y^m; \tag{7}$$

(b) *if*

$$\liminf_{n\to\infty} \Lambda_n \varepsilon^*(x,y) = 0, \qquad x \in E, y \in Y^m, \tag{8}$$

then ψ *is the unique fixed point of* $\mathcal{T}$ *satisfying*

$$\|\varphi(x) - \psi(x), y\| \le \varepsilon^*(x,y), \qquad x \in E, y \in Y^m. \tag{9}$$

Now we show some simple consequences of Theorem 3. Let us start with a result that corresponds to [10, Theorem 2.2] and [5, Theorem 2].

Corollary 4. *Let* $\emptyset \neq \mathcal{C} \subset Y^E$ *be pointwise closed,* $\Lambda\colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$, *and* $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$. *Assume also that* $\mathcal{T}$ *is* $\Lambda-$*contractive, and there exist functions* $\varepsilon \in {\mathbb{R}_+}^{E\times Y^m}$ *and* $\varphi \in \mathcal{C}$ *such that* (2) *holds,*

$$\varepsilon^*(x,y) := \sum_{i=0}^{\infty} \Lambda^i \varepsilon(x,y) < \infty, \qquad x \in E, y \in Y^m, \tag{10}$$

and

$$\liminf_{n\to\infty} \Lambda\Big(\sum_{i=n}^{\infty} \Lambda^i \varepsilon\Big)(x,y) = 0, \qquad x \in E, y \in Y^m. \tag{11}$$

Then limit (5) *exists for each* $x \in E$ *and the function* $\psi \in \mathcal{C}$, *defined in this way, is a unique fixed point of* $\mathcal{T}$ *with*

$$\|\mathcal{T}^n\varphi(x) - \psi(x), y\| \le \sum_{i=n}^{\infty} \Lambda^i \varepsilon(x,y), \qquad n \in \mathbb{N}_0, x \in E, y \in Y^m. \tag{12}$$

Moreover, the following two statements are valid:

(a) *for every sequence* $(k_n)_{n\in\mathbb{N}}$ *of positive integers with* $\lim_{n\to\infty} k_n = \infty$, ψ *is the unique fixed point of* $\mathcal{T}$ *such that*

$$\|\mathcal{T}^{k_n}\varphi(x) - \psi(x), y\| \le \sum_{i=k_n}^{\infty} \Lambda^i \varepsilon(x,y), \qquad n \in \mathbb{N}, x \in E, y \in Y^m; \tag{13}$$

(b) *if*

$$\liminf_{n\to\infty} \Lambda^n \varepsilon^*(x,y) = 0, \qquad x \in E, y \in Y^m, \tag{14}$$

then ψ *is the unique fixed point of* $\mathcal{T}$ *such that condition* (9) *holds true.*

Proof. Write $\Lambda_n := \Lambda^n$ for $n \in \mathbb{N}_0$. Clearly, $\mathcal{T}$ is Λ_1-contractive. Next, assume that $\mathcal{T}^n$ is Λ_n-contractive for a fixed $n \in \mathbb{N}$. Take $\xi, \mu \in \mathcal{C}$ and $\delta \in {\mathbb{R}_+}^{E\times Y^m}$ such that $\Delta(\xi,\mu) \le \delta$. Then, by the Λ_1-contractivity of $\mathcal{T}$, $\Delta(\mathcal{T}\xi, \mathcal{T}\mu) \le \Lambda\delta$ and therefore

$$\Delta(\mathcal{T}^{n+1}\xi, \mathcal{T}^{n+1}\mu) = \Delta(\mathcal{T}^n\mathcal{T}\xi, \mathcal{T}^n\mathcal{T}\mu)) \le \Lambda^n\Lambda\delta = \Lambda^{n+1}\delta.$$

Thus we have proved that $\mathcal{T}^n$ is Λ_n-contractive for every $n \in \mathbb{N}$. Consequently, Theorem 3 yields the assertions. $\square$

In what follows we need yet the following hypothesis concerning the operator $\Lambda\colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$.

($\mathcal{C}$) If $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of elements of ${\mathbb{R}_+}^{E\times Y^m}$ with

$$\lim_{n\to\infty} \delta_n(x,y) = 0, \qquad x \in E, y \in Y^m, \tag{15}$$

then

$$\liminf_{n\to\infty} \Lambda\delta_n(x,y) = 0, \qquad x \in E, y \in Y^m. \tag{16}$$

Remark 5. Note that if Λ_1 fulfils hypothesis ($\mathcal{C}$), then (3) results at once from (4). Analogously, (10) implies (11) when Λ satisfies ($\mathcal{C}$).

Remark 6. Let $j \in \mathbb{N}$. Fix $f_i\colon E \to E$ and $L_i\colon E \to \mathbb{R}$ for $i = 1,\dots,j$. Then the operator $\mathcal{T}\colon Y^E \to Y^E$, given by

$$\mathcal{T}\phi(x) := \sum_{i=1}^{j} L_i(x)\phi(f_i(x)), \qquad \phi \in Y^E, x \in E, \tag{17}$$

is $\Lambda-$contractive, with $\Lambda\colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$ defined by the formula

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} |L_i(x)|\delta(f_i(x),y), \qquad \delta \in {\mathbb{R}_+}^{E\times Y^m}, x \in E, y \in Y^m. \tag{18}$$

Moreover, it is easily seen that Λ fulfils ($\mathcal{C}$). Next, for every function $\varepsilon\colon E \times Y^m \to \mathbb{R}_+$ (with ε^* given by (10)) we have

$$\begin{aligned}\Lambda\varepsilon^*(x,y) &= \sum_{i=1}^{j} |L_i(x)| \sum_{k=0}^{\infty} (\Lambda^k\varepsilon)(f_i(x),y) = \sum_{k=0}^{\infty}\sum_{i=1}^{j} |L_i(x)|(\Lambda^k\varepsilon)(f_i(x),y)\\ &= \sum_{k=1}^{\infty} (\Lambda^k\varepsilon)(x,y), \qquad x \in E, y \in Y^m,\end{aligned}$$

and analogously, by induction, we get

$$\Lambda^n\varepsilon^*(x,y) = \sum_{k=n}^{\infty} (\Lambda^k\varepsilon)(x,y), \qquad x \in E, y \in Y^m, n \in \mathbb{N}_0.$$

Consequently, (10) implies (14). Therefore, one can easily derive from Corollary 4 an analogue of [4, Theorem 1] for n-Banach spaces.

Remark 7. Let $F : E \times Y^m \times \mathbb{R}_+ \to \mathbb{R}_+$ be subadditive and nondecreasing with respect to the third variable (i.e., $F(x, y, a + b) \leq F(x, y, a) + F(x, y, b)$ and $F(x, y, a) \leq F(x, y, c)$ for $a, b, c \in \mathbb{R}_+$ with $a \leq c$, $x \in E$ and $y \in Y^m$). Let $f : E \times Y^m \to E \times Y^m$ be given and $\Lambda\colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$ be defined by

$$\Lambda\varepsilon(x, y) = F(x, y, \varepsilon(f(x, y))), \qquad x \in E, y \in Y^m, \varepsilon \in {\mathbb{R}_+}^{E\times Y^m}.$$

We show that for such a Λ condition (10) yields (11) and (14).

So, assume that (10) holds for a suitable ε. Fix $x \in E$, $y \in Y^m$ and define $F_{xy} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F_{xy}(a) = F(x, y, a), \qquad a \in \mathbb{R}_+.$$

Since F_{xy} is nondecreasing and, for each $n \in \mathbb{N}_0$, $\Lambda^n\varepsilon(f(x, y)) \geq 0$, we have

$$\Lambda^{n+1}\varepsilon(x, y) = F_{xy}\big(\Lambda^n\varepsilon(f(x, y))\big) \geq F_{xy}(0).$$

Hence, by (10), we get $F_{xy}(0) = 0$.

Next, we show that either F_{xy} is continuous at 0 or there is an $l_0 \in \mathbb{N}$ such that $\Lambda^n\varepsilon(f(x, y)) = 0$ for $n > l_0$. So, suppose that F_{xy} is not continuous at 0 and there exists a strictly increasing sequence $\big(k_n\big)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $\Lambda^{k_n}\varepsilon(f(x, y)) \neq 0$ for $n \in \mathbb{N}$. Since F_{xy} is nondecreasing and $F_{xy}(0) = 0$, there is a $d > 0$ such that $F_{xy}(c) > d$ for $c > 0$ and therefore

$$\Lambda^{k_n+1}\varepsilon(x, y) = F_{xy}\big(\Lambda^{k_n}\varepsilon(f(x, y))\big) \geq d, \qquad n \in \mathbb{N},$$

which is a contradiction to (10).

We have thus proved that

$$\lim_{j\to\infty} F_{xy}\Big(\sum_{n=j}^{\infty} \Lambda^n\varepsilon(f(x, y))\Big) = 0.$$

Further, by the subadditivity of F_{xy}, for any $k, j \in \mathbb{N}_0$ with $j > k$, we get

$$F_{xy}\Big(\sum_{n=k}^{\infty} \Lambda^n\varepsilon(f(x, y))\Big) \leq \sum_{n=k}^{j} \Lambda^{n+1}\varepsilon(x, y) + F_{xy}\Big(\sum_{n=j+1}^{\infty} \Lambda^n\varepsilon(f(x, y))\Big),$$

whence letting $j \to \infty$ we obtain

$$\Lambda\Big(\sum_{n=k}^{\infty} \Lambda^n\varepsilon\Big)(x, y) = F_{xy}\Big(\sum_{n=k}^{\infty} \Lambda^n\varepsilon(f(x, y))\Big) \leq \sum_{n=k+1}^{\infty} \Lambda^n\varepsilon(x, y),$$

and consequently, by induction (with $k = 0$),

$$\Lambda^j\Big(\sum_{n=0}^{\infty} \Lambda^n\varepsilon\Big)(x, y) \leq \sum_{n=j}^{\infty} \Lambda^n\varepsilon(x, y), \qquad j \in \mathbb{N}.$$

It is easy to see that, using the last two inequalities, we can derive (11) and (14) from condition (10).

Now, consider a very special situation, when the set E has only one element, i.e. $E = \{s\}$. Then, actually, the set $E \times Y^m$ can be identified with Y^m and each $\mathcal{C} \subset Y^E$ can be considered as a subset $C := \{\phi(s) : \phi \in \mathcal{C}\}$ of Y. Define $\Delta : Y \times Y \to \mathbb{R}_+{}^{Y^m}$ by

$$\Delta(y_1, z_1)(y) := \|y_1 - z_1, y\|, \qquad y_1, z_1 \in Y, y \in Y^m.$$

Given $\Lambda \colon \mathbb{R}_+{}^{Y^m} \to \mathbb{R}_+{}^{Y^m}$ and $C \subset Y$, analogously as before, we say that $T \colon C \to C$ is $\Lambda-$contractive provided

$$\Delta(Ty_1, Tz_1)(y) \leq \Lambda\delta(y), \qquad y \in Y^m,$$

for any $y_1, z_1 \in Y$ and $\delta \in \mathbb{R}_+{}^{Y^m}$ such that $\Delta(y_1, z_1)(y) \leq \delta(y)$ for $y \in Y^m$.

Next, for $\Lambda_1 \colon \mathbb{R}_+{}^{Y^m} \to \mathbb{R}_+{}^{Y^m}$, hypothesis $(\mathcal{C})$ takes the following form.

$(\mathcal{C}_0)$ If $(\delta_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{R}_+{}^{Y^m}$ with

$$\lim_{n\to\infty} \delta_n(y) = 0, \qquad y \in Y^m, \tag{19}$$

then

$$\liminf_{n\to\infty} \Lambda_1\delta_n(y) = 0, \qquad y \in Y^m. \tag{20}$$

Finally, we say that a set $F \subset Y$ is closed provided

$$\lim_{n\to\infty} y_n \in F$$

for every convergent sequence $(y_n)_{n\in\mathbb{N}}$ of elements of F.

Theorem 3 (with $y_0 = \varphi(s)$ and $z_0 = \psi(s)$)) takes in this situation the following form.

Theorem 8. *Let $\emptyset \neq C \subset Y$ be closed, $T : C \to C$, $\Lambda_n \colon \mathbb{R}_+{}^{Y^m} \to \mathbb{R}_+{}^{Y^m}$ for $n \in \mathbb{N}$, and Λ_1 satisfy hypothesis $(\mathcal{C}_0)$. Let T^n be Λ_n –contractive for each $n \in \mathbb{N}$. Suppose also that there exist a $y_0 \in C$ and a function $\varepsilon \in \mathbb{R}_+{}^{Y^m}$ fulfilling the following two conditions:*

$$\big\|T(y_0) - y_0, y\big\| \leq \varepsilon(y), \qquad y \in Y^m, \tag{21}$$

$$\varepsilon^*(y) := \sum_{i=0}^{\infty} \Lambda_i\varepsilon(y) < \infty, \qquad y \in Y^m, \tag{22}$$

where $\Lambda_0\varepsilon(y) := \varepsilon(y)$ for $y \in Y^m$. Then the limit

$$z_0 := \lim_{n\to\infty} T^n(y_0) \tag{23}$$

exists and $z_0 \in C$ is a unique fixed point of T with

$$\|T^n(y_0) - z_0, y\| \leq \sum_{i=n}^{\infty} \Lambda_i\varepsilon(y), \qquad n \in \mathbb{N}_0, y \in Y^m. \tag{24}$$

Moreover, the following two statements are valid:

(a) *for every sequence* $(k_n)_{n\in\mathbb{N}}$ *of positive integers with* $\lim_{n\to\infty} k_n = \infty$, z_0 *is the unique fixed point of* T *with*

$$\|T^{k_n}(y_0) - z_0, y\| \leq \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon(y), \qquad n \in \mathbb{N}, y \in Y^m; \tag{25}$$

(b) *if*

$$\liminf_{n\to\infty} \Lambda_n \varepsilon^*(y) = 0, \qquad y \in Y^m, \tag{26}$$

then z_0 *is the unique fixed point of* T *such that*

$$\|y_0 - z_0, y\| \leq \varepsilon^*(y), \qquad y \in Y^m. \tag{27}$$

Observe that if T is a $\lambda-$contraction with a $\lambda \in (0,1)$, i.e.,

$$\|T(x) - T(z), y\| \leq \lambda\|x - z, y\|, \qquad x, z \in Y, y \in Y^m,$$

then taking $\Lambda_n\delta(y) = \lambda^n\delta(y)$ for $\delta \in {\mathbb{R}_+}^{Y^m}$, $y \in Y^m$ and $n \in \mathbb{N}$ we obtain from Theorem 8 an analogue of the Banach Contraction Principle for $(m+1)$–Banach spaces with

$$\varepsilon^*(y) = \frac{\varepsilon(y)}{1-\lambda}, \qquad y \in Y^m.$$

4. Further consequences of Corollary 4

From Corollary 4 we obtain the following corollary.

Corollary 9. *Let* $\emptyset \neq \mathcal{C} \subset Y^E$ *be pointwise closed,* $\Lambda\colon {\mathbb{R}_+}^{E\times Y^m} \to {\mathbb{R}_+}^{E\times Y^m}$, *and* $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$. *Assume also that* $\mathcal{T}$ *is* $\Lambda-$*contractive, and functions* $\varepsilon : E \times Y^m \to \mathbb{R}_+$, $\varphi \in \mathcal{C}$ *and* $q : E \times Y^m \to [0,1)$ *are such that* (2) *holds and*

$$\Lambda\phi(x,y) \leq q(x,y)\phi(x,y), \qquad x \in E, y \in Y^m, \phi \in {\mathbb{R}_+}^{E\times Y^m}. \tag{28}$$

Then limit (5) *exists for each* $x \in E$ *and the function* $\psi \in \mathcal{C}$ *so defined is a unique fixed point of* $\mathcal{T}$. *Moreover,*

$$\|\varphi(x) - \psi(x), y\| \leq \frac{1}{1-q(x,y)}\varepsilon(x,y), \qquad x \in E, y \in Y^m. \tag{29}$$

Proof. From (28) it follows that

$$\begin{aligned}\varepsilon^*(x,y) &= \sum_{l=0}^{\infty}(\Lambda^l\varepsilon)(x,y) \leq \sum_{l=0}^{\infty} q(x,y)^l\varepsilon(x,y)\\ &= \frac{1}{1-q(x,y)}\varepsilon(x,y), \qquad x \in E, y \in Y^m,\end{aligned}$$

so condition (10) holds. Next,

$$\liminf_{n\to\infty} \Lambda\Big(\sum_{i=n}^{\infty} \Lambda^i \varepsilon\Big)(x,y) \le q(x,y) \liminf_{n\to\infty} \sum_{i=n}^{\infty} \Lambda^i \varepsilon(x,y) = 0,$$
$$x \in E, y \in Y^m.$$

Consequently, also (11) is valid. Thus, by Corollary 4, limit (5) exists for each $x \in E$ and the function $\psi \in \mathcal{C}$ so defined is a fixed point of $\mathcal{T}$ such that (29) holds (in view of (12) with $n = 0$).

It remains to prove the uniqueness of ψ. So, suppose that $\xi \in \mathcal{C}$ is a fixed point of $\mathcal{T}$. Define $\phi : E \times Y^m \to \mathbb{R}_+$ by

$$\phi(x,y) := \|\psi(x) - \xi(x), y\|, \qquad x \in E, y \in Y^m.$$

Then, by (28), we have

$$\begin{aligned}\|\psi(x) - \xi(x), y\| = \|\mathcal{T}^n\psi(x) - \mathcal{T}^n\xi(x), y\| &\le \Lambda^n \phi(x,y) \\ &\le q(x,y)^n \phi(x,y), \qquad x \in E, y \in Y^m, n \in \mathbb{N}_0, \end{aligned} \tag{30}$$

which (with $n \to \infty$) implies that $\|\psi(x) - \xi(x), y\| = 0$ for any $x \in E$ and $y \in Y^m$, i.e., $\xi = \psi$. □

Clearly, the simplest situation, when (28) holds, occurs for Λ given by

$$\Lambda\delta(x,y) := q(x,y)\delta(x,y), \qquad \delta \in {\mathbb{R}_+}^{E\times Y^m}, x \in E, y \in Y^m.$$

For the next corollary, which also can be easily deduced from Corollary 4, we need the subsequent hypothesis.

(H1) $j \in \mathbb{N}$, $L_i : E \to \mathbb{R}_+$ for $i = 1, \dots, j$, $\Phi : E \times Y^j \to Y$, and

$$\|\Phi(x, w_1, ..., w_j) - \Phi(x, z_1, ..., z_j), y\| \le \sum_{k=1}^{j} L_k(x)\|w_k - z_k, y\| \tag{31}$$

for any $x \in E$, $y \in Y^m$ and $(w_1, ..., w_j), (z_1, ..., z_j) \in Y^j$.

Corollary 10. *Assume that* (H1) *is fulfilled,* $f_i : E \to E$ *for* $i = 1, \dots, j$, $\varepsilon : E \times Y^m \to \mathbb{R}_+$ *satisfies* (10) *with* $\Lambda : \mathbb{R}_+^{E\times Y^m} \to \mathbb{R}_+^{E\times Y^m}$ *given by*

$$\Lambda\delta(x,y) = \sum_{k=1}^{j} L_k(x)\delta(f_k(x), y), \qquad \delta \in \mathbb{R}_+^{E\times Y^m}, x \in E, y \in Y^m, \tag{32}$$

and $\varphi : E \to Y$ *is such that*

$$\|\varphi(x) - \Phi(x, \varphi(f_1(x)), ..., \varphi(f_j(x))), y\| \le \varepsilon(x,y), \qquad x \in E, y \in Y^m. \tag{33}$$

Then limit (5) *exists for each* $x \in E$ *with*

$$\mathcal{T}\varphi(x) := \Phi(x, \varphi(f_1(x)), ..., \varphi(f_j(x))), \qquad \varphi \in Y^E,\ x \in E, \tag{34}$$

and the function $\psi : E \to Y$ *defined by* (5) *is a unique solution of the functional equation*

$$\Phi(x, \psi(f_1(x)), ..., \psi(f_j(x))) = \psi(x), \qquad x \in E, \tag{35}$$

such that inequality (9) *holds.*

Proof. Let us note that inequality (33) implies (2). Next, (11) and (14) are valid in view of Remarks 5 and 6. Therefore, by Corollary 4, the function ψ defined by (5) is a unique fixed point of $\mathcal{T}$ (that is a solution of (35)) satisfying (9). □

Stability of functional equations of form (35) (or related to it) has been already studied by several authors and for further information we refer to survey papers [1, 6] and monograph [7]. A particular case of (35) is a linear functional equation of the form

$$\phi(x) = \sum_{i=1}^{j} L_i(x)\phi(f_i(x)), \qquad x \in E, \tag{36}$$

under the assumptions as in Remark 6 (some recent results concerning stability of less general cases of it can be found in [25, 26, 30]).

As an example of applications of Corollary 10 let us consider stability of the difference equation

$$\psi(i) = \Phi(i, \psi(i+1)), \qquad i \in \mathbb{N}, \tag{37}$$

where $\Phi : \mathbb{N} \times Y \to Y$ is given and $\psi : \mathbb{N} \to Y$ is unknown. Clearly, (37) is a very simple particular case of (35), with $E = \mathbb{N}$, $j = 1$ and $f_1(i) = i + 1$ for $i \in \mathbb{N}$.

Assume that $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers with

$$\sum_{k=1}^{\infty} \prod_{l=0}^{k-1} a_{i+l} < \infty, \qquad i \in \mathbb{N}. \tag{38}$$

For instance, if we take

$$a_{2n} = 2, \qquad a_{2n-1} = \frac{1}{4}, \qquad n \in \mathbb{N},$$

then

$$\prod_{l=0}^{2k} a_{i+l} = \frac{1}{2} \prod_{l=0}^{2k-2} a_{i+l}, \qquad \prod_{l=0}^{2k+1} a_{i+l} = \frac{1}{2} \prod_{l=0}^{2k-1} a_{i+l}, \qquad i, k \in \mathbb{N},$$

whence (38) holds.

Let $\Lambda : \mathbb{R}_+^{\mathbb{N}\times Y^m} \to \mathbb{R}_+^{\mathbb{N}\times Y^m}$ be given by

$$\Lambda\delta(i, y) = a_i \delta(i + 1, y), \qquad \delta \in \mathbb{R}_+^{\mathbb{N}\times Y^m}, i \in \mathbb{N}, y \in Y^m. \tag{39}$$

It is easily seen that, for every $\delta \in \mathbb{R}_+^{\mathbb{N}\times Y^m}$,

$$\Lambda^k \delta(i,y) = \delta(i+k,y) \prod_{l=0}^{k-1} a_{i+l}, \qquad i,k \in \mathbb{N},$$

and consequently

$$\sum_{k=1}^{n} \Lambda^k \delta(i,y) = \sum_{k=1}^{n} \delta(i+k,y) \prod_{l=0}^{k-1} a_{i+l}, \qquad i,n \in \mathbb{N}. \tag{40}$$

Suppose that $\gamma > 0$ and $\phi : \mathbb{N} \to Y$ fulfils (33) with an $\varepsilon : \mathbb{N} \times Y^m \to [0,\gamma]$, that is

$$\|\phi(i) - \Phi(i,\phi(i+1)), y\| \le \varepsilon(i,y), \qquad i \in \mathbb{N}, y \in Y^m. \tag{41}$$

Then, by (40),

$$\begin{aligned} \varepsilon^*(i,y) &:= \sum_{k=0}^{\infty} \Lambda^k \varepsilon(i,y) \\ &\le \gamma\Big(1 + \sum_{k=1}^{\infty} \prod_{l=0}^{k-1} a_{l+i}\Big) < \infty, \qquad i \in \mathbb{N}, y \in Y^m. \end{aligned}$$

Next, if

$$\|\Phi(i,z) - \Phi(i,w), y\| \le a_i \|z-w,y\|, \qquad w,z \in Y, y \in Y^m, i \in \mathbb{N},$$

(e.g., $\Phi(i,z) = a_i z$), then (H1) holds and the assumptions of Corollary 10 are fulfilled with $j = 1$ and

$$L_1(i) = a_i, \qquad f_1(i) = i+1, \qquad i \in \mathbb{N}.$$

Therefore, the limit

$$\psi(i) := \lim_{n\to\infty} \mathcal{T}^n \phi(i) \tag{42}$$

exists for each $i \in \mathbb{N}$, with

$$\mathcal{T}\xi(i) := \Phi(i,\xi(i+1)), \qquad \xi \in Y^{\mathbb{N}},\ i \in \mathbb{N}, \tag{43}$$

and the function $\psi : \mathbb{N} \to Y$, defined by (42), is a unique solution of difference equation (37) such that

$$\|\phi(i) - \psi(i), y\| \le \varepsilon^*(i,y), \qquad i \in \mathbb{N}, y \in Y^m. \tag{44}$$

For some earlier results and references concerning the Ulam type stability of difference equations of form (37) see [6, 7, 8, 9, 32, 33].

Finally, we present one more application of Corollary 4. Namely, if $\mathcal{T}$ is linear, then we can easily obtain from this theorem the following corollary concerning stability of eigenvectors, which corresponds to the investigations in, e.g., [15, 22].

Corollary 11. *Let* $\Lambda\colon \mathbb{R}_+{}^{E\times Y^m} \to \mathbb{R}_+{}^{E\times Y^m}$, $\mathcal{C}$ *be a pointwise closed linear subspace of* Y^E, $\mathcal{T}_0\colon \mathcal{C}\to\mathcal{C}$ *be linear,* $\gamma>0$ *and* $\mathcal{T}:=\gamma^{-1}\mathcal{T}_0$. *Assume also that* $\mathcal{T}$ *is* Λ*–contractive, and there exist functions* $\varepsilon\in\mathbb{R}_+{}^{E\times Y^m}$ *and* $\varphi\in\mathcal{C}$ *such that* (10) *and* (11) *are valid and*

$$\big\|\mathcal{T}_0\varphi(x)-\gamma\varphi(x),y\big\|\le\gamma\varepsilon(x,y),\qquad x\in E, y\in Y^m. \tag{45}$$

Then limit (5) *exists for each* $x\in E$ *and the function* $\psi\in\mathcal{C}$, *defined in this way, is an eigenvector of* $\mathcal{T}_0$, *with the eigenvalue* γ, *such that*

$$\|\varphi(x)-\psi(x),y\|\le\sum_{i=0}^{\infty}\Lambda^i\varepsilon(x,y),\qquad x\in E, y\in Y^m. \tag{46}$$

Moreover, ψ *is the unique eigenvector of* $\mathcal{T}_0$, *with the eigenvalue* γ, *such that*

$$\|\mathcal{T}_0^n\varphi(x)-\gamma^n\psi(x),y\|\le\gamma^n\sum_{i=n}^{\infty}\Lambda^i\varepsilon(x,y),\qquad n\in\mathbb{N}, x\in E, y\in Y^m. \tag{47}$$

Proof. It is enough to notice that (45) implies (2) and use Corollary 4. Clearly, (46) follows from (12) with $n=0$. Next, it is easily seen that (47) is just (13) with $k_n=n$ for $n\in\mathbb{N}$. □

5. Proof of Theorem 3

Note that, by (2) and (4), for any $k\in\mathbb{N}$, $n\in\mathbb{N}_0$, $x\in E$ and $y\in Y^m$ we have

$$\begin{aligned}\Delta(\mathcal{T}^n\varphi,\mathcal{T}^{n+k}\varphi)(x,y) &\le \sum_{i=0}^{k-1}\Delta(\mathcal{T}^{n+i}\varphi,\mathcal{T}^{n+i+1}\varphi)(x,y)\\ &\le \sum_{i=n}^{n+k-1}\Lambda_i\varepsilon(x,y)\le\varepsilon^*(x,y).\end{aligned} \tag{48}$$

Therefore, for each $x\in E$, $(\mathcal{T}^n\varphi(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Thus, the fact that Y is an $(m+1)$–Banach space implies that this sequence is convergent. Consequently, (5) defines a function $\psi\in\mathcal{C}$.

Letting $k\to\infty$ in (48), in view of Lemma 1 (iv) and (5) we get

$$\Delta(\mathcal{T}^n\varphi,\psi)\le\sum_{i=n}^{\infty}\Lambda_i\varepsilon,\qquad n\in\mathbb{N}_0, \tag{49}$$

which is (6). Moreover, using (49), we obtain

$$\Delta(\mathcal{T}\psi,\mathcal{T}^{n+1}\varphi)\le\Lambda_1\Big(\sum_{i=n}^{\infty}\Lambda_i\varepsilon\Big),\qquad n\in\mathbb{N}_0,$$

whence with $n \to \infty$ we get $\mathcal{T}\psi = \psi$, on account of (3) and Lemma 1 ((iii) and (iv)).

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers with $\lim_{n\to\infty} k_n = \infty$ and $\xi \in \mathcal{C}$ be such that

$$\|\mathcal{T}^{k_n}\varphi(x) - \xi(x), y\| \le \sum_{i=k_n}^{\infty} \Lambda_i \varepsilon(x, y), \qquad n \in \mathbb{N}, x \in E, y \in Y^m.$$

Then

$$\begin{aligned} \|\xi(x) - \psi(x), y\| &\le \|\xi(x) - \mathcal{T}^{k_n}\varphi(x), y\| + \|\mathcal{T}^{k_n}\varphi(x) - \psi(x), y\| \\ &\le 2\sum_{i=k_n}^{\infty} \Lambda_i \varepsilon(x, y), \qquad n \in \mathbb{N}, x \in E, y \in Y^m, \end{aligned}$$

whence letting $n \to \infty$ we get $\xi = \psi$.

It remains to prove the last statement on the uniqueness of ψ. So, assume that (8) holds and $\xi \in \mathcal{C}$ is a fixed point of $\mathcal{T}$ with

$$\|\varphi(x) - \xi(x), y\| \le \varepsilon^*(x, y), \qquad x \in E, y \in Y^m.$$

Then, for any $n \in \mathbb{N}$, $x \in E$ and $y \in Y^m$ we have

$$\begin{aligned} \|\psi(x) - \xi(x), y\| &\le \|\psi(x) - \mathcal{T}^n\varphi(x), y\| + \|\mathcal{T}^n\varphi(x) - \mathcal{T}^n\xi(x), y\| \\ &\le \|\psi(x) - \mathcal{T}^n\varphi(x), y\| + \Lambda_n \varepsilon^*(x, y), \end{aligned} \tag{50}$$

whence we can easily see that $\|\psi(x) - \xi(x), y\| = 0$ for any $x \in E$ and $y \in Y^m$, which means that $\xi = \psi$.

This completes the proof of Theorem 3. $\square$

[1] R.P. Agarwal, B. Xu, W. Zhang, Stability of functional equations in single variable, J. Math. Anal. Appl. 288 (2003), 852–869.

[2] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, On some recent developments in Ulam's type stability, Abstr. Appl. Anal. 2012, Art. ID 716936, 41 pp.

[3] J. Brzdęk, L. Cădariu, K. Ciepliński, Fixed point theory and the Ulam stability, J. Funct. Spaces 2014, Art. ID 829419, 16 pp.

[4] J. Brzdęk, J. Chudziak, Zs. Páles, A fixed point approach to stability of functional equations, Nonlinear Anal. 74 (2011), 6728–6732.

[5] J. Brzdęk, K. Ciepliński, A fixed point approach to the stability of functional equations in non-Archimedean metric spaces, Nonlinear Anal. 74 (2011), 6861–6867.

[6] J. Brzdęk, K. Ciepliński, Z. Leśniak, On Ulam's type stability of the linear equation and related issues, Discrete Dyn. Nat. Soc. 2014, Art. ID 536791, 14 pp.

[7] J. Brzdęk, D. Popa, I. Raşa, B. Xu, Ulam Stability of Operators, Academic Press, Elsevier, Oxford, 2018.

[8] J. Brzdęk, D. Popa, B. Xu, The Hyers-Ulam stability of nonlinear recurrences, J. Math. Anal. Appl. 335 (2007), 443–449.

[9] J. Brzdęk, D. Popa, B. Xu, On nonstability of the linear recurrence of order one, J. Math. Anal. Appl. 367 (2010), 146–153.

[10] L. Cădariu, L. Găvruţa, P. Găvruţa, Fixed points and generalized Hyers-Ulam stability, Abstr. Appl. Anal. 2012, Art. ID 712743, 10 pp.

[11] X.Y. Chen, M.M. Song, Characterizations on isometries in linear n-normed spaces, Nonlinear Anal. 72 (2010), 1895–1901.

[12] H.-Y. Chu, A. Kim, J. Park, On the Hyers-Ulam stabilities of functional equations on n-Banach spaces, Math. Nachr. 289 (2016), 1177–1188.

[13] H.-Y. Chu, K. Lee, C.-G. Park, On the Aleksandrov problem in linear n-normed spaces, Nonlinear Anal. 59 (2004), 1001–1011.

[14] K. Ciepliński, Applications of fixed point theorems to the Hyers-Ulam stability of functional equations - a survey, Ann. Funct. Anal. 3 (2012), 151–164.

[15] Z. Drmač, K. Veselić, Approximate eigenvectors as preconditioner, Linear Algebra Appl. 309 (2000), 191–215.

[16] H. Dutta, On some n-normed linear space valued difference sequences, J. Franklin Inst. 348 (2011), 2876–2883.

[17] N. Eghbali, J.M. Rassias, M. Taheri, On the stability of a k-cubic functional equation in intuitionistic fuzzy n-normed spaces, Results Math. 70 (2016), 233–248.

[18] S. Gähler, Lineare 2-normierte Räume, Math. Nachr. 28 (1964), 1–43.

[19] J. Gao, On the stability of the linear mapping in 2-normed spaces, Nonlinear Funct. Anal. Appl. 14 (2009), 801–807.

[20] H. Gunawan, M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (2001), 631–639.

[21] H. Gunawan, O. Neswan, E. Sukaesih, Fixed point theorems in bounded sets in an n-normed space, J. Math. Anal. 6 (2015), 51–58.

[22] D. Hecker, D. Lurie, Using least-squares to find an approximate eigenvector, Electron. J. Linear Algebra 16 (2007), 99–110.

[23] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Boston, 1998.

[24] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, New York, 2011.

[25] S.-M. Jung, M.Th. Rassias, A linear functional equation of third order associated to the Fibonacci numbers, Abstr. Appl. Anal. 2014, Art. ID 137468, 7 pp.

[26] S.-M. Jung, D. Popa, M.Th. Rassias, On the stability of the linear functional equation in a single variable on complete metric groups, J. Global Optim. 59 (2014), 165–171.

[27] Y. Ma, Isometry on linear n-normed spaces, Ann. Acad. Sci. Fenn. Math. 39 (2014), 973–981.

[28] Y. Ma, The Aleksandrov-Benz-Rassias problem on linear n-normed spaces, Monatsh. Math. 180 (2016), 305–316.

[29] A. Misiak, n-inner product spaces, Math. Nachr. 140 (1989), 299–319.

[30] C. Mortici, M.Th. Rassias, S.-M. Jung, On the stability of a functional equation associated with the Fibonacci numbers, Abstr. Appl. Anal. 2014, Art. ID 546046, 6 pp.

[31] Z. Moszner, Stability has many names, Aequationes Math. 90 (2016), 983–999.

[32] D. Popa, Hyers-Ulam-Rassias stability of a linear recurrence, J. Math. Anal. Appl. 309 (2005), 591–597.

[33] D. Popa, Hyers-Ulam stability of the linear recurrence with constant coefficients, Adv. Difference Equ. 2005, 101–107.

[34] T.Z. Xu, Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces, Abstr. Appl. Anal. 2013, Art. ID 648709, 12 pp.

[35] T.Z. Xu, J.M. Rassias, On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in n-Banach spaces, Abstr. Appl. Anal. 2012, Art. ID 926390, 23 pp.