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Microlocal inversion of a 3-dimensional restricted transverse ray transform on symmetric tensor fields

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ABSTRACT

We study the problem of inverting a restricted transverse ray transform on symmetric tensor fields in $\mathbb{R}^3$ using microlocal analysis techniques. We show that a symmetric m-tensor field can be recovered up to a known singular term and a smoothing term if its transverse ray transform is known along all lines intersecting a fixed smooth curve satisfying certain geometric conditions.

1. Introduction

The study of transverse ray transforms (TRT) of symmetric tensor fields is of interest in problems arising in polarization and diffraction tomography. We are interested in an approximate inversion of a TRT acting on symmetric tensor fields restricted to all lines passing through a fixed curve in $\mathbb{R}^3$. More precisely, we use techniques from microlocal analysis to construct a relative left parametrix for such restricted TRT.

We denote the space of covariant symmetric m-tensors in $\mathbb{R}^3$ by $S^m = S^m(\mathbb{R}^3)$. Let $C_c^\infty(S^m)$ be the space of smooth compactly supported symmetric m-tensor fields in $\mathbb{R}^3$. In $\mathbb{R}^3$, an element $f \in C_c^\infty(S^m)$ can be written as

$$f(x) = f_{i_1\cdots i_m}(x)\mathrm{d}x^{i_1}\cdots \mathrm{d}x^{i_m},$$

with $\{f_{i_1\cdots i_m}(x)\}$ symmetric in its indices, smooth and compactly supported. With repeating indices, Einstein summation convention will be assumed throughout this paper.

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Let $T\mathbb{S}^2 = \{(\omega, x) \in \mathbb{S}^2 \times \mathbb{R}^3 : \omega \cdot x = 0\}$ be the tangent bundle of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and let

$$T\mathbb{S}^2 \oplus T\mathbb{S}^2 = \{(\omega, x, y) \in \mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^3 : \omega \cdot x = 0, \omega \cdot y = 0\}$$

be the Whitney sum.

Definition 1.1 *([28]). The transverse ray transform $\mathcal{T} : C_c^\infty(S^m) \to C^\infty(T\mathbb{S}^2 \oplus T\mathbb{S}^2)$ is the bounded linear map defined by*

$$\mathcal{T}f(\omega, x, y) = \int_{\mathbb{R}} \langle f(x + t\omega), y^{\odot m} \rangle \mathrm{d}t,$$

where $y^{\odot m}$ denotes the mth symmetric tensor product of y and $\langle f(x), y^{\odot m}\rangle$ is defined by $f_{i_1 \cdots i_m}(x) y^{i_1} \cdots y^{i_m}$.

We will find it more convenient to work with an equivalent vectorial version of TRT which we define below. Let $\omega \in \mathbb{S}^2$ be represented in spherical coordinates by

$$\omega = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2)$$

where $0 \le \theta_1 < \pi$ and $0 \le \theta_2 < 2\pi$. Consider the orthonormal frame $\{\omega, \omega_1, \omega_2\}$ with ω_1 and ω_2 defined by

$$\omega_1 = (-\sin\theta_1, \cos\theta_1 \cos\theta_2, \cos\theta_1 \sin\theta_2) \text{ and } \omega_2 = (0, -\sin\theta_2, \cos\theta_2)\,. \tag{1}$$

We define the vectorial version of $\mathcal{T}$ as follows:

Definition 1.2. *For $0 \le i \le m$, define $\mathcal{T} = (\mathcal{T}_i) : C_c^\infty(S^m) \to \left(C^\infty(T\mathbb{S}^2)\right)^{m+1}$ by*

$$\mathcal{T}_i f(x, \omega) = \int_{\mathbb{R}} f_{j_1 j_2 \cdots j_m}(x + t\omega) \omega_1^{j_1} \cdots \omega_1^{j_{m-i}} \omega_2^{j_{m-(i-1)}} \cdots \omega_2^{j_m} \mathrm{d}t. \tag{2}$$

It is straightforward to see that these two definitions are equivalent.

In 2-dimensions, TRT and the standard ray transform [28], also called the longitudinal ray transform (LRT), give equivalent information and it is well-known that the latter transform on symmetric tensor fields has an infinite dimensional kernel. Hence it is not possible to reconstruct the full tensor field f from its transverse ray transform in 2-dimensions. Furthermore, the space of lines in $\mathbb{R}^n$ is $2n-2$ dimensional, and in dimensions $n \ge 3$, the problem of recovery of f from $\mathcal{T}f$ is over-determined. Therefore a natural question is to investigate the inversion of $\mathcal{T}$ restricted to an n-dimensional data set. We address this problem for the case of dimension $n = 3$ in this paper, and the 3-dimensional set of lines we choose is the set of all lines passing through a fixed curve $\boldsymbol{\gamma} \in \mathbb{R}^3$.

The inversion of TRT and of the corresponding non-linear problem appearing in polarization tomography has been considered in several prior works [28,26,29,14,15,24,6,9,19,22]. With respect to the study of restricted TRT, we refer to the works [23,6]. Recently a support theorem for TRT in the setting of analytic simple Riemannian manifolds was considered by [1].

We study the inversion of restricted TRT using microlocal analysis techniques. We are interested in the reconstruction of singularities of the symmetric tensor field f given its restricted TRT. The study of generalized Radon transforms in the framework of Fourier integral operators began with the fundamental work of Guillemin [11] and Guillemin-Sternberg [12]. Since then, microlocal analysis has become a very powerful tool in the study of tomography problems; see [10,8,4,5,31,16,21,27,32,33,30,35,17,2]. Of these works, the paper [8] is a fundamental work where Greenleaf and Uhlmann studied a restricted ray transform on functions in the setting of Riemannian manifolds. However, most of these works are done for LRT and to the best of our knowledge, other than the support theorem result [1], we are not aware of any prior work that studies a restricted TRT from the view point of microlocal analysis.

Specifically, we study the microlocal inversion of the Euclidean TRT on symmetric m-tensor fields given the restricted data set consisting of all lines passing through a fixed curve γ in $\mathbb{R}^3$. The transverse ray transform $\mathcal{T}$ defined in (2) restricted to lines passing through the curve γ will be denoted by $\mathcal{T}_\gamma$ and its formal L^2 adjoint by $\mathcal{T}_\gamma^*$. We determine the extent to which the wavefront set of a symmetric m-tensor field can be recovered from the wavefront set of its restricted TRT. We are motivated by the related works done for restricted LRT [8,20,21,27,18] and we mainly follow the techniques from these works.

The article is organized as follows. §2 is devoted to stating some preliminary results about the restricted TRT, to some fundamental results about distributions associated to two cleanly intersecting Lagrangians introduced in [25,13,8], the microlocal results relevant for the analysis of our transform, and the statement of the main result. We give the proof of the main results in §3 and §4.

2. Preliminaries and statement of the main result

We first state precisely the conditions imposed on the curve γ, and the wavefront set directions that are potentially recoverable based on microlocal analysis of the restricted transverse ray transform.

Let B be a ball in $\mathbb{R}^3$. Let γ be a smooth regular curve without self-intersections in $\mathbb{R}^3$ defined on a bounded interval and with its range in the complement of B. We assume that there are uniform upper and lower bounds on the number of intersection points of almost every hyperplane passing through the set B with the curve γ, and that the lower bound is at least $m+1$ (where m is the order of the tensor field under consideration). This condition on the lower bound is a modified form of so-called Kirillov-Tuy condition.

For our microlocal analysis approach to work, we need to restrict ourselves to certain wavefront set directions that we can potentially recover. The sets defined below (see [8,27,18]) are motivated by this restriction. Given $(x,\xi) \in T^*B \setminus \{0\}$, we denote by $H(x,\xi)$, the plane passing through x and perpendicular to ξ. The points $\{\gamma(t_i)\}$ on the curve γ below refer to the points of intersection of the curve γ and the hyperplane $H(x,\xi)$.

$$\begin{aligned}\Xi =&\Big\{(x,\xi) \in T^*B \setminus \{0\} : \text{there exists at least } m+1 \text{ points } \{\gamma(t_j)\}_{j=1}^{m+1} \text{ such that}\\ &\quad \text{all pairs of vectors from } \{(x-\gamma(t_j))\}_{j=1}^{m+1} \text{ are linearly independent}\Big\}. \qquad (3)\\ \Xi' =& \Big\{(x,\xi) \in \Xi : \text{all the intersection points of } H(x,\xi) \text{ with } \gamma \text{ are transverse}\Big\}.\\ \Xi'' =& \Big\{(x,\xi) \in \Xi : \text{ the tangential intersection points } \{\gamma(t_j))\} \text{ satisfy} \langle \gamma''(t_j), \xi\rangle \neq 0\Big\}.\end{aligned}$$

The potentially recoverable singularities belong to the union $\Xi' \cup \Xi''$ (see the statement of Theorem 2.2 for a more precise description). Therefore, without loss of generality, in all the analysis below we will restrict ourselves to the cotangent directions in this union. Below we give an example of a curve γ satisfying the Kirillov-Tuy condition for vector fields and also discuss the corresponding Ξ, Ξ', and Ξ''.

Example 2.1 *([34,36])*. For $m = 1$ (vector fields), consider the curve γ as the union of three orthogonal great circles on the sphere of radius 2 (the equator and the meridians of 0° and 90°) and center at the origin. Then every plane H intersecting the unit ball B will intersect the curve γ at least two different points γ_1 and γ_2. And for almost every $x \in H$, the vectors $x-\gamma_1$ and $x-\gamma_2$ are linearly independent.

With respect to above example, we have $\Xi = \Xi' = T^*B \setminus \{0\} = B \times \mathbb{R}^n \setminus \{0\}$ and Ξ'' is the empty set. In this case, the potentially recoverable singularities consist of the set Ξ'. Instead of three great circles, if we consider the curve $\boldsymbol{\gamma}$ to be union of two orthogonal great circles of radius 2, then the curve $\boldsymbol{\gamma}$ satisfies the Kirillov-Tuy condition of order one for almost every hyperplane. In that case, we have to exclude a set of measure zero from $\Xi = \Xi' = T^*B \setminus \{0\} = B \times \mathbb{R}^n \setminus \{0\}$ and Ξ'' is the same as above.

Next we state some preliminary microlocal results concerning the operators $\mathcal{T}_{\boldsymbol{\gamma}}$ and $\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}}$. The proofs of these statements follow by suitable adaptations of the arguments given in [20,18] and therefore we skip them.

Let us denote by $\mathcal{C}$, the line complex consisting of all lines passing through the curve $\boldsymbol{\gamma}$. Let ℓ be a line in $\mathcal{C}$ and

$$Z = \{(\ell, x) : x \in \ell\} \subset \mathcal{C} \times \mathbb{R}^3$$

be the point-line relation. For given t (in the domain of $\boldsymbol{\gamma}$) and $\omega = (\theta_1, \theta_2) \in \mathbb{S}^2$, we can define a unique line $\ell \in \mathcal{C}$ by $\ell = \{\boldsymbol{\gamma}(t) + s\omega : s \in \mathbb{R}\}$. Therefore, we have that (t, ω, s) is a local parametrization of Z. The conormal bundle of Z is given by

$$N^*Z = \left\{(\ell, x; \Gamma, \xi) : (\ell, x) \in Z \text{ and } (\Gamma, \xi)|_{T_{(\ell,x)}Z} = 0\right\}.$$

It has been shown in [20,18] that N^*Z can be parametrized by $\{(t, \omega, s, \Gamma, \xi)\}$ where

$$\xi = z_1\omega_1 + z_2\omega_2 \text{ for some } z_1 \text{ and } z_2 \in \mathbb{R}, \tag{4}$$

and ω_1, ω_2 are given by (1), and

$$\Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} = \begin{pmatrix} -\xi \cdot \boldsymbol{\gamma}'(t) \\ -sz_1 \\ -sz_2 \sin\theta_1 \end{pmatrix}. \tag{5}$$

Lemma 2.1. *The map*

$$\Phi : (t, \theta_1, \theta_2, s, z_1, z_2) \to (t, \theta_1, \theta_2, \Gamma; x, \xi)$$

with Γ as in (5), ξ as in (4) and $x = \boldsymbol{\gamma}(t) + s\omega$ gives a local parametrization of N^*Z at the points where $\theta_1 \neq 0, \pi$.

Proposition 2.1. *Each component of the operator $\mathcal{T}_{\boldsymbol{\gamma}}$ is a Fourier integral operator of order $-1/2$ with the associated canonical relation C given by $(N^*Z)'$ where $Z = \{(\ell, x) : x \in \ell\}$. The left and the right projections π_L and π_R from C drop rank simply by 1 on the set*

$$\Sigma := \{(t, \theta_1, \theta_2, s, z_1, z_2) : \boldsymbol{\gamma}'(t) \cdot \xi = 0\}, \tag{6}$$

where ξ is given by (4). The left projection π_L has a blowdown singularity along Σ and the right projection π_R has a fold singularity along Σ.

We refer the reader to [7] for the definitions of fold and blowdown singularities.

Lemma 2.2. *The wavefront set of the Schwartz kernel of $\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}}$ satisfies*

$$WF(\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}}) \subset \Delta \cup \Lambda,$$

where Δ and Λ are defined as follows:

$$\Delta = \left\{(x,\xi;x,\xi) : x = \boldsymbol{\gamma}(t) + s\theta, \xi \in \theta^\perp \setminus \{0\}\right\} \ \textit{and} \tag{7}$$

$$\Lambda = \left\{\left(x,\xi,y,\frac{\tau}{\tilde{\tau}}\xi\right) : x = \boldsymbol{\gamma}(t) + \tau\theta, y = \boldsymbol{\gamma}(t) + \tilde{\tau}\theta, \xi \in \theta^\perp \setminus \{0\}, \boldsymbol{\gamma}'(t)\cdot\xi = 0, \tau \neq 0 \neq \tilde{\tau}\right\}. \tag{8}$$

The condition imposed on the curve in the definition of Ξ'' entails the clean intersection of the sets Δ and Λ. We have

$$\Delta \cap \Lambda = \{(x,\xi;x,\xi) : x = \boldsymbol{\gamma}(t) + s\theta, \xi \in \theta^\perp \setminus \{0\}, \boldsymbol{\gamma}'(t)\cdot\xi = 0\}.$$

$\Delta \cap \Lambda$ is a smooth submanifold of codimension $k = 1$ in both Δ or Λ.

Lemma 2.3. *[20] The Lagrangian Λ defined in (8) arises as a flowout from the set $\pi_R(\Sigma)$.*

2.1. Paired Lagrangian distributions

We will analyze the operators $\mathcal{T}_\gamma$ and $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ in the framework of $I^{p,l}$ classes of distributions. We refer the reader to the three seminal works on this subject [25,13,8]. For the convenience of the reader, we give a quick summary of the properties of the $I^{p,l}$ class of distributions [13] that we require in this paper.

Let $u \in I^{p,l}(\Delta,\Lambda)$, where Δ and Λ are two cleanly intersecting Lagrangians with intersection $\Sigma = \Delta \cap \Lambda$. As an example, the reader may take Δ and Λ from (7) and (8).

Then

1. $WF(u) \subset \Delta \cup \Lambda$.
2. Microlocally, the Schwartz kernel of u equals the Schwartz kernel of a pseudodifferential operator of order $p + l$ on $\Delta \setminus \Lambda$ and that of a classical Fourier integral operator of order p on $\Lambda \setminus \Delta$.
3. $I^{p,l} \subset I^{p',l'}$ if $p \le p'$ and $l \le l'$.
4. $\cap_l I^{p,l}(\Delta,\Lambda) \subset I^p(\Lambda)$.
5. $\cap_p I^{p,l}(\Delta,\Lambda) \subset$ The class of smoothing operators.
6. The principal symbol $\sigma_0(u)$ on $\Delta \setminus \Sigma$ has the singularity on Σ as a conormal distribution of order $l - \frac{k}{2}$, where k is the codimension of Σ as a submanifold of Δ or Λ.
7. If the principal symbol $\sigma_0(u) = 0$ on $\Delta \setminus \Sigma$, then $u \in I^{p,l-1}(\Delta,\Lambda) + I^{p-1,l}(\Delta,\Lambda)$.
8. u is said to be elliptic if the principal symbol $\sigma_0(u) \neq 0$ on $\Delta \setminus \Sigma$ if $k \ge 2$, and for $k = 1$, if $\sigma_0(u) \neq 0$ on each connected component of $\Delta \setminus \Sigma$.

The Lagrangian Λ defined in (8) arises as a flowout, and the main tool in the construction of a relative left parametrix for our operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ is the following composition calculus due to Antoniano and Uhlmann [3]:

Theorem 2.1 *([3]). If $A \in I^{p,l}(\Delta,\Lambda)$ and $B \in I^{p',l'}(\Delta,\Lambda)$, then composition of A and B, $A \circ B \in I^{p+p'+\frac{k}{2},l+l'-\frac{k}{2}}(\Delta,\Lambda)$ and the principal symbol, $\sigma_0(A \circ B) = \sigma_0(A)\sigma_0(B)$, where, k is the codimension of Σ as a submanifold of either Δ or Λ.*

Let B be the ball that appears in the definition of the set Ξ (see (3)) above. Let $K \subset \Xi'$ be a closed conic subset. The space of compactly supported distributions in B whose wavefront set is contained in K will be denoted by $\mathcal{E}'_K(B)$. We now state the main result.

Theorem 2.2. *Let $\Xi_0 \subseteq \Xi'$ be such that $\overline{\Xi}_0 \subseteq \Xi' \cup \Xi''$ and K be a closed conic subset of Ξ_0. There exists an operator $\mathcal{B} \in I^{0,1}(\Delta,\Lambda)$ and an operator $\mathcal{A} \in I^{-1/2}(\Lambda)$ such that for any symmetric m-tensor field f with coordinates in $\mathcal{E}'_K(B)$,*

$$\mathcal{B}\mathcal{T}_\gamma^*\mathcal{T}_\gamma f = f + \mathcal{A}f + \textit{smoothing terms.}$$

Remark 2.1. The condition $\Xi_0 \subseteq \Xi'$ implies that the points $(x,\xi,x,\xi) \in \Delta \setminus \Sigma$ and also gives the ellipticity of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$; see (9). Furthermore, the condition $\overline{\Xi}_0 \subseteq \Xi' \cup \Xi''$ ensures the applicability of the functional calculus from [3] (see the statement of Theorem 2.1 above).

The proof of this result is based on a suitable adaptation of the techniques from [8,21,27,18] to the TRT setting. To this end, we compute the principal symbol of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ on the diagonal Δ away from the set Σ and use this to construct a relative left parametrix for this operator. Our inversion procedure introduces an additional error term (in addition to smoothing terms) because we are working with a restricted transverse ray transform. This error term is a Fourier integral operator associated to the known Lagrangian Λ; see (8).

3. Principal symbol of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$

In this section, we give the principal symbol matrix of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ and show that it is elliptic.

The operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ can be written as

$$\mathcal{T}_\gamma^*\mathcal{T}_\gamma = \sum_{i=0}^{m}\left[\mathcal{R}_\gamma^*\left(\omega_1^{j_1}\cdots\omega_1^{j_{m-i}}\omega_2^{j_{m-(i-1)}}\cdots\omega_2^{j_m}\omega_1^{l_1}\cdots\omega_1^{l_{m-i}}\omega_2^{l_{m-(i-1)}}\cdots\omega_2^{l_m}\right)\mathcal{R}_\gamma\right],$$

where $\mathcal{R}_\gamma$ is the restricted scalar ray transform (that is, ray transform of functions) and $\mathcal{R}_\gamma^*$ is its formal L^2 adjoint. The set K below is as in the statement of Theorem 2.2.

Proposition 3.1. *The principal symbol matrix $A_0(x,\xi)$ of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ for $(x,\xi) \in K$ is*

$$A_0(x,\xi) = \sum_j \sum_{i=0}^{m} \frac{2\pi\omega_1^{j_1}(t_j)\cdots\omega_1^{j_{m-i}}(t_j)\omega_2^{j_{m-(i-1)}}\cdots\omega_2^{j_m}(t_j)\omega_1^{l_1}(t_j)\cdots\omega_1^{l_{m-i}}(t_j)\omega_2^{l_{m-(i-1)}}(t_j)\cdots\omega_2^{l_m}(t_j)}{|\xi||(\gamma'(t_j(\xi_0))\cdot\xi_0)||(\gamma(t_j(\xi_0))-x)|}. \tag{9}$$

In (9) above, ξ_0 is the unit vector in the direction of ξ, j varies over the number of intersection points of the plane $H(x,\xi)$ with the given curve γ.

The derivation of this formula is similar to the one in [20,27,18] and therefore we do not give the details here.

Proposition 3.2. *For $(x,\xi) \in K$, the principal symbol matrix $A_0(x,\xi)$ for $\xi \neq 0$ is injective.*

Proof. For $(x,\xi) \in T^*\mathbb{R}^3 \setminus \{0\}$, without loss of generality, we choose a spherical coordinate system such that $\omega(\cdot)$ and $\omega_1(\cdot)$ are parallel to the plane $H(x,\xi)$ and $\omega_2(\cdot)$ is in the direction of ξ.

The plane $H(x,\xi)$ intersects the curve γ in at least $(m+1)$ points, say $t_1,\cdots,t_{m+1},\cdots,t_{j'}$.

Denote the collection of unit vectors in the directions $x-\gamma(t_1),\cdots,x-\gamma(t_{j'})$ by

$$\mathbb{A} = \left\{\omega(t_j) = \frac{x-\gamma_j}{|x-\gamma_j|} : \gamma_j = \gamma(t_j), 1 \le j \le j'\right\} \text{ where } j' \ge m+1.$$

Now any two of the vectors in $\mathbb{A}$ are linearly independent since $(x,\xi) \in \Xi$. This in turn implies that for almost all points x, any two of the vectors in the collection

$$\mathbb{A}' = \left\{\omega_1(t_j) : 1 \le j \le j'\right\}$$

where, recall, $\omega_1(t_j)$ corresponding to $\omega(t_j)$ defined in (1), is also linearly independent.

Denote the matrix $U_q = U_{\underbrace{1\cdots 1}_{q}\underbrace{2\cdots 2}_{m-q}}$, for $0 \le q \le m$, whose columns are

$$\left(\frac{2\pi}{|\xi||(\gamma'(t_j(\xi_0))\cdot\xi_0)||(\gamma(t_j(\xi_0))-x)|}\right)^{1/2}\omega_1(t_j)^{\odot q}\odot\omega_2(t_j)^{\odot m-q} \text{ for } 1 \le j \le j',$$

where $\odot$ denotes the symmetric tensor product. Let us denote the matrix P with column blocks $\{U_i\}$, $0 \le i \le m$:

$$P = \begin{pmatrix} U_m & U_{m-1} & \cdots & U_q & \cdots & U_0 \end{pmatrix}. \tag{10}$$

Note that the number of rows in P is $(m+2)(m+1)/2$.

We have

$$A_0(x,\xi) = PP^t,$$

with P defined in (10). In Lemma 3.4, we show that $\mathrm{Rank}(P) = (m+2)(m+1)/2$. Since P has real entries, $\mathrm{Rank}(PP^t) = \mathrm{Rank}(P)$. Therefore the principal symbol matrix $A_0(x,\xi)$ has full rank on $\Delta\setminus\Sigma$. $\square$

Lemma 3.1. *For $q \ge 1$, consider a collection of $q+1$ pair-wise independent vectors $v_1,\cdots,v_{q+1}$ in $\mathbb{R}^3$. Then the collection of vectors*

$$v_1^{\odot q},\cdots,v_{q+1}^{\odot q}$$

is also linearly independent.

Proof. We can write $v_i = c_{i1}v_1 + c_{i2}v_2$ for $i \ge 3$ and for two non-zero constants c_{i1} and c_{i2}.

Assume

$$\sum_{i=1}^{q+1} d_i v_i^{\odot q} = 0,$$

for some nonzero constants d_i. Then using the above, we have

$$d_1 v_1^{\odot q} + d_2 v_2^{\odot q} + \sum_{i=3}^{q+1} d_i\left(c_{i1}v_1 + c_{i2}v_2\right)^{\odot q} = 0$$

From this, we get,

$$\left(d_1 + \sum_{i=3}^{q+1} d_i c_{i1}^q\right)v_1^{\odot q} + \sum_{i=3}^{q+1}\left(\sum_{r=1}^{q-1}\binom{q}{r} d_i c_{i1}^{q-r}c_{i2}^r\right)v_1^{\odot q-r}\odot v_2^{\odot r} + \left(d_2 + \sum_{i=3}^{q+1} d_i c_{i2}^q\right)v_2^{\odot q} = 0.$$

Since v_1 and v_2 are linearly independent, the collection of tensors $\{v_1^{\odot q-r}\odot v_2^{\odot r} : 0 \le r \le q\}$ is also linearly independent. Thus

$$\left(d_1 + \sum_{i=3}^{q+1} d_i c_{i1}^q\right) = 0, \tag{11}$$

$$\left(d_2+\sum_{i=3}^{q+1} d_i c_{i2}^q\right)=0 \tag{12}$$

and

$$\left(\sum_{i=3}^{q+1}\binom{q}{r} d_i c_{i1}^{q-r} c_{i2}^r\right)=0, \quad \text{for} \quad 1 \leq r \leq q-1. \tag{13}$$

Since c_{i1} and c_{i2} are both non-zero constants for all i, by factoring out $c_{i1}c_{i2}$, the system of equations in (13) can be written as

$$\left(\sum_{i=3}^{q+1} d_i c_{i1}^{q-r-1} c_{i2}^{r-1}\right)=0, \quad \text{for} \quad 1 \leq r \leq q-1.$$

This can be written as a matrix system

$$BY=0,$$

where

$$B=\begin{pmatrix} c_{31}^{q-2} & c_{41}^{q-2} & \cdots & c_{q+1,1}^{q-2} \\ c_{31}^{q-3}c_{32} & c_{41}^{q-3}c_{42} & \cdots & c_{q+1,1}^{q-3}c_{q+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{32}^{q-2} & c_{4,2}^{q-2} & \cdots & c_{q+1,2}^{q-2} \end{pmatrix} \tag{14}$$

and $Y=(d_3,\cdots,d_{q+1})^t$.

Let

$$q_i=\frac{c_{i2}}{c_{i1}} \text{ and } b_i=(c_{i1},c_{i2}) \text{ for } 3 \leq i \leq q+1.$$

Since any two vectors from $\{v_i : 3 \leq i \leq q+1\}$ are linearly independent, we have that any two vectors from the set $\{b_i : 3 \leq i \leq q+1\}$ are also linearly independent. This gives that the ratios q_i's are all distinct.

We are interested in proving that Kernel$(B)=\{0\}$. It is enough to prove that Kernel$(B^t)=\{0\}$. Now

$$B^tX=0,$$

gives

$$\sum_{r=0}^{q-2} c_{i1}^{q-2-r} c_{i2}^r e_r=0,$$

for $3 \leq i \leq q+1$ and $X=(e_0,\cdots,e_{q-2})^t$. This in turn gives

$$\sum_{r=0}^{q-2} q_i^r e_r=0, \quad \text{for} \quad 3 \leq i \leq q+1. \tag{15}$$

We arrive at a Vandermonde matrix and hence $X=0$. This then implies that $Y=0$. Now going back to (11) and (12), we have that $d_1=d_2=0$. $\square$

Lemma 3.2. *The matrix U_q satisfies Rank$(U_q) \geq q+1$.*

Proof. We are interested in computing the principal symbol at points in Ξ. We have at least $m+1$ pairwise linearly independent vectors $\omega(t_1), \cdots, \omega(t_{m+1})$. The corresponding perpendicular vectors $\omega_1(t_1), \cdots, \omega_1(t_{m+1})$ are pairwise linearly independent and are also perpendicular to ξ. Now the collection of vectors $\{\omega_1(t_1)^{\odot q}, \cdots, \omega_1(t_{q+1})^{\odot q}\}$ has rank $q+1$ by Lemma 3.1. Therefore the rank of the matrix whose columns are $\omega_1(t_1)^{\odot q}, \cdots, \omega_1(t_{m+1})^{\odot q}$ is at least $q+1$. Finally, the rank of U_q is at least $q+1$ as well, since $\omega_2(t_k)$'s are in the direction of the nonzero vector ξ. $\square$

Lemma 3.3. *Consider an arbitrary U_s for $0 \leq s \leq m$. Assume that the values of t_k corresponding to $s+1$ linearly independent columns of U_s are $t_{j_1}, \cdots, t_{j_{s+1}}$. Any column among these $s+1$ linearly independent columns cannot be written as a linear combination of the columns of the matrices U_q for $0 \leq q \leq m, q \neq s$ and the remaining s linearly independent columns of the matrix U_s.*

Proof. After reordering, we may assume with loss of generality that $t_{j_i} = t_i$ for $1 \leq i \leq s+1$. Fix one of the linearly independent columns from U_s, say, $\omega_1(t_1)^{\odot s} \odot \xi^{\odot m-s}$ (note that since $\omega(t_1)$ and $\omega_1(t_1)$ are parallel to the plane $H(x,\xi)$, $\omega_2(t_1)$ is in the direction of ξ). Suppose there exists constants c_{qi}'s and d_j's such that

$$\omega_1(t_1)^{\odot s} \odot \xi^{\odot m-s} = \sum_{q=0, q\neq s}^{m} \sum_{i=1}^{j'} c_{qi}\omega_1(t_i)^{\odot q} \odot \xi^{\odot m-q} + \sum_{j=2}^{s+1} d_j \omega_1(t_j)^{\odot s} \odot \xi^{\odot m-s}. \tag{16}$$

We write $\omega_1(t_i) = \sum_{j=1}^{2} a_{ij}\omega_1(t_j)$ for $i \geq 3$ for some constants a_{ij}. Substituting this above, we have,

$$\omega_1(t_1)^{\odot s} \odot \xi^{\odot m-s} = \sum_{q=0, q\neq s}^{m} \left(c_{q1}\omega_1(t_1)^{\odot q} + c_{q2}\omega_1(t_2)^{\odot q} + \sum_{i=3}^{j'} c_{qi} \left(\sum_{j=1}^{2} a_{ij}\omega_1(t_j) \right)^{\odot q} \right) \odot \xi^{\odot m-q} \tag{17}$$

$$+ \sum_{j=3}^{s+1} d_j \left(a_{j1}\right)^s \omega_1(t_1)^{\odot s} \odot \xi^{\odot m-s} + \left(d_2 + \sum_{j=3}^{s+1} d_j \left(a_{j2}\right)^s \right) \omega_1(t_2)^{\odot s} \odot \xi^{\odot m-s} \tag{18}$$

$$+ \sum_{r=1}^{s-1} \sum_{j=3}^{s+1} \widetilde{d}_j \left(a_{j1}\right)^{s-r} \left(a_{j2}\right)^r \omega_1(t_1)^{\odot s-r} \odot \omega_1(t_2)^{\odot r} \odot \xi^{\odot m-s}. \tag{19}$$

In the sum above, (17) is the expansion of the first summand in (16) in terms of $\omega_1(t_1)$ and $\omega_1(t_2)$, (18) contains terms involving the powers of $\omega_1(t_1)^{\odot s}$ and $\omega_1(t_2)^{\odot s}$ when the second summand in (16) is expanded in terms of $\omega_1(t_1)$ and $\omega_1(t_2)$, and (19) consists of the remaining terms from the second summand in (16). Also $\widetilde{d}_j$ are certain new constants involving d_j's and binomial coefficients. This implies, for certain constants $c_{r_1 r_2}$,

$$\sum_{q=0, q\neq s}^{m} \sum_{r_1+r_2=q} c_{r_1 r_2}\omega_1(t_1)^{\odot r_1} \odot \omega_1(t_2)^{\odot r_2} \odot \xi^{\odot m-q}$$
$$+ \left(\sum_{j=3}^{s+1} d_j \left(a_{j1}\right)^s - 1 \right) \omega_1(t_1)^{\odot s} \odot \xi^{\odot m-s} + \left(d_2 + \sum_{j=3}^{s+1} d_j \left(a_{j2}\right)^s \right) \omega_1(t_2)^{\odot s} \odot \xi^{\odot m-s}$$
$$+ \sum_{r=1}^{s-1} \sum_{j=3}^{s+1} \widetilde{d}_j \left(a_{j1}\right)^{s-r} \left(a_{j2}\right)^r \omega_1(t_1)^{\odot s-r} \odot \omega_1(t_2)^{\odot r} \odot \xi^{\odot m-s} = 0.$$

The vectors $\{\omega_1(t_1), \omega_1(t_2), \xi\}$ are linearly independent. Therefore the collection of tensors $\{\omega_1(t_1)^{\odot j_1} \odot \omega_1(t_2)^{\odot j_2} \odot \xi^{\odot j_3} : j_1 + j_2 + j_3 = m\}$ is also linearly independent. Thus

$$c_{r_1 r_2} = 0 \tag{20}$$

$$\sum_{j=3}^{s+1} d_j \left(a_{j1}\right)^s - 1 = 0 \tag{21}$$

$$d_2 + \sum_{j=3}^{s+1} \widetilde{d}_j \left(a_{j2}\right)^s = 0 \tag{22}$$

$$\sum_{j=3}^{s+1} \widetilde{d}_j \left(a_{j1}\right)^{s-r} \left(a_{j2}\right)^r = 0 \quad \text{for } 1 \leq r \leq s-1. \tag{23}$$

Note that the product $a_{j1}a_{j2}$ appears as a factor in (23) and since a_{j1} and a_{j2} are both non-zero, we can cancel it out and after this write (23) as a matrix system:

$$AX = 0,$$

where

$$A = \begin{pmatrix} a_{31}^{s-2} & a_{41}^{s-2} & \cdots & a_{s+1,1}^{s-2} \\ a_{31}^{s-3}a_{32} & a_{41}^{s-3}a_{42} & \cdots & a_{s+1,1}^{s-3}a_{s+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{32}^{s-2} & a_{4,2}^{s-2} & \cdots & a_{s+1,2}^{s-2} \end{pmatrix} \tag{24}$$

and

$$X = (\widetilde{d}_3, \widetilde{d}_4, \cdots, \widetilde{d}_{s+1})^t.$$

Now the argument proceeds exactly as in Lemma 3.1. Therefore we have $\{\widetilde{d}_j = 0, 3 \leq j \leq s+1\}$, and this implies $d_j = 0$ for $3 \leq j \leq s+1$. However, this contradicts (21). $\square$

Lemma 3.4. *The rank of A_0 for $\xi \neq 0$ is $(m+2)(m+1)/2$.*

Proof. From the previous lemma, we have that $\mathrm{Rank}(P) \geq (m+2)(m+1)/2$. Since $A_0 = PP^t$ and $\mathrm{Rank}(P) = \mathrm{Rank}(PP^t)$, we have that $\mathrm{Rank}(A_0) \geq (m+2)(m+1)/2$. However A_0 has exactly $(m+2)(m+1)/2$ rows and columns. Hence $\mathrm{Rank}(A_0) = (m+2)(m+1)/2$. $\square$

Now going back to the proof of Lemma 3.2, we have that $\mathrm{Rank}(U_q)$ is exactly $q+1$ as well.

Remark 3.1. In the general case of fixing a spherical coordinate system independent of the plane $H(x,\xi)$, the arguments would follow similarly as above, except that, one would need to consider linear combinations of the components $\mathcal{T}_i$ of the TRT $\mathcal{T}$ in the proofs above.

4. Microlocal inversion

In this section, we give a relative left parametrix for the operator $\mathcal{T}_\gamma^* \mathcal{T}_\gamma$. This will complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Now that ellipticity of $A_0(x,\xi)$ is shown, the construction of the relative left parametrix follows the arguments of [27,18]. We sketch the proof.

Since $A_0(x,\xi)$ is a symmetric matrix of order $(m+1)(m+2)/2$, we diagonalize $A_0(x,\xi)$ by an orthogonal matrix $\mathcal{O}$ such that

$$A_0(x,\xi) = \mathcal{O}D\mathcal{O}^t,$$

where D is the diagonal matrix consisting eigenvalues of A_0 and $\mathcal{O}$ is an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalues of A_0. Since A_0 has full rank, all diagonal entries in D are non-zero. Let

$$B_0(x,\xi) = \mathcal{O}D^{-1}\mathcal{O}^t$$

where D^{-1} is the inverse of D. We have

$$B_0(x,\xi)A_0(x,\xi) = \mathrm{Id}.$$

Define the matrix b_0 as

$$b_0 = \begin{cases} B_0 \text{ if } (x,\xi) \in \Xi_0, \\ 0 \text{ otherwise}, \end{cases} \tag{25}$$

and $\mathcal{B}_0$ be the operator with symbol matrix $b_0(x,\xi)$. The entries of $B_0(x,\xi)$ belong to the symbol of an $I^{p,l}(\Delta,\Lambda)$ class, since the possible singularities of $\mathcal{O}$ and D^{-1} are only on Σ. Note that away from the intersection Σ, A_0 is a symbol of order -1 and since B_0 is formed by inverting A_0, B_0 is the symbol of a pseudodifferential operator of order 1 away from the intersection. Therefore the operator $\mathcal{B}_0 \in I^{0,1}(\Delta,\Lambda)$.

Now the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma \in I^{-1,0}(\Delta,\Lambda)$, and since the principal symbol of the composition $\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ on Δ away from the intersection $\Delta\cap\Lambda$ is the product of the respective principal symbols by [3], which by construction is the identity on Δ away from $\Delta\cap\Lambda$, we have that $\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma \in I^{-\frac{1}{2},\frac{1}{2}}(\Delta,\Lambda)$ using the composition calculus of Antoniano-Uhlmann; see Theorem 2.1.

Define $T_1 = \mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma - \mathrm{Id}$. By construction the principal symbol of T_1 is 0. Let us recall the symbol calculus for $I^{p,l}\Delta,\Lambda)$ which is given by the following exact sequence [13]:

$$0 \to I^{p,l-1}(\Delta,\Lambda) + I^{p-1,l}(\Delta,\Lambda) \to I^{p,l}(\Delta,\Lambda) \xrightarrow{\sigma_0} S^{p,l}(\Delta,\Sigma) \to 0$$

where $S^{p,l}(\Delta,\Sigma)$ denotes the space of product type symbols, see [18, Definition 2.3]. With the help of this exact sequence, we decompose T_1 as $T_1 = T_{11} + T_{12}$ where $T_{11} \in I^{-\frac{3}{2},\frac{1}{2}}$ and $T_{12} \in I^{-\frac{1}{2},-\frac{1}{2}}$.

Since A_0 has full rank, we can find two matrices t_{11} and t_{12} such that the principal symbol $\sigma_0(T_{1j}) = t_{1j}A_0$ for $j = 1,2$.

Let $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ be the operators having symbol matrices $-t_{11}$ and $-t_{12}$ respectively. For $\mathcal{B}_1 = \mathcal{B}_{11}+\mathcal{B}_{12}$, define $T_2 = (\mathcal{B}_0 + \mathcal{B}_1)\mathcal{T}_\gamma^*\mathcal{T}_\gamma - \mathrm{Id}$. We have

$$\begin{aligned} T_2 &= (\mathcal{B}_0 + \mathcal{B}_1)\mathcal{T}_\gamma^*\mathcal{T}_\gamma - \mathrm{Id} \\ &= \mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma + \mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma + \mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma - \mathrm{Id} \\ &= \underbrace{\mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma + T_{11}}_{K_1} + \underbrace{\mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma + T_{12}}_{K_2}. \end{aligned}$$

In the above expression $K_1 \in I^{-\frac{3}{2},\frac{1}{2}}$ and $K_2 \in I^{-\frac{1}{2},-\frac{1}{2}}$. Also, by construction, $\sigma_0(K_1) = 0$ and $\sigma_0(K_2) = 0$ because $\sigma_0(\mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma) = -\sigma_0(T_{11})$ and $\sigma_0(\mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma) = -\sigma_0(T_{12})$. Therefore we can again use the exact sequence to decompose K_1 and K_2 as follows:

$$\begin{aligned} K_1 &= K_{11} + K_{12}, \quad && \text{with } K_{11} \in I^{-\frac{5}{2},\frac{1}{2}}, K_{12} \in I^{-\frac{3}{2},-\frac{1}{2}} \\ K_2 &= K_{21} + K_{22}, \quad && \text{with } K_{21} \in I^{-\frac{3}{2},-\frac{1}{2}}, K_{22} \in I^{-\frac{1}{2},-\frac{3}{2}}. \end{aligned}$$

Putting this in T_2, we get

$$T_2 = \underbrace{K_{11}}_{T_{20}} + \underbrace{K_{12} + K_{21}}_{T_{21}} + \underbrace{K_{22}}_{T_{22}}$$

where $T_{20} \in I^{-\frac{5}{2},\frac{1}{2}}$, $T_{21} \in I^{-\frac{3}{2},-\frac{1}{2}}$, $T_{22} \in I^{-\frac{1}{2},-\frac{3}{2}}$. Therefore

$$T_2 \in \sum_{j=0}^{2} I^{-\frac{1}{2}-2+j,\frac{1}{2}-j}.$$

Proceeding recursively, we get a sequence of operators

$$T_N \in \sum_{j=0}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j}.$$

We write this as

$$T_N \in \sum_{j=0}^{[\frac{N}{2}]} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} + \sum_{j=[\frac{N}{2}]+1}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j}.$$

In the first sum $-\frac{1}{2} - N + j \leq -\frac{1}{2} - N + \left[\frac{N}{2}\right]$ and $\frac{1}{2} - j \leq \frac{1}{2}$. Similarly in the second sum, $-\frac{1}{2} - N + j \leq -\frac{1}{2}$ and $\frac{1}{2} - j \leq -\frac{1}{2} - \left[\frac{N}{2}\right]$. Now we use $I^{p,l} \subset I^{p',l'}$ for $p \leq p'$, $l \leq l'$ to get

$$\sum_{j=0}^{[\frac{N}{2}]} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} \in I^{-\frac{1}{2}-N+[\frac{N}{2}],\frac{1}{2}} \text{ and } \sum_{j=[\frac{N}{2}]+1}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} \in I^{-\frac{1}{2},-\frac{1}{2}-[\frac{N}{2}]}.$$

In the limit $N \to \infty$, the first term in the above expression is a smoothing term by the property that $\cap_p I^{p,l}(\Delta, \Lambda) \subset \mathcal{C}^\infty$ and the second term is an operator $\mathcal{A}$ in $I^{-\frac{1}{2}}(\Lambda)$ by the property $\cap_l I^{p,l}(\Delta, \Lambda) \subset I^p(\Lambda)$. Finally, we define $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \cdots$ and from the construction above, we get,

$$\mathcal{B}\mathcal{T}_\gamma^*\mathcal{T}_\gamma(f) = f + \mathcal{A}f + \mathcal{C}^\infty.$$

This completes the proof of Theorem 2.2. □

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