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The reducibility of $C_0(2)$ operators [☆]

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ABSTRACT

In this paper, we give a necessary and sufficient condition for the reducibility of a $C_0(2)$ operator by using its characteristic function. Moreover, we obtain the number of reducing subspaces of a $C_0(2)$ operator. As an application, we will restudy the reducibility of the truncated Toeplitz operator A_{z^2}.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk on the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the boundary of $\mathbb{D}$. The Hardy space H^2 consists of the holomorphic functions on $\mathbb{D}$ having square-summable Taylor coefficients at the origin. It is well known that H^2 can be identified with the subspace of $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ consisting of the functions whose Fourier coefficients with negative indices vanish (see [8]). A function $u \in H^2$ is called an inner function if $|u(\xi)| = 1$ *a.e.* for $\xi \in \mathbb{T}$. For a nonconstant inner function u, the model space $\mathcal{K}_u^2 = H^2 \ominus uH^2$ is invariant under the backward shift operator T_z^* on H^2. The truncated Toeplitz operator on $\mathcal{K}_u^2$ with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by

$$A_\varphi f = P_u(\varphi f), f \in \mathcal{K}_u^2,$$

where P_u is the orthogonal projection on $L^2(\mathbb{T}, \frac{d\theta}{2\pi})$ with range $\mathcal{K}_u^2$.

A conjugation C on a Hilbert space $\mathcal{H}$ is an anti-linear, isometric and involutive map, that is $C(\alpha f) = \overline{\alpha} Cf$, $C^2 = I$ and $(Cf, Cg) = (g, f)$, $\forall f, g \in \mathcal{H}, \alpha \in \mathbb{C}$. The model space $\mathcal{K}_u^2$ carries a natural conjugation given by

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$$Cf(\xi) = u(\xi)\overline{\xi f(\xi)} \ a.e. \text{ for } \xi \in \mathbb{T}, \text{ where } f \in \mathcal{K}_u^2. \tag{1.1}$$

For a completely nonunitary (c.n.u.) contraction T on the Hilbert space $\mathcal{H}$, the defect operators and defect spaces of T are defined by

$$\begin{aligned} D_T &= (I - T^*T)^{1/2}, \quad & D_{T^*} &= (I - TT^*)^{1/2}, \\ \mathfrak{D}_T &= \overline{D_T\mathcal{H}}, & \mathfrak{D}_{T^*} &= \overline{D_{T^*}\mathcal{H}}. \end{aligned}$$

The characteristic function of a c.n.u. contraction T is a purely contractive analytic operator-valued function $\Theta_T(z) : \mathfrak{D}_T \to \mathfrak{D}_{T^*}$ on $\mathbb{D}$ defined as follows:

$$\Theta_T(z) = [-T + zD_{T^*}(I - zT^*)^{-1}D_T]|_{\mathfrak{D}_T}, z \in \mathbb{D}. \tag{1.2}$$

The characteristic function is a useful tool in the study of model theory for the c.n.u. contractions, for instance, if Θ_T admits a nontrivial regular factorization, then T has a nontrivial invariant subspace generated by regular factorization. We call two characteristic functions $\Theta_1(z) : \mathcal{E}_1 \to \mathcal{E}_2$ and $\Theta_2(z) : \mathcal{F}_1 \to \mathcal{F}_2, z \in \mathbb{D}$ coincide if there exist a unitary operator U from $\mathcal{E}_1$ to $\mathcal{F}_1$ and a unitary operator V from $\mathcal{E}_2$ to $\mathcal{F}_2$ such that

$$V\Theta_1(z) = \Theta_2(z)U$$

for all $z \in \mathbb{D}$. It is known that two c.n.u. contractions T_1 and T_2 are unitarily equivalent if and only if Θ_{T_1} and Θ_{T_2} coincide. More information about the model theory on contractions can be found in [11].

For a positive integer N, a c.n.u. contraction T is called $C_0(N)$ operator if $\|T\| \le 1$, $T^n \to 0$ and $T^{*n} \to 0$ (in strong operator topology) as $n \to \infty$ and $rank(I - T^*T) = N$. Let u be an inner function such that $\dim \mathcal{K}_u^2 > 1$, then A_z and A_{z^2} on $\mathcal{K}_u^2$ are $C_0(1)$ operator and $C_0(2)$ operator, respectively. In particular, every $C_0(1)$ operator is unitarily equivalent to A_z on the model space $\mathcal{K}_u^2$ for some inner function u (see [1,11]). A good reference for studying the C_0 operators is the monograph [1]. A closed subspace M of $\mathcal{H}$ satisfying $TM \subset M$ is said to be an invariant subspace for T. If both M and $M^\perp$ are invariant subspaces for T, then M is called the reducing subspace for T. If T has a nontrivial reducing subspace ($M \neq \{0\}$ or $\mathcal{H}$), we say that T is reducible. Otherwise, we say that T is irreducible. The reducibility of T means to decide either T is reducible or irreducible. It is well known that A_z is irreducible and hence all $C_0(1)$ operators are irreducible. The reducibility of general $C_0(N)$ operators is complicated, for instance, in [2], it was shown that A_{z^2} can be reducible for some inner functions u. The authors in [10] provided a function theoretical based proof and then described the reducing subspaces of A_{z^2} explicitly. The classification of invariant subspaces and reducing subspaces of various operators on function spaces has proved to be very rewarding research problems in analysis. A lot of nice and deep work on the reducibility of multiplication operators induced by the finite Blaschke product on Bergman space can be seen in [3,9,12] and references therein. In general, $C_0(2)$ operator is not unitarily equivalent to A_{z^2}, and some examples are provided in section 3. So in this paper we focus on the reducibility of $C_0(2)$ operators and a good understanding of these cases that will shed light on the general picture.

The paper is organized as follows. In section 2, we give a necessary and sufficient condition for the reducibility of $C_0(2)$ operators by using the characteristic function and we obtain the number of the reducing subspaces of a $C_0(2)$ operator. In section 3, as an application, we also restudy the reducibility of the truncated Toeplitz operator A_{z^2} and we will provide examples of $C_0(2)$ operator which are not unitarily equivalent to A_{z^2} for any inner function u.

2. Reducibility of $C_0(2)$ operators

In this section, we will give a necessary and sufficient condition for the reducibility of a $C_0(2)$ operator by using the characteristic function.

A contractive analytic operator-valued function Θ is called inner if its boundary values $\Theta(e^{it})$ are isometries a.e. on $\mathbb{T}$. It is known that if $T \in C_0(N)$, then Θ_T is inner ([11]). For a $C_0(2)$ operator T, it follows from (1.2) that the characteristic function Θ_T of T is a 2×2 matrix-valued analytic function. Let

$$\Theta(z) = \begin{pmatrix} a(z) & -b(z) \\ c(z) & d(z) \end{pmatrix} \tag{2.1}$$

be a 2×2 matrix-valued inner function, then $a, b, c, d \in \mathcal{K}^2_{z\varphi}$, where $\varphi = \det(\Theta)$ (see [5]). The following lemma gives a parametrization of 2×2 matrix-valued inner functions.

Lemma 2.1 *(Theorem 1 in [5]). Let φ be a nonconstant inner function and Θ be defined in (2.1). Then Θ is unitary a.e. on $\mathbb{T}$ and $\det(\Theta) = \varphi$ if and only if*

(1) *a, b, c, d belong to $\mathcal{K}^2_{z\varphi}$.*
(2) *$d = \varphi\overline{a}$ and $c = \varphi\overline{b}$.*
(3) *$|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$.*

Lemma 2.2 *(Theorem 8.16 in [4]). Suppose φ is an inner function. Then $u \in \mathcal{K}^2_{z\varphi}$ is an inner function if and only if u is the inner factor of φ.*

For each conjugation C on a Hilbert space $\mathcal{H}$, there are many fixed points of C, for example, there exists an orthonormal basis $\{e_n\}$ such that $Ce_n = e_n$ (see [7]). For an inner function φ, let $\xi \in \mathbb{T}$ such that both φ and φ' have nontangential limit at ξ, then we have the kernel function

$$k_\xi(z) = \frac{1 - \overline{\varphi(\xi)}\varphi(z)}{1 - \overline{\xi}z} \in \mathcal{K}^2_\varphi.$$

It is shown in [6] that $(\overline{\xi}\varphi(\xi))^{1/2}k_\xi$ is a fixed point of C. The following lemma is to characterize the fixed points of C.

Lemma 2.3. *Let C be a conjugation on Hilbert space $\mathcal{H}$. For two vectors $a, b \in \mathcal{H}$, the following statements hold.*

(1) *If there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $C(a) = \alpha a$, then there exists $\beta \in \mathbb{C}$ with $\beta \neq 0$ such that βa is a fixed point of C.*
(2) *There exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| \neq 0$ such that $a = \alpha b + \beta C(b)$ if and only if there exist $\sigma, \delta, \gamma \in \mathbb{C}$ with $\sigma\delta\gamma \neq 0$ and $\sigma\overline{\delta} \notin \mathbb{R}$ such that σa and $\delta a + \gamma b$ are fixed points of C.*
(3) *There exist $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$ and $|\alpha| \neq |\beta|$ such that $a = \alpha b + \beta C(b)$ if and only if there exist at least two pairs $(\delta_i, \gamma_i) \in \mathbb{C}^2$ with $\delta_i\gamma_i \neq 0$ $\delta_1\overline{\delta_2} \notin \mathbb{R}$, $\delta_2\gamma_1 - \delta_1\gamma_2 \neq 0$ and $|\delta_2\gamma_1 - \delta_1\gamma_2| \neq |\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1|$, such that $\delta_i a + \gamma_i b$ are fixed points of C.*

Proof. (1) Choosing β such that $\beta/|\beta| = \alpha^{1/2}$, it is not hard to see that βa is a fixed point of C.

(2) If $a = \alpha b + \beta C(b)$, choosing σ such that $\sigma/|\sigma| = (\alpha^{-1}\overline{\beta})^{1/2}$, then $\sigma\alpha = \overline{\sigma\beta}$. We have

$$\begin{aligned} C(\sigma a) &= C(\sigma\alpha b + \sigma\beta C(b)) \\ &= \overline{\sigma\alpha}C(b) + \overline{\sigma\beta}b \\ &= \sigma a. \end{aligned}$$

To obtain the second fixed point, for any $\eta \in \mathbb{C}, \eta \neq \overline{\alpha}$, we calculate

$$\begin{aligned} C(a + \overline{\eta}b) &= \overline{\alpha}C(b) + \overline{\beta}b + \eta C(b) \\ &= \overline{\beta}b + (\overline{\alpha} + \eta)C(b) \\ &= \overline{\beta}b + (\overline{\alpha} + \eta)\beta^{-1}(a - \alpha b) \\ &= (\overline{\alpha} + \eta)\beta^{-1}(a + (\frac{|\beta|^2}{\overline{\alpha} + \eta} - \alpha)b). \end{aligned} \tag{2.2}$$

To get the desired result, we first solve the following equations

$$\begin{cases} |(\overline{\alpha} + \eta)\beta^{-1}| = 1, \\ \frac{|\beta|^2}{\overline{\alpha}+\eta} - \alpha = \overline{\eta}. \end{cases} \tag{2.3}$$

It is not hard to see that the above equations have solutions as follows:

$$\eta = \beta e^{ix} - \overline{\alpha},\ x \in \mathbb{R}. \tag{2.4}$$

It is clear that we can choose $x_0 \in \mathbb{R}$ such that $\eta_0 = \beta e^{ix_0} - \overline{\alpha} \neq 0$. Since $|\alpha| = |\beta|$, we have $\beta e^{ix_0}\alpha \notin \mathbb{R}$.

Putting η_0 into (2.2), we get

$$C(e^{\frac{x_0}{2}i}a + (\overline{\beta}e^{-\frac{x_0}{2}i} - \alpha e^{\frac{x_0}{2}i})b) = e^{\frac{x_0}{2}i}a + (\overline{\beta}e^{-\frac{x_0}{2}i} - \alpha e^{\frac{x_0}{2}i})b.$$

Let

$$\delta = e^{\frac{x_0}{2}i} \ \text{ and } \ \gamma = \overline{\beta}e^{-\frac{x_0}{2}i} - \alpha e^{\frac{x_0}{2}i},$$

then $\delta a + \gamma b$ is a fixed point of C. In this case,

$$\sigma\overline{\delta} = |\sigma|(\alpha^{-1}\overline{\beta})^{1/2}e^{-\frac{x_0}{2}i} = |\sigma|(\alpha^{-1}\overline{\beta}e^{-x_0 i})^{1/2} \notin \mathbb{R}.$$

On the other hand, if σa and $\delta a + \gamma b$ are both fixed points of C, then

$$\begin{aligned} \delta a + \gamma b &= C(\delta a + \gamma b) \\ &= \frac{\overline{\delta}}{\overline{\sigma}}\sigma a + \overline{\gamma}C(b). \end{aligned}$$

Since $\overline{\delta}\sigma \notin \mathbb{R}$, then $\overline{\delta}\sigma - \delta\overline{\sigma} \neq 0$. Let

$$\alpha = \frac{\gamma\overline{\sigma}}{\overline{\delta}\sigma - \delta\overline{\sigma}} \ \text{ and } \ \beta = -\frac{\overline{\sigma\gamma}}{\overline{\delta}\sigma - \delta\overline{\sigma}},$$

then $\alpha\beta \neq 0$, $|\alpha| = |\beta|$ and $a = \alpha b + \beta C(b)$. The proof of (2) is completed.

(3) Suppose that there exist $\alpha,\ \beta \in \mathbb{C}$ with $\beta \neq 0$ and $|\alpha| \neq |\beta|$ such that $a = \alpha b + \beta C(b)$, then $C(a) = \overline{\alpha}C(b) + \overline{\beta}b$ and $C(b) = \beta^{-1}(a - \alpha b)$. For any $\delta,\ \gamma \in \mathbb{C}$ with $\delta\gamma \neq 0$, we have

$$\begin{aligned} C(\delta a + \gamma b) &= \overline{\delta}C(a) + \overline{\gamma}C(b) \\ &= (\overline{\delta}\overline{\alpha} + \overline{\gamma})\beta^{-1}a + (\overline{\delta\beta} - (\overline{\delta}\overline{\alpha} + \overline{\gamma})\beta^{-1}\alpha)b. \end{aligned}$$

Let

$$\begin{cases} \delta = (\overline{\delta}\overline{\alpha} + \overline{\gamma})\beta^{-1}, \\ \gamma = \overline{\delta\beta} - (\overline{\delta}\overline{\alpha} + \overline{\gamma})\beta^{-1}\alpha, \end{cases}$$

we get that

$$\gamma = \overline{\delta\beta} - \delta\alpha.$$

Then

$$\overline{\alpha}\gamma = \overline{\delta\alpha\beta} - \delta|\alpha|^2 \text{ and } \overline{\beta\gamma} = \delta|\beta|^2 - \overline{\delta\alpha\beta}.$$

Since $|\alpha| \neq |\beta|$, the above equations are equivalent to

$$\begin{cases} \delta = \frac{\overline{\alpha}\gamma + \overline{\beta}\overline{\gamma}}{|\beta|^2 - |\alpha|^2}, \\ \gamma = \overline{\delta\beta} - \delta\alpha. \end{cases}$$

The non-zero solutions of above equations can be written as

$$\begin{cases} \delta \in \mathbb{C} \text{ with } \delta \neq 0 \\ \gamma = \overline{\delta\beta} - \delta\alpha \end{cases} \quad \text{or} \quad \begin{cases} \gamma \in \mathbb{C} \text{ with } \gamma \neq 0, \\ \delta = \frac{\overline{\alpha}\gamma + \overline{\beta}\overline{\gamma}}{|\beta|^2 - |\alpha|^2}. \end{cases}$$

By the above solutions, there exist two pairs (δ_i, γ_i) with $\delta_i\gamma_i \neq 0,\ (i = 1, 2)$ such that $\delta_1\overline{\delta_2} \notin \mathbb{R}$, $\delta_2\gamma_1 - \delta_1\gamma_2 = (\overline{\delta_1}\delta_2 - \delta_1\overline{\delta_2})\overline{\beta} \neq 0$ and $\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1 = (\delta_1\overline{\delta_2} - \overline{\delta_1}\delta_2)\alpha$. Then $|\delta_2\gamma_1 - \delta_1\gamma_2| \neq |\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1|$.

On the other hand, if there are two pairs $(\delta_1, \gamma_1) \neq (\delta_2, \gamma_2)$ with $\delta_i\gamma_i \neq 0$ such that $\delta_i a + \gamma_i b (i = 1, 2)$ are fixed points of C, then

$$\begin{cases} C(\delta_1 a + \gamma_1 b) = \delta_1 a + \gamma_1 b, \\ C(\delta_2 a + \gamma_2 b) = \delta_2 a + \gamma_2 b. \end{cases}$$

By the first equation we get that

$$C(a) = \overline{\delta_1}^{-1}(\delta_1 a + \gamma_1 b - C(\gamma_1 b)).$$

Replacing $C(a)$ in second equation, we obtain

$$\begin{aligned} \delta_2 a + \gamma_2 b &= C(\delta_2 a + \gamma_2 b) \\ &= \overline{\delta_2}\ \overline{\delta_1}^{-1}(\delta_1 a + \gamma_1 b - C(\gamma_1 b)) + \overline{\gamma_2}C(b). \end{aligned}$$

Since $\delta_1\overline{\delta_2} \notin \mathbb{R}$, we have

$$a = \frac{\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1}{\delta_1\overline{\delta_2} - \overline{\delta_1}\delta_2} b + \frac{\overline{\delta_2}\,\overline{\gamma_1} - \overline{\delta_1}\,\overline{\gamma_2}}{\delta_1\overline{\delta_2} - \overline{\delta_1}\delta_2} C(b).$$

Since $\delta_2\gamma_1 - \delta_1\gamma_2 \neq 0$, let

$$\alpha = \frac{\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1}{\delta_1\overline{\delta_2} - \overline{\delta_1}\delta_2}, \quad \beta = \frac{\overline{\delta_2}\,\overline{\gamma_1} - \overline{\delta_1}\,\overline{\gamma_2}}{\delta_1\overline{\delta_2} - \overline{\delta_1}\delta_2},$$

then $|\alpha| \neq |\beta|$, $\beta \neq 0$ and $a = \alpha b + \beta C(b)$. Therefore we complete the proof. □

The following lemma comes from [2], which is key to studying the reducibility of $C_0(2)$ operators.

Lemma 2.4 *(Lemma 2 in [2]). Let u be a nonconstant inner function. Then A_{z^2} is reducible on $\mathcal{K}_u^2$ if and only if there exist orthogonal projections Q_1 and Q_2 in $\mathcal{L}(\mathbb{C}^2)$ so that*

$$\Theta_{A_{z^2}}(z)Q_2 = Q_1\Theta_{A_{z^2}}(z), z \in \mathbb{D} \tag{2.5}$$

and $0 \neq Q_i \neq I_{\mathbb{C}^2}$ $(i = 1, 2)$.

It is not hard to see that Lemma 2.4 also holds for $C_0(2)$ operators. From Lemma 2 in [2], we know that Q_i has the form

$$Q_i = \begin{pmatrix} q_i & r_i\overline{\xi_i} \\ r_i\xi_i & 1 - q_i \end{pmatrix}, \tag{2.6}$$

where $0 \leq q_i \leq 1,\ \xi_i \in \mathbb{C},\ |\xi_i| = 1,$ and $r_i = (q_i(1 - q_i))^{1/2},$ for $i = 1, 2$. By the proof of Lemma 2.4, a $C_0(2)$ operator T is reducible if and only if Θ_T coincides with

$$\Theta(z) = \begin{pmatrix} \theta_1(z) & 0 \\ 0 & \theta_2(z) \end{pmatrix}, \tag{2.7}$$

where θ_1 and θ_2 are inner functions.

Let $\mathrm{Aut}(\mathbb{D})$ denote the automorphism group of $\mathbb{D}$. Note that if T is reducible, then $\det\Theta_T \notin \mathrm{Aut}(\mathbb{D})$. In what follows, we assume that $\det\Theta_T \notin \mathrm{Aut}(\mathbb{D})$.

Lemma 2.5. *Let T be a $C_0(2)$ operator and Θ_T be the characteristic function of T with $\det(\Theta_T) = \varphi$. Then T is reducible if and only if one of the following conditions holds:*

(1) *$a = \alpha u$ and $b = \beta u$, where $u \in \mathcal{K}_{z\varphi}^2$ is inner and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.*
(2) *$a = \alpha u$ and $b = \beta C(u)$, where $u \in \mathcal{K}_{z\varphi}^2$ is inner and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.*
(3) *There exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ such that $C(a) = \alpha a$ and $C(b) = \beta b$.*
(4) *There exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| \neq 0$ such that*

$$b = \alpha a + \beta C(a)$$

or

$$a = \alpha b + \beta C(b).$$

(5) *There exist* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \neq |\beta|$ *and* $\alpha\beta \neq 0$ *such that*

$$a = \alpha b + \beta C(b).$$

Proof. If T is reducible, by Lemma 2.4, there exist Q_1 and Q_2 with the form (2.6), such that

$$\Theta_T(z)Q_2 = Q_1\Theta_T(z), z \in \mathbb{D}.$$

By some calculations we obtain that

$$\begin{cases} (q_2 - q_1)a = r_1\overline{\xi_1}\varphi\bar{b} + r_2\xi_2 b, \\ (q_2 - q_1)\varphi\overline{a} = r_1\xi_1 b + r_2\overline{\xi_2}\varphi\bar{b}, \\ (1 - q_1 - q_2)b = -r_1\overline{\xi_1}\varphi\overline{a} + r_2\overline{\xi_2}a, \\ (1 - q_1 - q_2)\varphi\bar{b} = -r_1\xi_1 a + r_2\xi_2\varphi\overline{a}, \end{cases}$$

which is equivalent to

$$\begin{cases} (q_2 - q_1)a = r_1\overline{\xi_1}\varphi\bar{b} + r_2\xi_2 b, \\ (1 - q_1 - q_2)b = -r_1\overline{\xi_1}\varphi\overline{a} + r_2\overline{\xi_2}a. \end{cases} \tag{2.8}$$

The necessity of Lemma 2.5 will be proved in five cases.

Case I: $q_1 = 0$ or $q_1 = 1$, then $r_1 = (q_1(1 - q_1))^{1/2} = 0$. We only need to discuss the case $q_1 = 0$, since the argument of the case $q_1 = 1$ is similar. The equations (2.8) give that

$$\begin{cases} q_2 a = r_2\xi_2 b, \\ (1 - q_2)b = r_2\overline{\xi_2}a. \end{cases} \tag{2.9}$$

(i) If $q_2 = 0$ or 1, then $r_2 = 0$. The equations (2.9) show that $b \equiv 0$ or $a \equiv 0$. Lemma 2.1 yields that

$$a \in \mathcal{K}^2_{z\varphi}$$

or

$$b \in \mathcal{K}^2_{z\varphi},$$

which are inner functions.

(ii) If $q_2 \in (0, 1)$, it follows from equations (2.9) and $r_2 = (q_2(1 - q_2))^{1/2}$ that

$$a = q_2^{-1/2}(1 - q_2)^{1/2}\xi_2 b. \tag{2.10}$$

Since $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$, we have

$$q_2^{-1}(1 - q_2)|b|^2 + |b|^2 = 1,$$

hence $|b|^2 = q_2$ a.e. on $\mathbb{T}$. So there exists an inner function $u \in \mathcal{K}^2_{z\varphi}$ such that $b = q_2^{1/2}u$. By equation (2.10), we know that

$$a = (1 - q_2)^{1/2}\xi_2 u,$$

which proves (1) in Lemma 2.5.

Case II: $q_2 = 0$ or $q_2 = 1$ and $0 < q_1 < 1$. The similar argument as Case I, we will obtain that (2) in Lemma 2.5 holds.

Case III: $q_i \neq 0, 1$ and $q_1 = q_2 = \frac{1}{2}$. Then $r_1 = r_2 \neq 0$, and the equations (2.8) imply that

$$\begin{cases} \overline{\xi_1}\varphi\bar{b} + \xi_2 b = 0, \\ -\overline{\xi_1}\varphi\bar{a} + \overline{\xi_2} a = 0. \end{cases}$$

Let

$$\alpha = \xi_1\overline{\xi_2} \text{ and } \beta = -\xi_1\xi_2,$$

then

$$C(a) = \alpha a \text{ and } C(b) = \beta b.$$

Case IV: $q_i \neq 0, \frac{1}{2}, 1$ and $q_1 = q_2$ or $q_1 + q_2 = 1$. If $q_1 = q_2$, set

$$\alpha = \frac{r_2\overline{\xi_2}}{1 - q_2 - q_1} \text{ and } \beta = -\frac{r_1\overline{\xi_1}}{1 - q_2 - q_1}.$$

The second formula in equations (2.8) gives that

$$b = \alpha a + \beta C(a).$$

If $q_1 + q_2 = 1$, set

$$\alpha = \frac{r_2\xi_2}{q_2 - q_1} \text{ and } \beta = \frac{r_1\overline{\xi_1}}{q_2 - q_1}.$$

The first formula in equations (2.8) gives that

$$a = \alpha b + \beta C(b).$$

Case V: $q_i \neq 0, 1$, $q_1 \neq q_2$ and $q_1 + q_2 \neq 1$. Then the two formulas in (4) in Lemma 2.5 are indeed the same. By the similar argument as Case IV, we know that (5) in Lemma 2.5 will hold.

To prove the sufficiency, we first suppose that (1) in Lemma 2.5 holds.

(i) If $\beta = 0$, then $b \equiv 0$ and a is inner. Let

$$Q_1 = Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $\Theta_T Q_2 = Q_1 \Theta_T$ and T is reducible. The argument for $\alpha = 0$ is similar and T is also reducible.

(ii) If $\alpha\beta \neq 0$, let

$$\xi_2 = \frac{|\beta|\alpha}{|\alpha|\beta}, \quad q_2 = |\beta|^2,$$

then $1 - q_2 = |\alpha|^2,\ r_2 = |\alpha\beta|$.

$$\begin{aligned}\Theta_T Q_2 &= \begin{pmatrix} a & -b \\ \varphi\bar{b} & \varphi\bar{a} \end{pmatrix} \begin{pmatrix} |\beta|^2 & |\beta|^2\overline{(\frac{\alpha}{\beta})} \\ |\beta|^2\frac{\alpha}{\beta} & |\alpha|^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \varphi\bar{b} & \varphi\bar{a} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & -b \\ \varphi\bar{b} & \varphi\bar{a} \end{pmatrix} \\ &= Q_1\Theta_T,\end{aligned}$$

so that T is reducible.

The argument for (2) in Lemma 2.5 is similar to (1), we also know that T is reducible.

If there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$ such that $C(a) = \alpha a$ and $C(b) = \beta b$. Let $q_1 = q_2 = \frac{1}{2}$, $\xi_1 = (-\alpha\beta)^{1/2}$ and $\xi_2 = (-\overline{\alpha}\beta)^{1/2}$. It is easy to check that

$$\Theta_T(z)Q_2 = Q_1\Theta_T(z), z \in \mathbb{D},$$

which means that T is reducible.

Suppose (4) or (5) in Lemma 2.5 holds. Without loss of generality, assume that there exist α and β with $\alpha\beta \neq 0$ such that $a = \alpha b + \beta C(b)$. For finding two projections Q_1, Q_2 with the form (2.6) such that $\Theta_T Q_2 = Q_1\Theta_T$, one only needs to solve the following equations

$$\begin{cases} r_2\xi_2 = \alpha(q_2 - q_1), \\ r_1\overline{\xi_1} = \beta(q_2 - q_1). \end{cases} \tag{2.11}$$

It is easy to see that $\xi_1 = \overline{\beta}/|\beta|$ and $\xi_2 = \alpha/|\alpha|$. From (2.11), we get

$$\begin{cases} r_2^2 = |\alpha|^2(q_2 - q_1)^2, \\ r_1^2 = |\beta|^2(q_2 - q_1)^2. \end{cases} \tag{2.12}$$

Since $r_2^2 - r_1^2 = (q_2 - q_1)(1 - q_1 - q_2)$, and we also assume that $q_1 \neq q_2$, then we have that

$$(|\alpha|^2 - |\beta|^2)(q_2 - q_1) = 1 - q_1 - q_2.$$

If $|\alpha| = |\beta|$, then we have $q_1 + q_2 = 1$, which shows that when the second condition in (4) of the Lemma 2.5 implies that T is reducible.

If $|\alpha|^2 - |\beta|^2 - 1 = 0$, then $|\alpha| = \sqrt{|\beta|^2 + 1} > 1$, so that

$$q_2 = \frac{1}{2} \text{ and } q_1 = \frac{1}{2} \pm \frac{1}{2|\alpha|}. \tag{2.13}$$

If $|\alpha|^2 - |\beta|^2 + 1 = 0$, then $|\beta| = \sqrt{|\alpha|^2 + 1} > 1$, so that

$$q_1 = \frac{1}{2} \text{ and } q_2 = \frac{1}{2} \pm \frac{1}{2|\beta|}. \tag{2.14}$$

For other cases, we get

$$q_1 = \frac{q_2(|\alpha|^2 - |\beta|^2 + 1) - 1}{|\alpha|^2 - |\beta|^2 - 1} \tag{2.15}$$

and

$$q_2 = \frac{q_1(|\alpha|^2 - |\beta|^2 - 1) + 1}{|\alpha|^2 - |\beta|^2 + 1}. \tag{2.16}$$

It is easy to see that

$$q_2 - q_1 = \frac{1 - 2q_2}{|\alpha|^2 - |\beta|^2 - 1}. \tag{2.17}$$

Let

$$\delta_2 = \frac{|\alpha|^2}{(|\alpha|^2 - |\beta|^2 - 1)^2}.$$

Replacing $q_2 - q_1$ in the first equation in (2.12) by (2.17), we obtain

$$(4\delta_2 + 1)q_2^2 - (4\delta_2 + 1)q_2 + \delta_2 = 0.$$

So

$$q_2 = \frac{1}{2} \pm \frac{1}{2\sqrt{4\delta_2 + 1}}.$$

Similarly, we can solve for q_1 such that

$$q_1 = \frac{1}{2} \pm \frac{1}{2\sqrt{4\delta_1 + 1}},$$

where $\delta_1 = \frac{|\beta|^2}{(|\alpha|^2 - |\beta|^2 + 1)^2}$. We proved that (5) in Lemma 2.5 implies that T is reducible.

The case $b = \alpha a + \beta C(a)$ is similar to above argument, and will show that the first condition in (4) or (5) implies that T is reducible. □

Now we can prove the main theorem of this paper.

Theorem 2.1. *Let T be a $C_0(2)$ operator and Θ_T be the characteristic function of T with $\det(\Theta_T) = \varphi \notin Aut(\mathbb{D})$. Let C denote the natural conjugation on $\mathcal{K}^2_{z\varphi}$ defined in (1.1). Then T is reducible if and only if one of the following conditions holds:*

(1) *$a = \alpha u$ and $b = \beta u$ or $b = \beta\varphi\overline{u}$, where u is an inner factor of φ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.*
(2) *There exist $\alpha_i,\ \beta_i,\ (i = 1, 2) \in \mathbb{C},$ with $\prod_{i=1}^{2} \alpha_i\beta_i \neq 0,\ \alpha_i\overline{\beta_i} \notin \mathbb{R}$ such that at least two of $\{\alpha_1 a,\ \alpha_2 b,\ \beta_1 a + \beta_2 b\}$ are fixed points of C.*
(3) *There exist at least two pairs $(\delta_i, \gamma_i) \in \mathbb{C}^2,$ with $\delta_i\gamma_i \neq 0,\ i = 1, 2,\ \delta_1\overline{\delta_2} \notin \mathbb{R},\ \delta_2\gamma_1 - \delta_1\gamma_2 \neq 0,\ \overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1 \neq 0$ and $|\delta_2\gamma_1 - \delta_1\gamma_2| \neq |\overline{\delta_1}\gamma_2 - \overline{\delta_2}\gamma_1|$, such that $\delta_i a + \gamma_i b$ are fixed points of C.*

Proof. Combining the Lemma 2.2, Lemma 2.3 and Lemma 2.5, we proved the theorem. □

Theorem 2.1 shows that characteristic function is important for the reducibility of T. By (1.2), $\Theta_T(z) : \mathfrak{D}_T \to \mathfrak{D}_{T^*}$, by (2.7), T is reducible if and only if Θ_T coincides with a diagonal matrix. Next, we will give a description of reducing subspaces by its defect spaces.

Lemma 2.6. *Suppose T is a $C_0(N)$ operator on the Hilbert space $\mathcal{H}$. The following statements hold.*

(1) $\overline{span}\{T^n\mathfrak{D}_{T^*}: \; n=0,1,2,\ldots\}=\mathcal{H}$;
(2) $\overline{span}\{T^{*n}\mathfrak{D}_{T}: \; n=0,1,2,\ldots\}=\mathcal{H}$,

where $\overline{span}$ denotes the closed linear span.

Proof. It suffices to prove (1). Assume that $x\perp \overline{span}\{T^n\mathfrak{D}_{T^*}: n=0,1,2,\ldots\}$, then

$$0=(x,T^n\mathfrak{D}_{T^*})=(T^{*n}x,\mathfrak{D}_{T^*}),\;\; for\; n=0,1,2,\ldots,$$

it follows that $T^{*n}x\in\mathfrak{D}_{T^*}^{\perp}$ for all $n\geq 0$. Since $\mathfrak{D}_{T^*}^{\perp}=\ker D_{T^*}$ and

$$\ker\mathfrak{D}_{T^*}=\{x\in\mathcal{H}: \;\|T^*x\|=\|x\|\},$$

we have that $\|T^{*n}x\|=\|x\|$, for $n=0,1,2,\ldots$. Since T is a $C_0(N)$ operator, $T^{*n}\to 0$ (in strong operator topology) as $n\to\infty$, hence $x=0$ and this completes the proof. □

Theorem 2.2. *Suppose that T is a $C_0(2)$ operator on the Hilbert space $\mathcal{H}$ with characteristic function*

$$\Theta_T(z)=\begin{pmatrix}\theta_1(z) & 0\\ 0 & \theta_2(z)\end{pmatrix}, z\in\mathbb{D}, \tag{2.18}$$

where θ_1 and θ_2 are inner functions.

(1) *If θ_1 and θ_2 coincide, then T has infinitely many reducing spaces.*
(2) *If θ_1 and θ_2 don't coincide, then T has only two reducing spaces.*

Proof. It follows from (2.18) that T is unitarily equivalent to

$$S=S_1\oplus S_2$$

on $\mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2$, where the characteristic function of S_i is θ_i, and $S_i=S\mid_{\mathcal{K}_i}, i=1,2$.

Since T is a $C_0(2)$ operator, then S_1 and S_2 are $C_0(1)$ operators. Choosing $x\in\mathfrak{D}_{S_1}$, $y\in\mathfrak{D}_{S_2}$ and x,y are unit vectors with $x\perp y$, by Lemma 2.6, we get

$$\mathcal{K}_1=\overline{span}\{S^{*n}x: n=0,1,2,\ldots\}=\overline{span}\{S_1^{*n}x: n=0,1,2,\ldots\}$$

and

$$\mathcal{K}_2=\overline{span}\{S^{*n}y: n=0,1,2,\ldots\}=\overline{span}\{S_2^{*n}y: n=0,1,2,\ldots\}.$$

It is clear that $\mathcal{K}_1$ and $\mathcal{K}_2$ are reducing subspaces of S, hence it suffices to show that θ_1 and θ_2 coincide if and only if S has infinitely many reducing spaces.

For any $(\alpha_i,\beta_i)\neq(0,0), i=1,2$ such that

$$\alpha_1\overline{\alpha_2}+\beta_1\overline{\beta_2}=0, \tag{2.19}$$

let

$$M_i = \overline{span}\{S^{*n}(\alpha_i x + \beta_i y) : n = 0, 1, 2, \ldots\}, i = 1, 2.$$

It is clear that $S^* M_i \subset M_i, i = 1, 2$. It follows from

$$S^{*n}x = \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1} S^{*n}(\alpha_1 x + \beta_1 y) + \frac{\beta_1}{\alpha_2\beta_1 - \alpha_1\beta_2} S^{*n}(\alpha_2 x + \beta_2 y)$$

and

$$S^{*n}y = \frac{\alpha_2}{\alpha_2\beta_1 - \alpha_1\beta_2} S^{*n}(\alpha_1 x + \beta_1 y) + \frac{\alpha_1}{\alpha_1\beta_2 - \alpha_2\beta_1} S^{*n}(\alpha_2 x + \beta_2 y)$$

that $\overline{span}\{M_1, M_2\} = \mathcal{K}$. $M_i (i = 1, 2)$ are reducing subspaces if and only if $M_1 \perp M_2$. It concludes that S has infinitely many reducing spaces if and only if $M_1 \perp M_2$.

Claim: $M_1 \perp M_2$ if and only if S_1 is unitarily equivalent to S_2, and in this case, θ_1 and θ_2 coincide.

Proof of the claim. Firstly, suppose S_1 is unitarily equivalent to S_2, i.e., there exists a unitary map $U : \mathcal{K}_1 \to \mathcal{K}_2$, such that $US_1 = S_2 U$. It is not hard to see $UD^2_{S_1} = D^2_{S_2} U$ and hence $UD^{2n}_{S_1} = D^{2n}_{S_2} U$. Let $p_n(x)$ be the polynomial and $p_n(x) \to \sqrt{x}$ uniformly on the interval $0 \le x \le 1$, then $p_n(D^2_{S_i}) \to D_{S_i},\ i = 1, 2$ in norm and $Up_n(D^2_{S_1}) = p_n(D^2_{S_2})U$. Hence $UD_{S_1} = D_{S_2} U$ and $U\mathfrak{D}_{S_1} = \mathfrak{D}_{S_2}$. We can choose $x \in \mathfrak{D}_{S_1}$, $y \in \mathfrak{D}_{S_2}$ such that $Ux = y$, then

$$\begin{aligned}(S^{*n}y, S^{*m}y) &= (S_2^{*n}y, S_2^{*m}y) \\ &= (S_2^{*n}Ux, S_2^{*m}Ux) \\ &= (S_1^{*n}x, S_1^{*m}x).\end{aligned} \tag{2.20}$$

For $n, m = 0, 1, 2, \ldots$, it follows from (2.19) and (2.20) that

$$\begin{aligned}(S^{*n}(\alpha_1 x + \beta_1 y), S^{*m}(\alpha_2 x + \beta_2 y)) &= \alpha_1\overline{\alpha_2}(S^{*n}x, S^{*m}x) + \alpha_1\overline{\beta_2}(S^{*n}x, S^{*m}y) \\ &\quad + \beta_1\overline{\alpha_2}(S^{*n}y, S^{*m}x) + \beta_1\overline{\beta_2}(S^{*n}y, S^{*m}y) \\ &= \alpha_1\overline{\alpha_2}(S^{*n}x, S^{*m}x) + \beta_1\overline{\beta_2}(S^{*n}y, S^{*m}y) \\ &= 0,\end{aligned}$$

therefore, $M_1 \perp M_2$.

Conversely, suppose that $M_1 \perp M_2$. By above equation, we have

$$\alpha_1\overline{\alpha_2}(S^{*n}x, S^{*m}x) + \beta_1\overline{\beta_2}(S^{*n}y, S^{*m}y) = 0, \tag{2.21}$$

for $n, m = 0, 1, 2, \ldots$. Let $n = m = 0$ in (2.21), we get $\alpha_1\overline{\alpha_2} + \beta_1\overline{\beta_2} = 0$, this yields that

$$(S^{*n}x, S^{*m}x) = (S^{*n}y, S^{*m}y) \tag{2.22}$$

for $n, m = 0, 1, 2, \ldots$. Let $VS_1^{*n}x = S_2^{*n}y$. By (2.22),

$$(VS_1^{*n}x, VS_1^{*m}x) = (S_2^{*n}y, S_2^{*m}y) = (S^{*n}x, S^{*m}x),$$

which shows that V is isometric on a dense linear manifold $\{S^{*n}x : n = 0, 1, 2, \ldots\}$ of $\mathcal{K}_1$, so V can be extended to be an isometry from $\mathcal{K}_1$ to $\mathcal{K}_2$. From $\mathcal{K}_2 = \overline{span}\{S^{*n}y : n = 0, 1, 2, \ldots\}$, we know that V is surjective, hence V is unitary. It is clear that

$$VS_1^* S_1^{*n} x = S_2^{*n+1} y = S_2^* V S_1^{*n} x, \quad \forall n = 0, 1, 2, \dots,$$

this shows that $VS_1^* = S_2^* V$ and therefore $VS_1 = S_2 V$. We obtain that S_1 is unitarily equivalent to S_2 and this completes the proof. □

3. Some examples

Note that A_{z^2} is an $C_0(2)$ operator, in this section, we will apply the Theorem 2.1 and Theorem 2.2 to restudy the reducibility of A_{z^2}. Let u be an inner function such that $u \notin \mathrm{Aut}(\mathbb{D})$. In [2], it is shown that the characteristic function of A_{z^2} coincides with

$$\Theta(z) = \frac{1}{2} \begin{pmatrix} d(z) & ze(z) \\ e(z) & d(z) \end{pmatrix} z \in \mathbb{D}, \tag{3.1}$$

where $d(z) = u(\sqrt{z}) + u(-\sqrt{z})$ and

$$e(z) = \begin{cases} \frac{u(\sqrt{z}) - u(-\sqrt{z})}{\sqrt{z}}, & \text{if } \ z \neq 0, \\ 2u'(0), & \text{if } \ z = 0. \end{cases}$$

Let $\varphi(z) = \det \Theta(z) = u(\sqrt{z})u(-\sqrt{z})$, $a(z) = \frac{1}{2}d(z)$ and $b(z) = -\frac{1}{2}ze(z)$. It is clear that a is a fixed point of conjugation operator C on $\mathcal{K}^2_{z\varphi}$ and b is a fixed point if and only if $e \equiv 0$. Using Theorem 2.1, we will study the reducibility of A_{z^2}.

Theorem 3.1 *(Theorem 1 in [2]). A_{z^2} is reducible on $\mathcal{K}^2_u$ if and only if either*

$$u(z) \equiv u(-z), z \in \mathbb{D} \tag{3.2}$$

or there exists $\mu \in \mathbb{D}$ such that

$$u(z) \equiv p(z)\frac{z + \mu}{1 + \overline{\mu}z}, z \in \mathbb{D}, \tag{3.3}$$

where $p \in H^\infty$ satisfies

$$p(z) \equiv p(-z), z \in \mathbb{D}. \tag{3.4}$$

Proof. If A_{z^2} is reducible, then one of the cases in the Theorem 2.1 holds.

Case 1. $a = \alpha\psi$ and $b = \beta\psi$ with $|\alpha|^2 + |\beta|^2 = 1$, where ψ is an inner function, then

(1) if $\alpha = 0$, then u is odd;
(2) if $\beta = 0$, then u is even.

For $\alpha\beta \neq 0$, then $\beta a = \alpha b$. Replacing a and b, we have

$$(\alpha\sqrt{z} + \beta)u(\sqrt{z}) = (\alpha\sqrt{z} - \beta)u(-\sqrt{z}).$$

Therefore $|\alpha\xi + \beta| = |\alpha\xi - \beta|$ for almost every $\xi \in \mathbb{T}$ and hence for all $\xi \in \mathbb{T}$. Since

$$|\alpha\xi + \beta|^2 = |\alpha|^2 + |\beta|^2 + 2Re(\xi\alpha\overline{\beta})$$

and

$$|\alpha\xi - \beta|^2 = |\alpha|^2 + |\beta|^2 - 2Re(\xi\alpha\overline{\beta}),$$

we obtain that

$$Re(\xi\alpha\overline{\beta}) = 0 \ \text{ for all } \xi \in \mathbb{T}.$$

Let $\alpha = |\alpha|e^{i\theta_\alpha},\ \beta = |\beta|e^{i\theta_\beta}$ and $\xi = e^{i\theta_\xi}$, then

$$Re(\xi\alpha\overline{\beta}) = |\alpha\beta|\cos(\theta_\alpha + \theta_\xi - \theta_\beta) = 0 \ \text{ for all } \xi \in \mathbb{T}.$$

This is a contradiction and hence $\alpha = 0$ or $\beta = 0$, which means that either u is odd or u is even.

Case 2. $a = \alpha\psi$ and $b = \beta\varphi\overline{\psi}$, this case is similar to Case 1.

Case 3. By Theorem 2.1, one of the following cases holds.

(1) $\alpha_1 a$ and $\alpha_2 b$ are fixed points of C;
(2) $\alpha_1 a$ and $\beta_1 a + \beta_2 b$ are fixed points of C;
(3) $\alpha_2 b$ and $\beta_1 a + \beta_2 b$ are fixed points of C.

It is clear that $C(b) = \alpha b$ if and only if $b \equiv 0$, and $C(a) = a$, so if (1) or (3) holds, u is even.

Suppose (2) holds, that is $\beta_1 a+\beta_2 b$ is a fixed point of C. Since a is also a fixed point, by some calculations, we have

$$(\beta_2 z - (\beta_1 - \overline{\beta_1})\sqrt{z} + \overline{\beta_2})u(\sqrt{z}) = (\beta_2 z + (\beta_1 - \overline{\beta_1})\sqrt{z} + \overline{\beta_2})u(-\sqrt{z}).$$

The following analysis is taken from [2], however, for the reader's convince, we put it here.

If $\beta_1 \in \mathbb{R}$, then $C(\beta_1 a + \beta_2 b) = \beta_1 a + C(\beta_2 b)$, this implies $C(\beta_2 b) = \beta_2 b$, by above argument we have $b \equiv 0$. Then u is even. If $b \not\equiv 0$, then $Im\beta_1 \neq 0$, let $x = 2Im\beta_1$ and $\beta_2 = |\beta_2|\xi$, where $|\xi| = 1$. Let

$$n(z) = (\beta_2 z^2 - (\beta_1 - \overline{\beta_1})z + \overline{\beta_2})u(z),$$

then $n(z) = n(-z)$. By some calculations, we obtain

$$\beta_2 z^2 - (\beta_1 - \overline{\beta_1})z + \overline{\beta_2} = \beta_2(z - i\overline{\xi}\delta_-)(z - i\overline{\xi}\delta_+),$$

where

$$\delta_\pm = \frac{1 \pm \sqrt{1+\rho^2}}{\rho}, \ \text{ where } \rho = \frac{2|\beta_2|}{x}.$$

It is clear that $\delta_+\delta_- = -1$ and $|\delta_-| < 1$. Let $\mu = i\overline{\xi}\delta_- \in \mathbb{D}$, then $z - i\overline{\xi}\delta_+ = -i\overline{\xi}\delta_+(1 + \overline{\mu}z)$, we have

$$n(z) = -i|\beta_2|\delta_+(z - \mu)(1 + \overline{\mu}z)u(z).$$

Since $n(z) = n(-z)$, we obtain $u(-\mu) = 0$, then

$$u(z) = p(z)\frac{z + \mu}{1 + \overline{\mu}z}.$$

Using $n(z) = n(-z)$ again, we have $p(z) = p(-z)$. See [2] for more details.

To show that A_{z^2} is reducible, we only need to show that one of the conditions in Theorem 3.1 implies that one of the conditions in Theorem 2.1 holds.

If u satisfies (3.2), then

$$u(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}.$$

Let $\psi(z) = \sum_{n=0}^{\infty} a_{2n} z^n$, then $u(z) = \psi(z^2)$ and ψ is inner. By (3.1), $a(z) = \psi(z)$, $b(z) \equiv 0$ and $\det \Theta_{A_{z^2}} = \psi^2$, (1) in Theorem 2.1 holds.

If u satisfies (3.3), then $u(z) = \psi(z^2)\frac{z+\mu}{1+\overline{\mu}z}$. By (3.1), $a(z) = \psi(z)\frac{\mu-\overline{\mu}z}{1-\overline{\mu}^2 z}$, $b(z) = -\psi(z)\frac{(1-|\mu|^2)z}{1-\overline{\mu}^2 z}$ and $\det \Theta_{A_{z^2}} = \psi^2 \frac{\mu^2 - z}{1-\overline{u}^2 z}$.

If $\mu = 0$, then (1) in Theorem 2.1 holds.

If $\mu \neq 0$. Let $\varphi = \det \Theta_{A_{z^2}}$, C be the conjugation on $\mathcal{K}^2_{z\varphi}$ defined in (1.1). It is not hard to check that a and $\alpha a + \beta b$ are fixed points of C, where $\alpha \in \mathbb{C}, Im\alpha \neq 0$ and $\beta = \frac{(\overline{\alpha}-\alpha)\overline{\mu}}{1-|\mu|^2}$. Then (2) in Theorem 2.1 holds. Then we finish the proof. $\square$

Now we can determine the number of reducing subspaces of A_{z^2} by using the Theorem 2.2. If u satisfies (3.2), the proof in Theorem 3.1 implies that the characteristic function $\Theta_{A_{z^2}}$ coincides with

$$\Theta(z) = \begin{pmatrix} a(z) & 0 \\ 0 & a(z) \end{pmatrix}, z \in \mathbb{D}.$$

By the Theorem 2.2, A_{z^2} has infinitely many reducing subspaces. The same result can also be found in [10], where the authors used different method to show it. If u satisfies (3.3), the characteristic function $\Theta_{A_{z^2}}$ coincides with

$$\Theta(z) = \begin{pmatrix} \frac{q}{1-2q}\alpha b + \frac{1-q}{1-2q}\overline{\alpha}\varphi\overline{b} & 0 \\ 0 & \frac{1-q}{1-2q}\alpha b + \frac{q}{1-2q}\overline{\alpha}\varphi\overline{b} \end{pmatrix} z \in \mathbb{D},$$

where $0 < q < 1$ and $q \neq \frac{1}{2}$, $|\alpha| = 1$, $b = -\frac{1}{2}ze$ and $\varphi = \det(\Theta)$ in (3.1). In this case, $\frac{q}{1-2q}\alpha b + \frac{1-q}{1-2q}\overline{\alpha}\varphi\overline{b}$ does not coincide with $\frac{1-q}{1-2q}\alpha b + \frac{q}{1-2q}\overline{\alpha}\varphi\overline{b}$, so A_{z^2} has only two reducing subspaces.

In the following, we give examples of $C_0(2)$ operator which are not unitarily equivalent to A_{z^2}. Let $\mathcal{E}$ be a Hilbert space, the $\mathcal{E}$-valued Hardy space denotes by $H^2(\mathcal{E})$ defined by

$$H^2(\mathcal{E}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : a_n \in \mathcal{E}, z \in \mathbb{D}, \|f\|^2 = \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty\}.$$

Suppose $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. Let

$$\Theta(z) = \begin{pmatrix} \alpha z & -\beta z \\ \overline{\beta} z^3 & \overline{\alpha} z^3 \end{pmatrix}. \tag{3.5}$$

By Lemma 2.1, Θ defined in (3.5) is a characteristic function of a $C_0(2)$ operator.

Example 3.1. Suppose Θ is defined by (3.5). Let $\mathcal{K}_\Theta = H^2(\mathbb{C}^2) \ominus \Theta H^2(\mathbb{C}^2)$, the compressed shift operator S_z on $\mathcal{K}_\Theta$ is defined by

$$S_z f = P_\Theta T_z f, f \in \mathcal{K}_\Theta,$$

where P_Θ denotes the orthogonal projection onto $\mathcal{K}_\Theta$. Then S_z is not unitarily equivalent to A_{z^2} on $\mathcal{K}_u^2$ for any inner function u.

Proof. Since Θ is a pure contraction, the characteristic function of S_z is $\Theta_{S_z} = \Theta$ (see [11]). It has been shown [2] that $\Theta_{A_{z^2}}$ coincides with characteristic function defined in (3.1). Assume S_z is unitarily equivalent to A_{z^2}, then Θ_{S_z} and $\Theta_{A_{z^2}}$ coincide. Therefore, $\xi \det \Theta_{A_{z^2}} = \det \Theta_{S_z} = z^4$, where $|\xi| = 1$. By calculation, we see that $\det \Theta_{A_{z^2}} = u(\sqrt{z})u(-\sqrt{z}) = \overline{\xi} z^4$, which implies that $u(z) = \overline{\xi}^{1/2} z^4$. In this case, by (3.1), $\Theta_{A_{z^2}}$ coincides with

$$\Theta_1(z) = \begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix}. \tag{3.6}$$

It is also not hard to see that Θ_1 and Θ don't coincide. This is a contradiction and we complete the proof. $\square$

By Theorem 2.1, S_z in Example 3.1 is reducible, and it follows from Theorem 2.2 that S_z has only two reducing subspaces. The following example gives a $C_0(2)$ operator with infinitely many reducing subspaces.

Example 3.2. For $a \in \mathbb{D}$ and $a \neq 0$, let $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ and $u = (\varphi_a)^2$, then by Theorem 3.1, A_{z^2} is irreducible on $\mathcal{K}_u^2$ and $\det \Theta_{A_{z^2}} = (\varphi_{a^2})^2$. Let

$$\Theta(z) = \begin{pmatrix} \varphi_{a^2}(z) & 0 \\ 0 & \varphi_{a^2}(z) \end{pmatrix}, z \in \mathbb{D}. \tag{3.7}$$

Using Θ in (3.7) to replace the characteristic function in Example 3.1, by Theorem 2.1, S_z is reducible, and we also see that S_z has infinitely many reducing subspaces by Theorem 2.2. Similar argument as in Example 3.1 shows that S_z is not unitarily equivalent to A_{z^2} for any inner function u.

The following example gives an irreducible $C_0(2)$ operator.

Example 3.3. Let $\varphi(z) = z^2$, $a = \frac{1}{2} + \frac{1}{2}z$ and $b = -\frac{1}{2} + \frac{1}{2}z$. It is easy to check that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. Let C denote the conjugation on $\mathcal{K}_{z\varphi}^2$ defined in (1.1), then we have $C(a) = \frac{1}{2}z^2 + \frac{1}{2}z$ and $C(b) = -\frac{1}{2}z^2 + \frac{1}{2}z$. Let

$$\Theta(z) = \begin{pmatrix} a & -b \\ \varphi\overline{b} & \varphi\overline{a} \end{pmatrix}. \tag{3.8}$$

Using Θ in (3.8) replace the characteristic function in Example 3.1. It is easy to check that none of conditions in Theorem 2.1 holds, then S_z is irreducible. Since $\det \Theta = \varphi = z^2$, similar argument in Example 3.1 shows that $u = z^2$. By Theorem 3.1, A_{z^2} is reducible on $\mathcal{K}_u^2$, thus S_z is not unitarily equivalent to A_{z^2} for any inner function u.

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