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# Weighted Finsler trace Hardy inequality on half spaces

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A B S T R A C T

We prove the sharp weighted trace Hardy inequality on the half space in the Finsler context. The obtained inequality extends the results obtained in [23,29] in the Euclidean context to the Finsler context. It also generalizes the recent result in [5] to the weight context.



## 1. Introduction

In this paper, we are interested in studying the weighted trace Hardy inequality on the half spaces endowed with a Finsler norm. Let $n \geq 2$, we denote by $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ the half space, i.e.,

$$\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} \,:\, x \in \mathbb{R}^n,\, t > 0\}.$$

Its boundary is $\partial\mathbb{R}^{n+1}_+ = \{(x,0) \,:\, x \in \mathbb{R}^n\}$ which we identify with $\mathbb{R}^n$. Given $s \in (-1,1)$, let us denote $n_s = n+1+s$ which plays the role of dimension in our analysis and define the weight function $w_s(x,t) := t^s$ on $\mathbb{R}^{n+1}_+$. Let $\dot{W}(\mathbb{R}^{n+1}_+, w_s)$ denote the weighted Sobolev space on $\mathbb{R}^{n+1}_+$ which is the completion of $C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ under the Dirichlet norm

$$\|u\|_{\dot{W}(\mathbb{R}^{n+1}_+, w_s)} := \left(\int\limits_{\mathbb{R}^{n+1}_+} |\nabla u|^2 w_s dxdt\right)^{\frac{1}{2}},$$

here we say a function $u \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$ if it is the restriction to $\mathbb{R}^{n+1}_+$ of a function in $C_0^\infty(\mathbb{R}^{n+1})$. In [4,29], the following weighted trace Hardy inequalities were established for functions $u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s)$

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$$\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 t^s dxdt \geq \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{|x|^2+t^2} t^s dxdt + H(n,s,\beta) \int_{\mathbb{R}^n} \frac{u(x,0)^2}{|x|^{1-s}} dx, \tag{1.1}$$

for any $2 \leq \beta < n_s$, where $H(n,s,\beta)$ is given by

$$H(n,s,\beta) = 2\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{n_s+\beta-2-2s}{4})\Gamma(\frac{n_s-\beta+2-2s}{4})}{\Gamma(\frac{1-s}{2})\Gamma(\frac{n_s+\beta-4}{4})\Gamma(\frac{n_s-\beta}{4})}. \tag{1.2}$$

Notice that the constant $H(n,s,\beta)$ is sharp and never achieved. We refer the reader to [23] for a generalization of (1.1) to the polyhedral convex cone. The interest of (1.1) is that it contains two important inequalities which are the sharp weighted Hardy inequality and the sharp weighted trace Hardy inequality. Indeed, we have $H(n,s,\beta) \to 0$ as $\beta \to n_s$, so by letting $\beta \to n_s$, the inequality (1.1) implies the sharp weighted Hardy inequality in half space

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 t^s dxdt \geq \frac{(n_s-2)^2}{4} \int_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{|x|^2+t^2} t^s dxdt. \tag{1.3}$$

Again, the constant $\frac{(n_s-2)^2}{4}$ is sharp in (1.3) and is never achieved. Taking $\beta = 2$, the inequality (1.1) reduces to the sharp weighted trace Hardy inequality

$$\int_{\mathbb{R}^{n+1}_+} |\nabla u|^2 t^s dxdt \geq 2\left(\frac{\Gamma(\frac{n+1-s}{4})}{\Gamma(\frac{n-1+s}{4})}\right)^2 \int_{\mathbb{R}^n} \frac{u(x,0)^2}{|x|^{1-s}} dx. \tag{1.4}$$

Again, the constant $2\left(\frac{\Gamma(\frac{n+1-s}{4})}{\Gamma(\frac{n-1+s}{4})}\right)^2$ in (1.4) is sharp and is never attained. The case $s = 0$, the inequality (1.4) becomes Kato's inequality (see [4,13,15]). By a result of Caffarelli and Silvestre on the extension problem concerning to the fractional Laplacian [11], the inequality (1.4) is equivalent to the Hardy inequality for fractional Laplacian on $\mathbb{R}^n$ or fractional Hardy inequality [21] (see also [7,19] for alternative proofs).

Both the Hardy type inequality and the trace Hardy type inequality have many applications to boundary value problems in partial differential equations and nonlinear analysis. They have been developed by many authors in may different setting by many different methods. Here we just recall some recent papers and references therein [3–5,16–18,23,29].

The aim of this paper is to extend the inequality (1.1) to Finsler context. The interest in the so-called Finsler geometry arose from the works of G. Wulff on crystal shapes and minimization of anisotropic surface tensions in 1901 and it is becoming increasing important in different contexts, as in the field of phase changes and phase of separation in multiphase materials (see, e.g., [6,8]). This justifies the necessity to extend to Finsler case many of the classical tools, which are useful in classical variational problems. The basic idea is to endow the space $\mathbb{R}^N$ with the distance obtained by a Finsler metric and to extend classical results to such a new geometrical context. For example, the Finsler Hardy inequalities were recently established in [22] (We also refer the reader to seminal works of Ruzhansky [24,26–28] in which the Hardy inequalities were established in the more general context of homogeneous groups). The interest reader may consult [2,30] for the symmetrization method related to the Finsler norm with applications to Sobolev type inequalities and to comparison results for several partial differential equations.

Let $H$ be a Finsler norm on $\mathbb{R}^n, n \geq 2$ and $H^\circ$ is its dual norm (we refer the reader to Section 2 for the precise definition of Finsler norm and the dual norm). We then define a new norm $\Phi$ on $\mathbb{R}^{n+1}$ by

$$\Phi(x,t) = (H(x)^2 + t^2)^{\frac{1}{2}}, \quad (x,t) \in \mathbb{R}^{n+1}. \tag{1.5}$$

The dual norm of $\Phi$ is given by

$$\Phi^\circ(x,t) = (H^\circ(x)^2 + t^2)^{\frac{1}{2}}, \quad (x,t) \in \mathbb{R}^{n+1}. \tag{1.6}$$

The main result in this paper is the following sharp weighted Finsler trace Hardy inequality on the half space.

**Theorem 1.1.** *Given $n \geq 2$ and $s \in (-1,1)$. Let $H$ be a Finsler norm on $\mathbb{R}^n$ and $H^\circ$ be its dual norm. For any $2 \leq \beta < n_s$, the following inequality*

$$\int_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 t^s dxdt \geq \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{\Phi^\circ(x,t)^2} t^s dxdt + H(n,s,\beta) \int_{\mathbb{R}^n} \frac{u(x,0)^2}{H^\circ(x)^{1-s}} dx \tag{1.7}$$

*holds for any $u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s)$, where $\Phi$ and $\Phi^\circ$ are given by (1.5) and (1.6) respectively, and $H(n,s,\beta)$ is given by (1.2). Furthermore, the constant $H(n,s,\beta)$ is sharp and is never attained.*

Let us make some comments on Theorem 1.1. By taking $H$ to be the Euclidean norm in $\mathbb{R}^n$, Theorem 1.1 yields the inequality (1.1). In the case $s = 0$, Theorem 1.1 implies the Finsler Hardy–Kato's inequality which recently was established by Alvino et al. (see [5]),

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 dxdt \geq& \frac{(\beta-2)^2}{4} \int_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{\Phi^\circ(x,t)^2} dxdt \\ &+ 2\frac{\Gamma(\frac{n-1+\beta}{4})\Gamma(\frac{n+3-\beta}{4})}{\Gamma(\frac{n-3+\beta}{4})\Gamma(\frac{n+1-\beta}{4})} \int_{\mathbb{R}^n} \frac{u(x,0)^2}{H^\circ(x)} dx.\end{aligned} \tag{1.8}$$

The Euclidean version of (1.8) was previously proved by Alvino, Ferone and Volpicelli [4]. By taking $\beta = 2$ in Theorem 1.1, we obtain the following sharp Finsler weighted Hardy trace inequality on the half space

$$\int_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 t^s dxdt \geq 2\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1-s}{2})} \left(\frac{\Gamma(\frac{n_s-2s}{4})}{\Gamma(\frac{n_s-2}{4})}\right)^2 \int_{\mathbb{R}^n} \frac{u(x,0)^2}{H^\circ(x)^{1-s}} dx \tag{1.9}$$

for any $u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s)$. The inequality (1.9) extends the weighted trace Hardy inequality (1.4) to the Finsler context. Finally, by letting $\beta \uparrow n_s$, we get a sharp Finsler weighted Hardy inequality on half space as follows

$$\int_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 t^s dxdt \geq \frac{(n_s-2)^2}{4} \int_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{\Phi^\circ(x,t)^2} t^s dxdt, \tag{1.10}$$

for any $u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s)$.

The proof of Theorem 1.1 follows the method developed in [23,29] which is based on the factorization technique. This method is different with the one used in [5] to prove Theorem 1.1 in the case $s = 0$. Indeed, in [5] the authors obtained their result by using a classical method of Calculus of Variations introduced by Weierstrass and developed by Schwartz, Lichtenstein and Morrey (see [20] and references therein for more details). More precisely, they construct a special free divergence vector field on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$ and then using divergence theorem to obtain their result. One important ingredient in our method is the construction of a singular solution $\varphi$ for the equation

$$\Delta_\Phi \varphi(x,t) + s\frac{\partial_t \varphi(x,t)}{t} + \frac{(\beta-2)^2}{4}\frac{\varphi(x,t)}{\Phi^\circ(x,t)^2} = 0, \quad \text{in} \quad \mathbb{R}^{n+1}_+ \tag{1.11}$$

such that $\varphi(x,0) = H^\circ(x)^{-\frac{n_s-2}{2}}$, here $\Delta_\Phi$ is Finsler Laplacian with respect to $\Phi$ (see Section 2 for definition). An explicit form of $\varphi$ and its properties are given in Section 2 below. For a function $u \in \dot W(\mathbb{R}^{n+1}_+, w_s)$, we factorize it as $u = v\varphi$ and replace this in to $\int_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 t^s dxdt$ and using integration by parts to derive Theorem 1.1. The detail proof of Theorem 1.1 is given in Section 4 below.

The rest of this paper is organized as follows. In Section 2 we recall the definition of Finsler norm and list some its useful properties. In Section 3, we construct an explicit singular solution for the equation (1.11) which plays an important role in our proof. The proof of Theorem 1.1 is given in Section 4.

## 2. Preliminaries

In this section, we recall the definition of the Finsler norm and some its properties. Let $N$ be an integer. A function $H : \mathbb{R}^N \to [0,\infty)$ is called a Finsler norm if it is a norm on $\mathbb{R}^N$ and is strongly convex in the sense that $H \in C^2(\mathbb{R}^N \setminus \{0\})$ and the Hessian matrix of $H^2$ (i.e., $\nabla^2 H^2 = (\partial^2_{ij} H^2)_{1\le i,j\le N}$) is positive define. From the definition of $H$, there exist $a, b > 0$ such that

$$a|x| \le H(x) \le b|x|, \qquad x \in \mathbb{R}^N, \tag{2.1}$$

here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. The dual norm (or polar function) $H^\circ$ of $H$ is defined by

$$H^\circ(x) = \sup_{y\in\mathbb{R}^N, H(y)\le 1} \langle x, y\rangle, \tag{2.2}$$

here $\langle \cdot,\cdot\rangle$ denotes the canonical scalar product on $\mathbb{R}^N$. Notice that $H^\circ : \mathbb{R}^N \to [0,\infty)$ is also a Finsler norm on $\mathbb{R}^N$ and the following equality

$$(H^\circ)^\circ = H, \tag{2.3}$$

holds. From the definition of $H^\circ$, we have

$$\frac{1}{b}|x| \le H^\circ(x) \le \frac{1}{a}|x|, \qquad x \in \mathbb{R}^N, \tag{2.4}$$

and the following Schwartz inequality holds true

$$|\langle x, y\rangle| \le H(x)\, H^\circ(y), \qquad x, y \in \mathbb{R}^n. \tag{2.5}$$

For references on Finsler norms (or, more general, on Finsler metric), we refer the reader to [6,8].

We next recall some useful and further properties concerning to $H$ and $H^\circ$ whose proof can be found in [8, Lemma 2.1 and 2.2] and in [30, Proposition 6.2] (see also [22, Proposition 2.1] for a review).

**Proposition 2.1.** *Let $H$ be a norm on $\mathbb{R}^N$ which is $C^2(\mathbb{R}^N \setminus \{0\})$. Then the following properties hold true*

(i) $\langle \nabla H(x), x\rangle = H(x)$ *for* $x \ne 0$.
(ii) $\nabla H(tx) = \frac{t}{|t|}\nabla H(x)$ *for* $t \ne 0$ *and* $x \ne 0$.
(iii) $H^\circ(\nabla H(x)) = 1$ *for* $x \ne 0$.
(iv) $H^\circ(x)\nabla H(\nabla H^\circ(x)) = x$ *for* $x \ne 0$.

Given a Finsler norm $H$ on $\mathbb{R}^N$, the Finsler–Laplace operator $\Delta_H$ with respect to $H$ is defined by

$$\Delta_H u(x) = \operatorname{div}(H(\nabla u)\nabla H(\nabla u))(x)$$

for any function $u \in C^2(\mathbb{R}^N)$. Note that unlike the classical Laplace operator, the Finsler–Laplacian is a nonlinear operator. However, the strongly convexity of $H$ ensures that $\Delta_H$ is a uniformly elliptic operator locally. The Finsler Laplacian has been widely investigated in literature and its notion goes back to the work of G. Wulff who considered it to describe the theory of crystals. Since then, many authors developed the related theory in several settings, considering both analytic and geometric points of view (see, e.g., [8–10,12,14] and references therein).

## 3. Construction of extremals

Given $n \geq 2$, $s \in (-1,1)$ and $\beta \in [2, n_s)$. For a function $u$ on $\mathbb{R}^{n+1}_+$, its gradient is denoted by $\nabla u = (\nabla_x u, \partial_t u)$ where $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ is the gradient in variable $x$ of $u$. Let $H$ be a Finsler norm on $\mathbb{R}^n$. We define a new Finsler norm $\Phi$ on $\mathbb{R}^{n+1}$ by (1.5). Its dual norm $\Phi^\circ$ is given by (1.6). Considering the functional

$$J(u) = \int\limits_{\mathbb{R}^{n+1}_+} \Phi(\nabla u)^2 t^s dxdt - \frac{(\beta-2)^2}{4} \int\limits_{\mathbb{R}^{n+1}_+} \frac{u(x,t)^2}{\Phi^\circ(x,t)^2} t^s dxdt, \quad u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s).$$

The Euler–Lagrange equation associated with functional $J$ is given by

$$\Delta_\Phi u(x,t) + s\frac{\partial_t u(x,t)}{t} + \frac{(\beta-2)^2}{4}\frac{u(x,t)}{\Phi^\circ(x,t)^2} = 0, \quad \text{in} \quad \mathbb{R}^{n+1}_+, \tag{3.1}$$

which is nothing the equation (1.11). Our aim is to construct a singular solution $\varphi(x,t)$ of (3.1) such that $\varphi(x,0) = H^\circ(x)^{-\frac{n_s-2}{2}}$. To do this, we find the function $\varphi$ in the form

$$\varphi(x,t) = \Phi^\circ(x,t)^{-\frac{n_s-2}{2}} \omega\left(\frac{t^2}{\Phi^\circ(x,t)^2}\right) = (H^\circ(x)^2 + t^2)^{-\frac{n_s-2}{4}} \omega\left(\frac{t^2}{H^\circ(x)^2 + t^2}\right). \tag{3.2}$$

The next proposition shows that $\omega$ is a solution of an ordinary differential equation which can be solved explicitly.

**Proposition 3.1.** *The function $\omega$ is the solution of the following ordinary differential equation*

$$z(z-1)\omega''(z) + \left(\frac{n_s}{2}z - \frac{1+s}{2}\right)\omega'(z) + \left(\frac{(n_s-2)^2}{16} - \frac{(\beta-2)^2}{16}\right)\omega(z) = 0, \tag{3.3}$$

*for $z \in (0,1]$ with the initial condition $\omega(0) = 1$ such that there exits the limit $\lim_{z\uparrow 1}\omega(z)$.*

**Proof.** Recall that $\Phi(x,t)^2 = H(x)^2 + t^2$ and $\Phi^\circ(x,t)^2 = H^\circ(x)^2 + t^2$, hence for any $(x,t) \in \mathbb{R}^{n+1}$ it holds

$$\Phi(x,t)\nabla\Phi(x,t) = (H(x)\nabla H(x), t)$$

and then $\Delta_\Phi = \Delta_H + \partial^2_{tt}$, here $\Delta_H$ is the Finsler Laplace operator on $\mathbb{R}^n$ with respect to $H$. Therefore, we have

$$\Delta_\Phi \varphi(x,t) = \Delta_H \varphi(x,t) + \partial^2_{tt}\varphi(x,t).$$

By direct computations, we have

$$\nabla_x\varphi(x,t) = -\frac{n_s-2}{2}(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}H^\circ(x)\nabla H^\circ(x)\,\omega\left(\frac{t^2}{H^\circ(x)^2+t^2}\right) - 2(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\frac{t^2}{H^\circ(x)^2+t^2}H^\circ(x)\nabla H^\circ(x)\,\omega'\left(\frac{t^2}{H^\circ(x)^2+t^2}\right), \tag{3.4}$$

and

$$\partial_t\varphi(x,t) = -\frac{n_s-2}{2}(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}t\,\omega\left(\frac{t^2}{H^\circ(x)^2+t^2}\right) + 2(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\frac{tH^\circ(x)^2}{H^\circ(x)^2+t^2}\,\omega'\left(\frac{t^2}{H^\circ(x)^2+t^2}\right). \tag{3.5}$$

By Proposition 2.1, we get

$$H(\nabla_x\varphi(x,t))\nabla H(\nabla_x\varphi(x,t)) = -(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\left(\frac{n_s-2}{2}\omega\left(\frac{t^2}{H^\circ(x)^2+t^2}\right)+2\frac{t^2}{H^\circ(x)^2+t^2}\omega'\left(\frac{t^2}{H^\circ(x)^2+t^2}\right)\right)x.$$

Using Proposition 2.1 (part (i) for $H^\circ$) and denote $z = \frac{t^2}{H^\circ(x)^2+t^2} \in (0,1]$, we have

$$\begin{aligned}\Delta_H\varphi(x,t) =& -n(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\left(\frac{n_s-2}{2}\omega(z)+2z\omega'(z)\right)\\ &+\frac{n_s+2}{2}(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}(1-z)\left(\frac{n_s-2}{2}\omega(z)+2z\omega'(z)\right)\\ &+(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\left((n_s+2)\omega'(z)z(1-z)+4z^2(1-z)\omega''(z)\right).\end{aligned} \tag{3.6}$$

Differentiating $\partial_t\varphi$ in $t$ once, we get

$$\begin{aligned}\partial^2_{tt}\varphi(x,t) =& +\frac{n_s+2}{2}(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}z\left(\frac{n_s-2}{2}\omega(z)-2(1-z)\omega'(z)\right)\\ &-(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\left(\frac{n_s-2}{2}\omega(z)-2(1-z)\omega'(z)+(n_s+2)z(1-z)\omega'(z)\right)\\ &+(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}4z(1-z)^2\omega''(z),\end{aligned} \tag{3.7}$$

with $z = \frac{t^2}{H^\circ(x)^2+t^2} \in (0,1]$.

Combining (3.5), (3.6) and (3.7) together, we arrive

$$\begin{aligned}\Delta_\Phi\varphi(x,t)+s\frac{\partial_t\varphi(x,t)}{t} &= \Delta_H\varphi(x,t)+\partial^2_{tt}\varphi(x,t)+s\frac{\partial_t\varphi(x,t)}{t}\\ &= -(H^\circ(x)^2+t^2)^{-\frac{n_s+2}{4}}\Big(4z(1-z)\omega''(z)-2(n_sz-(1+s))\omega'(z)\\ &\qquad -\frac{(n_s-2)^2}{4}\omega(z)\Big),\end{aligned} \tag{3.8}$$

with $z = \frac{t^2}{H^\circ(x)^2+t^2} \in (0,1]$.

The conclusion of this lemma follows from (3.8), (3.2) and (3.1). This finishes our proof. $\square$

It was proved in [23] that the function $\omega$ which is solution of (3.3) can be expressed in terms of hypergeometric functions and has the following form (see [23, Formula (3.5)])

$$\begin{aligned}\omega(z) =& F\left(\frac{n_s+\beta-4}{4}, \frac{n_s-\beta}{4}, \frac{1+s}{2}; z\right)\\ &- \frac{H(n,s,\beta)}{1-s} z^{\frac{1-s}{2}} F\left(\frac{n_s+\beta}{4} - \frac{1+s}{2}, \frac{n_s-\beta}{4} + \frac{1-s}{2}, \frac{3-s}{2}; z\right),\end{aligned} \tag{3.9}$$

where $F(a,b,c;\cdot)$ denotes the hypergeometric function (see [1,25] for the definition and further properties of hypergeometric functions). Inserting (3.9) into (3.2), we get

$$\begin{aligned}\varphi(x,t) =& \frac{F\left(\frac{n_s+\beta-4}{4}, \frac{n_s-\beta}{4}, \frac{1+s}{2}; \frac{t^2}{H^\circ(x)^2+t^2}\right)}{(H^\circ(x)^2+t^2)^{\frac{n_s-2}{4}}}\\ &- \frac{H(n,s,\beta)}{1-s} \frac{t^{1-s} F\left(\frac{n_s+\beta}{4} - \frac{1+s}{2}, \frac{n_s-\beta}{4} + \frac{1-s}{2}, \frac{3-s}{2}; \frac{t^2}{H^\circ(x)^2+t^2}\right)}{(H^\circ(x)^2+t^2)^{\frac{n_s-2s}{4}}}.\end{aligned} \tag{3.10}$$

It follows from Proposition 3.2 in [23] that $\omega(z) > 0$ for $z \in [0,1]$, $\omega'(z) < 0$ for $z \in (0,1)$ and

$$\lim_{z\downarrow 0} z^{1+s}\omega'(z^2) = -\frac{H(n,s,\beta)}{2},$$

hence it holds $\varphi(x,t) > 0$ on $\mathbb{R}^{n+1}_+$,

$$|\omega'(z)| \le C s^{-\frac{1+s}{2}}, \qquad s \in (0,1] \tag{3.11}$$

for some constant $C > 0$, and

$$\lim_{t\downarrow 0} \frac{t^s \partial_t \varphi(x,t)}{(H^\circ(x)^2+t^2)^{\frac{s-1}{2}}\varphi(x,t)} = \begin{cases} H(n,s,\beta) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \tag{3.12}$$

## 4. Proof of Theorem 1.1

In this section, we provide the proof of Theorem 1.1. Our proof is based on the factorization method which is very familiar in proving the Hardy type inequalities (for examples, see [19,23,29] and references therein).

**Proof of Theorem 1.1.** By density argument, it is enough to prove Theorem 1.1 for function $u \in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$. Define the function $v(x,t) = \frac{u(x,t)}{\varphi(x,t)}$. Since $\varphi > 0$ on $\mathbb{R}^{n+1}_+$ then $v(x,t)$ is well-defined. We have

$$\nabla u = v\nabla\varphi + \varphi\nabla v.$$

For any $x, y \in \mathbb{R}^{n+1}$, we have

$$H(x+y)^2 = H(x)^2 + 2\langle H(x)\nabla H(x), y\rangle + \int_0^1 \langle \nabla^2 H^2(x+\theta y)y, y\rangle (1-\theta) d\theta.$$

Applying the preceding equality to $\nabla u = v\nabla\varphi + \varphi\nabla v$, we get

$$\Phi(\nabla u)^2 = v^2\Phi(\nabla\varphi)^2 + \varphi\Phi(\nabla\varphi)\langle\nabla\Phi(\nabla\varphi), \nabla v^2\rangle + \varphi^2\int_0^1 \langle\nabla H^2(v\nabla\varphi + \theta\varphi\nabla v)\nabla v, \nabla v\rangle(1-\theta)d\theta. \tag{4.1}$$

For any $\epsilon > 0$ and $\delta > 0$, we denote

$$\Omega_{\delta,\epsilon} = \{(x,t) \in \mathbb{R}^{n+1} \,:\, t > \delta\} \setminus \{(x,t) \in \mathbb{R}^{n+1}_+ \,:\, H^\circ(x) \le \epsilon\}.$$

Using (4.1) and integration by parts, we have

$$\begin{aligned}\int_{\Omega_{\delta,\epsilon}} \Phi(\nabla u)^2 t^s dxdt &= \int_{\Omega_{\delta,\epsilon}} v^2\Phi(\nabla\varphi)^2 t^s dxdt + \int_{\Omega_{\delta,\epsilon}} \varphi\Phi(\nabla\varphi)\langle\nabla\Phi(\nabla\varphi), \nabla v^2\rangle t^s dxdt\\ &\quad + \int_{\Omega_{\delta,\epsilon}} \varphi^2 \int_0^1 \langle\nabla H^2(v\nabla\varphi + \theta\varphi\nabla v)\nabla v, \nabla v\rangle(1-\theta)d\theta t^s dxdt\\ &= \int_{\Omega_{\delta,\epsilon}} v^2\Phi(\nabla\varphi)^2 t^s dxdt - \int_{\Omega_{\delta,\epsilon}} t^{-s}\mathrm{div}(t^s\varphi\Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi))v^2 t^s dxdt\\ &\quad + \int_{\partial\Omega_{\delta,\epsilon}} \varphi\Phi(\nabla\varphi)\langle\nabla\Phi(\nabla\varphi), \vec{n}(x,t)\rangle v^2 t^s d\mathcal{H}^n(x,t)\\ &\quad + \int_{\Omega_{\delta,\epsilon}} \varphi^2 \int_0^1 \langle\nabla H^2(v\nabla\varphi + \theta\varphi\nabla v)\nabla v, \nabla v\rangle(1-\theta)d\theta t^s dxdt,\end{aligned} \tag{4.2}$$

here $\vec{n}$ denotes the outer normal unit vector on $\partial\Omega_{\delta,\epsilon}$ and $\mathcal{H}^n$ denotes the $n-$dimensional Hausdorff measure on $\partial\Omega_{\delta,\epsilon}$. Note that

$$\begin{aligned}t^{-s}\mathrm{div}(t^s\varphi\Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)) &= \varphi\left(\Delta_\Phi\varphi + s\frac{\partial_t\varphi}{t}\right) + \Phi(\nabla\varphi)^2\\ &= -\frac{(\beta-2)^2}{4}\frac{\varphi^2}{\Phi^\circ(x,t)^2} + \Phi(\nabla\varphi)^2,\end{aligned} \tag{4.3}$$

here we have used (3.1) and part (i) of Proposition 2.1 for $H^\circ$. In other hand, we have

$$\partial\Omega_{\delta,\epsilon} = \{(x,\delta) \,:\, H^\circ(x) \ge \epsilon\} \cup \{(x,t) \in \mathbb{R}^{n+1} \,:\, H^\circ(x) = \epsilon, \text{ and } \ t \ge \delta\},$$

and

$$\vec{n}(x,t) = \begin{cases}(0,\dots,0,-1) & \text{on } \{(x,\delta) \,:\, H^\circ(x) \ge \epsilon\},\\ (-\frac{\nabla_x H^\circ(x)}{|\nabla_x H^\circ(x)|}, 0) & \text{on } \{(x,t) \in \mathbb{R}^{n+1} \,:\, H^\circ(x) = \epsilon, \text{ and } \ t \ge \delta\}.\end{cases} \tag{4.4}$$

From the definition of $\Phi$ and $\varphi$ and Proposition 2.1, we have

$$\Phi(x,t)\nabla\Phi(x,t) = (H(x)\nabla H(x), t)$$

and hence

$$\begin{aligned}\Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi) &= (H(\nabla_x\varphi(x,t))\nabla H(\nabla_x\varphi(x,t),\partial_t\varphi)\\ &= \left(-\left(\frac{n_s-2}{2}+2\frac{z\omega'(z)}{\omega(z)}\right)\frac{\varphi(x,t)H^\circ(x)}{\Phi^\circ(x,t)^2}\nabla H(\nabla H^\circ(x)),\partial_t\varphi\right),\end{aligned} \tag{4.5}$$

with $z=\frac{t^2}{\Phi^\circ(x,t)^2}$. Inserting (4.3), (4.4) and (4.5) into (4.2), we get

$$\begin{aligned}\int_{\Omega_{\delta,\epsilon}}\Phi(\nabla u)^2t^s dxdt &= \frac{(\beta-2)^2}{4}\int_{\Omega_{\delta,\epsilon}}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt-\int_{\{H^\circ\ge\epsilon\}}\delta^s\frac{\partial_t\varphi(x,\delta)}{\varphi(x,\delta)}u(x,\delta)^2dx\\ &\quad+\int_\delta^\infty\frac{\epsilon\left(\frac{n_s-2}{2}+2\frac{z\omega'(z)}{\omega(z)}\Big|_{z=\frac{t^2}{\epsilon^2+t^2}}\right)}{\epsilon^2+t^2}\int_{\{H^\circ=\epsilon\}}\frac{u(x,t)^2}{|\nabla H^\circ(x)|}d\mathcal{H}^{n-1}(x)t^s dt\\ &\quad+\int_{\Omega_{\delta,\epsilon}}\varphi^2\int_0^1\langle\nabla H^2(v\nabla\varphi+\theta\varphi\nabla v)\nabla v,\nabla v\rangle(1-\theta)d\theta t^s dxdt.\end{aligned} \tag{4.6}$$

Notice that $\frac{z\omega'(z)}{\omega(z)}$ is bounded in $[0,1]$. Moreover, since $H(\nabla H^\circ(x))=1$ for $x\neq 0$ and by (2.1), it holds $b^{-1}\le|\nabla H^\circ(x)|\le a^{-1}$ for any $x\neq 0$. Hence, it is true that

$$\lim_{\epsilon\to 0}\int_\delta^\infty\left(\frac{n_s-2}{2}+2\frac{z\omega'(z)}{\omega(z)}\Big|_{z=\frac{t^2}{\epsilon^2+t^2}}\right)\frac{\epsilon}{\epsilon^2+t^2}\int_{\{H^\circ=\epsilon\}}\frac{u(x,t)^2}{|\nabla H^\circ(x)|}d\mathcal{H}^{n-1}(x)t^s dt=0,$$

for any $\delta>0$ (here we used $u\in C_0^\infty(\overline{\mathbb{R}^{n+1}_+})$). Since $u$ has compact support, then $\frac{\partial_t\varphi(x,\delta)}{\varphi(x,\delta)}$ is bounded on the support of $u(x,\delta)$. Therefore, letting $\epsilon\to 0$ in (4.6) and using the Lebesgue's monotone and dominated convergence theorems, we get

$$\begin{aligned}\int_{\{t>\delta\}}\Phi(\nabla u)^2t^s dxdt &=\frac{(\beta-2)^2}{4}\int_{\{t>\delta\}}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt-\int_{\mathbb{R}^n}\delta^s\frac{\partial_t\varphi(x,\delta)}{\varphi(x,\delta)}u(x,\delta)^2dx\\ &\quad+\int_{\{t>\delta\}}\varphi^2\int_0^1\langle\nabla H^2(v\nabla\varphi+\theta\varphi\nabla v)\nabla v,\nabla v\rangle(1-\theta)d\theta t^s dxdt,\end{aligned} \tag{4.7}$$

for any $\delta>0$. Our next aim is to let $\delta\to 0$. Notice that, by Lebesgue's monotone convergence theorem, we have

$$\lim_{\delta\to 0}\int_{\{t>\delta\}}\Phi(\nabla u)^2t^s dxdt=\int_{\mathbb{R}^{n+1}_+}\Phi(\nabla u)^2t^s dxdt,$$

and

$$\lim_{\delta\to 0}\int_{\{t>\delta\}}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt=\int_{\mathbb{R}^{n+1}_+}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt.$$

By (3.5), we have

$$\frac{\partial_t\varphi(x,\delta)}{\varphi(x,\delta)}=-\left(\frac{n_s-2}{2}-2(1-z)\frac{\omega'(z)}{\omega(z)}\Big|_{z=\frac{\delta^2}{H^\circ(x)^2+\delta^2}}\right)\frac{\delta}{H^\circ(x)^2+\delta^2}.$$

It follows from the preceding equality and (3.11) that

$$\delta^s \left|\frac{\partial_t \varphi(x,\delta)}{\varphi(x,\delta)}\right| \leq \left(\frac{n_s-2}{2} + 2\frac{H^\circ(x)^2}{H^\circ(x)^2+\delta^2}\frac{C}{\min_{z\in[0,1]}\omega(z)}\left(\frac{\delta^2}{H^\circ(x)^2+\delta^2}\right)^{-\frac{1+s}{2}}\right)\frac{\delta^{1+s}}{H^\circ(x)^2+\delta^2}$$
$$\leq \left(\frac{n_s-2}{2}\left(\frac{\delta^2}{H^\circ(x)^2+\delta^2}\right)^{\frac{1+s}{2}} + \frac{2C}{\min_{z\in[0,1]}\omega(z)}\frac{H^\circ(x)^2}{H^\circ(x)^2+\delta^2}\right)\left(H^\circ(x)^2+\delta^2\right)^{\frac{s-1}{2}}$$
$$\leq C' H^\circ(x)^{s-1},$$

for some $C' > 0$, and for any $\delta > 0$. Since $u$ is bounded and has compact support, then by Lebesgue's dominated convergence theorem and (3.12), we have

$$\lim_{\delta\to 0}\int_{\mathbb{R}^n} \delta^s \frac{\partial_t\varphi(x,\delta)}{\varphi(x,\delta)} u(x,\delta)^2 dx = -H(n,s,\beta)\int_{\mathbb{R}^n}\frac{u(x,0)^2}{H^\circ(x)^{1-s}}dx.$$

Putting all the previous limits together and letting $\delta \to 0$ in (4.7), we obtain

$$\int_{\mathbb{R}^{n+1}_+}\Phi(\nabla u)^2 t^s dxdt = \frac{(\beta-2)^2}{4}\int_{\mathbb{R}^{n+1}_+}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt + H(n,s,\beta)\int_{\mathbb{R}^n}\frac{u(x,0)^2}{H^\circ(x)^{1-s}}dx$$
$$+ \int_{\mathbb{R}^{n+1}_+}\varphi^2\int_0^1\langle \nabla H^2(v\nabla\varphi+\theta\varphi\nabla v)\nabla v,\nabla v\rangle(1-\theta)d\theta t^s dxdt, \tag{4.8}$$

which immediately implies our desired result (1.7).

It remains to check the sharpness of the constant $H(n,s,\beta)$ in (1.7). Let $K(n,s,\beta)$ denote the best constant for which (1.7) holds, i.e.,

$$K(n,s,\beta) = \inf_{u\in \dot W(\mathbb{R}^{n+1}_+,w_s), u\not\equiv 0}\frac{\int_{\mathbb{R}^{n+1}_+}\Phi(\nabla u)^2 t^s dxdt - \frac{(\beta-2)^2}{4}\int_{\mathbb{R}^{n+1}_+}\frac{u(x,t)^2}{\Phi^\circ(x,t)^2}t^s dxdt}{\int_{\mathbb{R}^n}\frac{u(x,0)^2}{H^\circ(x)^{1-s}}dx}.$$

Obviously, $K(n,s,\beta) \geq H(n,s,\beta)$. We next show that $K(n,s,\beta) \leq H(n,s,\beta)$. Let $\psi \in C^2([0,\infty))$ such that $\psi(t) = 1$ if $t \leq 1$ and $\psi(t) = 0$ if $t \geq 2$. Define the function $\phi(x,t) = \psi(\Phi^\circ(x,t))$ and $\phi_a(x,t) = \phi(x/a, t/a)$ for $a > 0$. For $\epsilon > 0$, we consider the functions

$$\varphi_\epsilon(x,t) = \phi_{\frac{1}{\epsilon}}(x,t)(1-\phi_\epsilon(x,t))\varphi(x,t).$$

Using triangle inequality and the convexity of the function $f(t) = t^2$, we have

$$\Phi(z_1+z_2)^2 \leq (1+\delta)\Phi(z_1)^2 + C_\delta\Phi(z_2)^2, \tag{4.9}$$

for any $\delta > 0$, $z_1, z_2 \in \mathbb{R}^{n+1}$ with $C_\delta = (1+\delta)/\delta$. Applying (4.9) to $\nabla\varphi_\epsilon$, we get

$$\int_{\mathbb{R}^{n+1}_+}\Phi(\nabla\varphi_\epsilon)^2 t^s dxdt \leq (1+\delta)\int_{\mathbb{R}^{n+1}_+}\phi_{\frac{1}{\epsilon}}^2(1-\phi_\epsilon)^2\Phi(\nabla\varphi)^2 t^s dxdt$$
$$+ C_\delta\int_{\mathbb{R}^{n+1}_+}\varphi^2\Phi(\nabla(\phi_{\frac{1}{\epsilon}}(1-\phi_\epsilon)))^2 t^s dxdt. \tag{4.10}$$

Notice that $\Phi(\nabla\varphi)^2 = \langle \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi), \nabla\varphi\rangle$. Using integration by parts and the approximation argument in the proof of (1.7), we easily obtain

$$\int_{\mathbb{R}^{n+1}_+} \phi_{\frac{1}{\epsilon}}^2(1-\phi_\epsilon)^2\Phi(\nabla\varphi)^2 t^s dxdt = \frac{(\beta-2)^2}{4}\int_{\mathbb{R}^{n+1}_+} \frac{\varphi_\epsilon^2}{(\Phi^\circ)^2} t^s dxdt + H(n,s,\beta)\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}}dx - \int_{\mathbb{R}^{n+1}_+} \varphi\langle \nabla(\phi_{\frac{1}{\epsilon}}^2(1-\phi_\epsilon)^2), \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)\rangle t^s dxdt.$$

Combining the preceding equality and (4.10) together implies

$$\begin{aligned} K(n,s,\beta) \le& \frac{(\beta-2)^2}{4}\delta \frac{\int_{\mathbb{R}^{n+1}_+} \frac{\varphi_\epsilon^2}{(\Phi^\circ)^2} t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}}dx} + (1+\delta)H(n,s,\beta) \\ & - (1+\delta)\frac{\int_{\mathbb{R}^{n+1}_+} \varphi\langle \nabla(\phi_{\frac{1}{\epsilon}}^2(1-\phi_\epsilon)^2), \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)\rangle t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}}dx} \\ & + C_\delta \frac{\int_{\mathbb{R}^{n+1}_+} \varphi^2\Phi(\nabla(\phi_{\frac{1}{\epsilon}}(1-\phi_\epsilon)))^2 t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}}dx}. \end{aligned} \tag{4.11}$$

We next estimate the integrals in the right-hand side of (4.11). Denote

$$S_{\Phi^\circ,+} = \{(x,t)\in\mathbb{R}^{n+1}_+ \,:\, \Phi^\circ(x,t) = 1\}.$$

Recall that $\varphi(x,t) = \omega(\frac{t^2}{\Phi^\circ(x,t)^2})\Phi^\circ(x,t)^{-\frac{n_s-2}{2}}$ (by (3.2)) and $\varphi_\epsilon(x,t) = 0$ if $\Phi^\circ(x,t)\le\epsilon$ or $\Phi^\circ(x,t)\ge\frac{2}{\epsilon}$. Using the co-area formula, we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} \frac{\varphi_\epsilon^2}{(\Phi^\circ)^2} t^s dxdt \le& \int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le\frac{2}{\epsilon}\}} \omega\left(\frac{t^2}{\Phi^\circ(x,t)^2}\right)^2 \Phi^\circ(x,t)^{-n_s} t^s dxdt \\ =& \int_{S_{\Phi^\circ,+}} \omega(t^2)^2 \frac{t^s}{|\nabla\Phi^\circ(x,t)|} d\mathcal{H}^n \int_\epsilon^{\frac{2}{\epsilon}} r^{-1}dr \\ =& \int_{S_{\Phi^\circ,+}} \omega(t^2)^2 \frac{t^s}{|\nabla\Phi^\circ(x,t)|} d\mathcal{H}^n \ln\frac{2}{\epsilon^2}, \end{aligned} \tag{4.12}$$

where $\mathcal{H}^n$ is the surface area measure on $S_{\Phi^\circ,+}$. Moreover, for $\epsilon>0$ small enough (such as $2\epsilon^2<1$), we have $(1-\phi_\epsilon(x,t))\phi_{\frac{1}{\epsilon}}(x,t) = 1$ if $2\epsilon\le\Phi^\circ(x,t)\le\frac{1}{\epsilon}$, and hence $\varphi_\epsilon(x,t) = \varphi(x,t)$ if $2\epsilon\le\Phi^\circ(x,t)\le\frac{1}{\epsilon}$. By the co-are formula, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}}dx \ge& \int_{2\epsilon\le H^\circ(x)\le\frac{1}{\epsilon}} H^\circ(x)^{-n}dx \\ =& \int_{\{H^\circ=1\}} |\nabla H^\circ|^{-1}\mathcal{H}^{n-1}\int_{2\epsilon}^{\frac{1}{\epsilon}} r^{-1}dr \end{aligned}$$

$$= \int_{\{H^\circ=1\}} |\nabla H^\circ|^{-1} \mathcal{H}^{n-1} \ln \frac{1}{2\epsilon^2}, \tag{4.13}$$

here we use $\omega(0) = 1$ and $\mathcal{H}^{n-1}$ is the surface are measure on the set $\{H^\circ = 1\}$.

Notice that the supports of $\nabla\phi_\epsilon$ and $\nabla\varphi_{\frac{1}{\epsilon}}$ are disjoint for $\epsilon > 0$ small enough. Therefore, for $\epsilon > 0$ small enough, we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \varphi^2 \Phi(\nabla(\phi_{\frac{1}{\epsilon}}(1-\phi_\epsilon)))^2 t^s dxdt &= \int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}} \varphi^2\Phi(\nabla\phi_\epsilon)^2 t^s dxdt \\ &+ \int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \frac{1}{\epsilon}\le\Phi^\circ(x,t)\le\frac{2}{\epsilon}\}} \varphi^2\Phi(\nabla\phi_{\frac{1}{\epsilon}})^2 t^s dxdt. \end{aligned}\tag{4.14}$$

Furthermore, by using the co-are formula, we have

$$\begin{aligned}&\int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}} \varphi^2\Phi(\nabla\phi_\epsilon)^2 t^s dxdt \\ &= \frac{1}{\epsilon^2}\int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}} \varphi^2\Phi\left(\nabla\phi\left(\frac{(x,t)}{\epsilon}\right)\right)^2 t^s dxdt \\ &\le \frac{\sup_{\mathbb{R}^{n+1}}\Phi^\circ(\nabla\phi)^2}{\epsilon^2}\int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}} \omega\left(\frac{t^2}{\Phi^\circ(x,t)^2}\right)^2 \Phi^\circ(x,t)^{-n_s+2} t^s dxdt \\ &\le \frac{\sup_{\mathbb{R}^{n+1}}\Phi^\circ(\nabla\phi)^2}{\epsilon^2}\int_{S_{\Phi^\circ,+}} \omega(t^2)^2\frac{t^s}{|\nabla\Phi^\circ|}\mathcal{H}^n \int_\epsilon^{2\epsilon} rdr \\ &= \frac{\sup_{\mathbb{R}^{n+1}}\Phi^\circ(\nabla\phi)^2}{2}\int_{S_{\Phi^\circ,+}} \omega(t^2)^2\frac{t^s}{|\nabla\Phi^\circ|}\mathcal{H}^n. \end{aligned}\tag{4.15}$$

By the same argument, we get

$$\int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \frac{1}{\epsilon}\le\Phi^\circ(x,t)\le\frac{2}{\epsilon}\}} \varphi^2\Phi(\nabla\phi_{\frac{1}{\epsilon}})^2 t^s dxdt \le \frac{\sup_{\mathbb{R}^{n+1}}\Phi^\circ(\nabla\phi)^2}{2}\int_{S_{\Phi^\circ,+}} \omega(t^2)^2\frac{t^s}{|\nabla\Phi^\circ|}\mathcal{H}^n. \tag{4.16}$$

Inserting (4.16) and (4.15) into (4.14), we obtain

$$\int_{\mathbb{R}^{n+1}_+} \varphi^2 \Phi(\nabla(\phi_{\frac{1}{\epsilon}}(1-\phi_\epsilon)))^2 t^s dxdt \le \sup_{\mathbb{R}^{n+1}}\Phi^\circ(\nabla\phi)^2 \int_{S_{\Phi^\circ,+}} \omega(t^2)^2\frac{t^s}{|\nabla\Phi^\circ|}\mathcal{H}^n. \tag{4.17}$$

For $\epsilon > 0$ small enough, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+} \varphi\langle\nabla(\phi^2_{\frac{1}{\epsilon}}(1-\phi_\epsilon)^2), \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)\rangle t^s dxdt \\ &\qquad = \frac{1}{2}\int_{\{(x,t)\in\mathbb{R}^{n+1}_+, \epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}} \varphi\langle\nabla(1-\phi_\epsilon)^2, (\nabla\Phi^2)(\nabla\varphi)\rangle t^s dxdt\end{aligned}$$

$$+\frac{1}{2}\int\limits_{\{(x,t)\in\mathbb{R}^{n+1}_+,\frac{1}{\epsilon}\le\Phi^\circ(x,t)\le\frac{2}{\epsilon}\}}\varphi\langle\nabla(\phi_{\frac{1}{\epsilon}})^2,(\nabla\Phi^2)(\nabla\varphi)\rangle t^s dxdt. \tag{4.18}$$

The straightforward computations show that $\nabla\Phi^2(x,t) = 2(H\nabla H, t)$ and

$$\nabla\varphi = (\Phi^\circ)^{-\frac{n_s+2}{2}}\left(-\left(\frac{2t^2}{(\Phi^\circ)^2}\omega' + \frac{n_s-2}{2}\omega\right)H^\circ\nabla H^\circ, \left(\frac{2(H^\circ)^2}{(\Phi^\circ)^2}\omega' - \frac{n_s-2}{2}\omega\right)t\right).$$

It follows from the preceding equalities, the parts (ii), (iii) and (iv) of Proposition 2.1 that

$$\frac{1}{2}(\nabla\Phi^2)(\nabla\varphi) = (\Phi^\circ)^{-\frac{n_s+2}{2}}\left(-\left(\frac{2t^2}{(\Phi^\circ)^2}\omega' + \frac{n_s-2}{2}\omega\right)x, \left(\frac{2(H^\circ)^2}{(\Phi^\circ)^2}\omega' - \frac{n_s-2}{2}\omega\right)t\right).$$

Notice that $\phi_\epsilon(x,t) = \psi(\Phi^\circ(x,t)/\epsilon)$ hence it holds

$$\nabla(1-\phi_\epsilon)^2 = ((1-\psi)^2)'\left(\frac{\Phi^\circ}{\epsilon}\right)\frac{(H^\circ\nabla H^\circ, t)}{\epsilon\Phi^\circ}.$$

Using part (i) of Proposition 2.1, we get

$$\frac{1}{2}\langle\nabla(1-\phi_\epsilon)^2,(\nabla\Phi^2)(\nabla\varphi)\rangle = -\frac{n_s-2}{2\epsilon}((1-\psi)^2)'\left(\frac{\Phi^\circ}{\epsilon}\right)\Phi^\circ(x,t)^{-\frac{n_s}{2}}.$$

Consequently, we have

$$\begin{aligned}
&\left|\frac{1}{2}\int\limits_{\{(x,t)\in\mathbb{R}^{n+1}_+,\epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}}\varphi\langle\nabla(1-\phi_\epsilon)^2,(\nabla\Phi^2)(\nabla\varphi)\rangle t^s dxdt\right| \\
&\le \frac{(n_s-2)\sup_{\mathbb{R}}|((1-\psi)^2)'|}{2\epsilon}\int\limits_{\{(x,t)\in\mathbb{R}^{n+1}_+,\epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}}\varphi(x,t)\Phi^\circ(x,t)^{-\frac{n_s}{2}}t^s dxdt \\
&= \frac{(n_s-2)\sup_{\mathbb{R}}|((1-\psi)^2)'|}{2\epsilon}\int\limits_{\{(x,t)\in\mathbb{R}^{n+1}_+,\epsilon\le\Phi^\circ(x,t)\le 2\epsilon\}}\omega\left(\frac{t^2}{\Phi^\circ(x,t)^2}\right)\Phi^\circ(x,t)^{-n_s+1}t^s dxdt \\
&= \frac{(n_s-2)\sup_{\mathbb{R}}|((1-\psi)^2)'|}{2\epsilon}\int\limits_{S_{\Phi^\circ,+}}\omega(t^2)t^s d\mathcal{H}^n\int\limits_{\epsilon}^{2\epsilon}dr \\
&= \frac{(n_s-2)\sup_{\mathbb{R}}|((1-\psi)^2)'|}{2}\int\limits_{S_{\Phi^\circ,+}}\omega(t^2)t^s d\mathcal{H}^n,
\end{aligned} \tag{4.19}$$

here we used again the co-area formula. By the same argument, we obtain

$$\left|\frac{1}{2}\int\limits_{\{(x,t)\in\mathbb{R}^{n+1}_+,\frac{1}{\epsilon}\le\Phi^\circ(x,t)\le\frac{2}{\epsilon}\}}\varphi\langle\nabla(\phi_{\frac{1}{\epsilon}})^2,(\nabla\Phi^2)(\nabla\varphi)\rangle t^s dxdt\right| \le \frac{(n_s-2)\sup_{\mathbb{R}}|(\psi^2)'|}{2}\int\limits_{S_{\Phi^\circ,+}}\omega(t^2)t^s d\mathcal{H}^n. \tag{4.20}$$

Inserting (4.20) and (4.19) into (4.18), we get

$$\left| \int_{\mathbb{R}^{n+1}_+} \varphi \langle \nabla(\phi_{\frac{1}{\epsilon}}^2 (1-\phi_\epsilon)^2), \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)\rangle t^s dxdt \right|$$

$$\leq \frac{n_s - 2}{2} \left( \sup_{\mathbb{R}} |((1-\psi)^2)'| + \sup_{\mathbb{R}} |(\psi^2)'| \right) \int_{S_{\Phi^\circ,+}} \omega(t^2) t^s d\mathcal{H}^n. \tag{4.21}$$

It follows from (4.13), (4.17) and (4.21) that

$$\lim_{\epsilon\to 0} \frac{\int_{\mathbb{R}^{n+1}_+} \varphi \langle \nabla(\phi_{\frac{1}{\epsilon}}^2 (1-\phi_\epsilon)^2), \Phi(\nabla\varphi)\nabla\Phi(\nabla\varphi)\rangle t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}} dx} = 0, \tag{4.22}$$

and

$$\lim_{\epsilon\to 0} \frac{\int_{\mathbb{R}^{n+1}_+} \varphi^2 \Phi(\nabla(\phi_{\frac{1}{\epsilon}}(1-\phi_\epsilon)))^2 t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}} dx} = 0. \tag{4.23}$$

Furthermore, from (4.12) and (4.13), we see that

$$\frac{\int_{\mathbb{R}^{n+1}_+} \frac{\varphi_\epsilon^2}{(\Phi^\circ)^2} t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}} dx}$$

is bounded as $\epsilon \to 0$. Consequently, letting $\epsilon \to 0$ in (4.11) and using the limits (4.22) and (4.23), we get

$$K(n,s,\beta) \leq C' \frac{(\beta-2)^2}{4} \delta + (1+\delta) H(n,s,\beta), \tag{4.24}$$

with

$$C' = \liminf_{\epsilon\to 0} \frac{\int_{\mathbb{R}^{n+1}_+} \frac{\varphi_\epsilon^2}{(\Phi^\circ)^2} t^s dxdt}{\int_{\mathbb{R}^n} \frac{\varphi_\epsilon(x,0)^2}{H^\circ(x)^{1-s}} dx} < \infty.$$

Since (4.24) holds for any $\delta > 0$, by letting $\delta \to 0$ we obtain $K(n,s,\beta) \leq H(n,s,\beta)$. Consequently, we have $K(n,s,\beta) = H(n,s,\beta)$. This proves the sharpness of $H(n,s,\beta)$.

Finally, we prove the non-attainability of $H(n,s,\beta)$. Indeed, suppose that there exists $u \in \dot{W}(\mathbb{R}^{n+1}_+, w_s) \setminus \{0\}$ which realizes the equality in (1.7). By (4.8) we must have

$$\int_{\mathbb{R}^{n+1}_+} \varphi^2 \int_0^1 \langle \nabla H^2(v\nabla\varphi + \theta\varphi\nabla v)\nabla v, \nabla v\rangle (1-\theta) d\theta t^s dxdt = 0.$$

Nevertheless, the matrix $\nabla^2 H^2$ is positive define since $H$ is strongly convex. Hence, $\nabla v = 0$ on $\mathbb{R}^{n+1}_+$. Consequently, $v$ is constant function which implies $u(x,t) = c\varphi(x,t)$ for some constant $c \neq 0$. This is impossible since $\varphi \notin \dot{W}(\mathbb{R}^{n+1}_+, w_s)$. Therefore, $H(n,s,\beta)$ is not attained.

The proof of Theorem 1.1 is then completed. $\square$

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