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Some Families of Generating Functions for the Jacobi and Related Orthogonal Polynomials

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By making use of the familiar group-theoretic (Lie algebraic) method of Louis Weisner (1899–1988), many authors recently proved various single- and multiple-series generating functions for the so-called extended Jacobi polynomials. The main object of the present sequel to these earlier works is to show how easily each of such generating functions can be derived from the corresponding known result for the classical Jacobi polynomials. Many general families of bilinear, bilateral, or mixed multilateral generating functions for the Jacobi and related orthogonal polynomials, which are seemingly relevant to the present investigation, are also considered here.

Key Words: Jacobi polynomials; orthogonality property; Rodrigues formula; generating functions; generalized hypergeometric function; group-theoretic method.

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1. INTRODUCTION AND DEFINITIONS

Let $(\lambda)_n$ denote the Pochhammer symbol (or the *shifted factorial*, since $(1)_n = n!$) defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\ldots\}). \end{cases} \tag{1.1}$$

Also, as usual, denote by ${}_pF_q$ a generalized hypergeometric function with p numerator and q denominator parameters.

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, of order (α, β) and degree n in x, defined (in terms of the Gauss hypergeometric ${}_2F_1$ function) by

$$P_n^{(\alpha,\beta)}(x) := \binom{\alpha+n}{n} {}_2F_1\left(-n, \alpha+\beta+n+1; \alpha+1; \frac{1-x}{2}\right) \tag{1.2}$$

or, equivalently, by the Rodrigues formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\, n!} \cdot D_x^n\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\} \qquad \left(D_x := \frac{d}{dx}\right), \tag{1.3}$$

are orthogonal over the interval $(-1, 1)$ with respect to the weight function:

$$w(x) := (1-x)^{\alpha}(1+x)^{\beta}; \tag{1.4}$$

in fact, we have (cf., e.g., Szegö [24])

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\, \delta_{m,n} \qquad \left(\min\{\Re(\alpha), \Re(\beta)\} > -1; m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right), \tag{1.5}$$

where $\delta_{m,n}$ denotes the Kronecker delta.

In recent years, a great deal of attention seems to have been paid to an obvious variant of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. These so-called extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x; a, b, c)$, studied by (among others) Izuru Fujiwara (1928–1985) in an attempt to give a unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre, and Hermite polynomials), are defined by the Rodrigues formula:

$$F_n^{(\alpha,\beta)}(x;a,b,c) := \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} \cdot D_x^n\{(x-a)^{\alpha+n}(b-x)^{\beta+n}\} \qquad \left(c := \frac{\lambda}{b-a} > 0\right) \tag{1.6}$$

and are orthogonal over the interval (a, b) with respect to the weight function [cf. Eq. (1.4)]:

$$w(x;a,b) := (x-a)^{\alpha}(b-x)^{\beta}. \tag{1.7}$$

The polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are essentially those that were considered by Szegö [24, p. 58], who showed (by means of a simple linear transformation) that these polynomials are just a constant multiple of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. In fact, by comparing the Rodrigues formulas (1.3) and (1.6), it is not difficult to rewrite Szegö's observation [24, p. 58, Eq. (4.1.2)] in the form (cf., e.g., Srivastava and Manocha [22, p. 388, Problem 11]):

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \tag{1.8}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n} F_n^{(\alpha,\beta)}\left(\tfrac{1}{2}\{a+b+(a-b)x\};a,b,c\right). \tag{1.9}$$

Thus, as already pointed out by Srivastava and Manocha [22], the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ may by looked upon as being equivalent to (and *not* as a generalization of) the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

Furthermore, by recourse to certain limiting processes, it is easily seen that the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ would give rise to the Laguerre and Hermite polynomials (and indeed also to the Bessel polynomials) just as the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ do. Consequently, the *main* purpose of Fujiwara's investigation [8] is already served by the classical Jacobi polynomials *themselves*.

Even after the aforementioned observation by Szegö [24] and others (cf., e.g., Srivastava and Manocha [22]), the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ have been (and are still being) made, in recent years, a tool for the purpose of generalizing what is already known in the context of the classical Jacobi polynomials. For example, by applying the familiar group-theoretic (Lie algebraic) method of Louis Weisner (1899–1988), which is described fairly adequately in the works of Miller [13], McBride [12, Chaps. 2 and 3], and Srivastava and Manocha [22, Chap. 6], many authors have proved various single- and multiple-series generating functions for the so-called extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$. The main object of this paper is to show how easily each of these generating functions can be derived from the corresponding known result for the classical Jacobi polynomials. We also consider many general families of bilinear, bilateral, or mixed multilateral generating functions for the Jacobi and related orthogonal polynomials, which are seemingly relevant to the present investigation.

2. A SET OF LINEAR GENERATING FUNCTIONS

One of the latest works on the subject of generating functions for the extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$, which are derived by the group-theoretic (Lie algebraic) method already referred to in Section 1, is by Chongdar *et al.* [7], who obtained these generating functions by suitably interpreting the degree n. We choose first to recall here the main results of Chongdar *et al.* [7] in the following modified forms:

$$\sum_{k=0}^{n} \binom{k-\beta-n-1}{k} F_{n-k}^{(\alpha+k,\beta)}(x;a,b,c)t^k = \left(1-\frac{t}{\lambda}\right)^n F_n^{(\alpha,\beta)}\left(\frac{\lambda x-bt}{\lambda-t};a,b,c\right) \tag{2.1}$$

or, equivalently,

$$\sum_{k=0}^{n} \binom{k-\alpha-n-1}{k} F_{n-k}^{(\alpha,\beta+k)}(x;a,b,c)t^k = \left(1+\frac{t}{\lambda}\right)^n F_n^{(\alpha,\beta)}\left(\frac{\lambda x+at}{\lambda+t};a,b,c\right), \tag{2.2}$$

which was proven similarly by Mukherjee [14, p. 7, Eq. (3.4)];

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} F_{n+k}^{(\alpha-k,\beta)}(x;a,b,c)t^k &= (1-\lambda t)^{\alpha}\{1-c(x-a)t\}^{-\alpha-\beta-n-1} \\ &\quad\cdot F_n^{(\alpha,\beta)}\left(\frac{x-bc(x-a)t}{1-c(x-a)t};a,b,c\right) \\ &\qquad\left(|t|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}\right)\end{aligned} \tag{2.3}$$

or, equivalently,

$$\sum_{k=0}^{\infty} \binom{n+k}{k} F_{n+k}^{(\alpha,\,\beta-k)}(x;a,b,c)t^k$$
$$= (1+\lambda t)^{\beta}\{1-c(x-b)t\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-ac(x-b)t}{1-c(x-b)t};a,b,c\right)$$
$$\left(|t| < \min\{\lambda^{-1}, |c(x-b)|^{-1}\right), \quad (2.4)$$

which was proven similarly by Mukherjee [14, p. 8, Eq. (3.5)];

$$\sum_{k=0}^{\infty} \binom{n+k}{k} t^k \sum_{j=0}^{n+k} \binom{j-\beta-n-k-1}{j} F_{n-j+k}^{(\alpha+j-k,\,\beta)}(x;a,b,c)\,\tau^j$$
$$= \left(1-\frac{\tau}{\lambda}\right)^n (1-\lambda t+t\tau)^{\alpha}\{1-c(x-a)t+t\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{\lambda x-b\tau-b(\lambda-\tau)\{c(x-a)-\tau\}t}{(\lambda-\tau)\,\{1-c(x-a)t+t\tau\}};a,b,c\right)$$
$$\left(|t| < \min\{|\lambda-\tau|^{-1}, |c(x-a)-\tau|^{-1}\}\right), \quad (2.5)$$

which appears erroneouly in the work of Chongdar *et al*. [7, p. 375, Eq. (3.7)]; and

$$\sum_{k=0}^{\infty} \binom{n+k}{k} t^k \sum_{j=0}^{n+k} \binom{j-\alpha-n-k-1}{j} F_{n-j+k}^{(\alpha,\,\beta+j-k)}(x;a,b,c)\,\tau^j$$
$$= \left(1+\frac{\tau}{\lambda}\right)^n (1+\lambda t+t\tau)^{\beta}\{1-c(x-b)t+t\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{\lambda x+at-a(\lambda+\tau)\{c(x-b)-\tau\}t}{(\lambda+\tau)\,\{1-c(x-b)t+t\tau\}};a,b,c\right)$$
$$(|t| < \min\{|\lambda+\tau|^{-1}, |c(x-b)-\tau|^{-1}\}), \quad (2.6)$$

which was given earlier by Mukherjee [14, p. 11, Eq. (4.3)] and appears erroneously in the work of Chongdar *et al*. [7, p. 376, Eq. (3.12)].

The equivalence of (2.1) and (2.2), as also of (2.3) and (2.4), can be exhibited by appealing to the relationship:

$$F_n^{(\alpha,\,\beta)}(a+b-x;a,b,c) = (-1)^n F_n^{(\beta,\,\alpha)}(x;a,b,c), \quad (2.7)$$

which follows readily from the well-known relationship [24, p. 59, Eq. (4.1.3)]:

$$P_n^{(\alpha,\,\beta)}(-x) = (-1)^n P_n^{(\beta,\,\alpha)}(x), \quad (2.8)$$

by means of (1.8).

Earlier, just as we have indicated above, by suitably interpreting the degree n or one of the parameters α and β (or both the degree n and one of the parameters α and β *simultaneously*) in the aforementioned group-theoretic (Lie algebraic) method, essentially the same generating functions as some of the above, and several additional generating functions for $F_n^{(\alpha,\,\beta)}(x; a, b, c)$, were derived in many other works on this subject. For the sake of ready reference, we also recall these other generating functions in their (corrected and/or modified) forms:

$$\sum_{k=0}^{n} \binom{\alpha+\beta+n+k}{k} F_{n-k}^{(\alpha+k,\,\beta+k)}(x; a, b, c)t^k = F_n^{(\alpha,\,\beta)}\left(x + \frac{t}{c}; a, b, c\right), \tag{2.9}$$

which first appeared in the work of Shrivastava and Dhillon [18, p. 133, Eq. (3.5)];

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} F_{n+k}^{(\alpha-k,\,\beta-k)}(x; a, b, c)t^k &= \{1 + c(x-b)t\}^{\alpha}\{1 + c(x-a)t\}^{\beta} \\ &\quad \cdot F_n^{(\alpha,\,\beta)}\Big(x + c(x-a)(x-b)t; a, b, c\Big) \\ &\qquad \left(|t| < \min\{|c(x-a)|^{-1}, |c(x-b)|^{-1}\}\right),\end{aligned} \tag{2.10}$$

which also appeared first in the work of Shrivastava and Dhillon [18, p. 133, Eq. (3.8)];

$$\begin{aligned}&\sum_{k=0}^{\infty} \binom{n+k}{k} t^k \sum_{j=0}^{n+k} \binom{\alpha+\beta+n-k+j}{j} F_{n+k-j}^{(\alpha-k+j,\,\beta-k+j)}(x; a, b, c)\tau^j \\ &= \left\{1 + c\left(x - b + \frac{\tau}{c}\right)t\right\}^{\alpha}\left\{1 + c\left(x - a + \frac{\tau}{c}\right)t\right\}^{\beta} \\ &\quad \cdot F_n^{(\alpha,\,\beta)}\left(x + \frac{\tau}{c} + c\left(x - a + \frac{\tau}{c}\right)\left(x - b + \frac{\tau}{c}\right)t; a, b, c\right) \\ &\qquad \left(|t| < \min\{|c(x-a)+\tau|^{-1}, |c(x-b)+\tau|^{-1}\}\right),\end{aligned} \tag{2.11}$$

which first appeared in the work of Shrivastava and Dhillon [18, p. 134, Eq. (3.11)] (see also [18, p. 135, Eq. (3.14)] for an obviously erroneous version of (2.11) above);

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} F_n^{(\alpha+k,\,\beta-k)}(x; a, b, c)t^k = (1-t)^{\beta} F_n^{(\alpha,\,\beta)}(x - (x-b)t; a, b, c) \qquad (|t| < 1), \tag{2.12}$$

which appeared in the work of Sen and Chongdar [16, p. 85, Eq. (3.4)] (and *also* in the identical work of Sen and Chongdar [17]; see also Chongdar and Majumdar [6, p. 32, Eq. (3.4)]);

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} F_n^{(\alpha-k,\, \beta+k)}(x; a, b, c) t^k$$
$$= (1-t)^{\alpha} F_n^{(\alpha,\, \beta)}(x-(x-a)t; a, b, c) \qquad (|t| < 1), \tag{2.13}$$

which appeared in the work of Sen and Chongdar [16, p. 86, Eq. (3.6)] (and also in the work of Chongdar and Majumdar [6, p. 33, Eq. (3.5)]); and

$$\sum_{k,\, j=0}^{\infty} \binom{k-\alpha-n-j-1}{k}\binom{j-\beta-n-1}{j} F_n^{(\alpha-k+j,\, \beta-k+j)}(x; a, b, c)\, t^k \tau^j$$
$$= (1-t)^{\alpha}\{1-(1-t)\tau\}^{\beta}$$
$$\cdot F_n^{(\alpha,\, \beta)}\big(\{x-(x-a)t\}\{1-(1-t)\tau\}+b(1-t)\tau; a, b, c\big)$$
$$(|t| < 1\,; |\tau| < |1-t|^{-1}), \tag{2.14}$$

which appeared in the work of Sen and Chongdar [16, p. 86, Eq. (3.7)] (see also Chongdar and Majumdar [6, p. 33, Eq. (3.6)] for an obviouly erroneous version of (2.14) above).

In view of the relationship (2.7), the generating functions (2.12) and (2.13) are equivalent. Furthermore, since

$$F_n^{(\alpha,\, \beta)}(x; a, b, c) = \left(-\frac{x-a}{a-b}\right)^n F_n^{(-\alpha-\beta-2n-1,\, \beta)}\left(\frac{ax-2ab+b^2}{x-a}; a, b, c\right)$$
$$= \left(\frac{x-b}{a-b}\right)^n F_n^{(\alpha,\, -\alpha-\beta-2n-1)}\left(\frac{bx-2ab+a^2}{x-b}; a, b, c\right), \tag{2.15}$$

which would follow easily from the known relationships [24, p. 64, Eq. (4.22.1)]:

$$P_n^{(\alpha,\, \beta)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\, \beta)}\left(\frac{x+3}{x-1}\right)$$
$$= \left(\frac{1+x}{2}\right)^n P_n^{(\alpha,\, -\alpha-\beta-2n-1)}\left(\frac{3-x}{1+x}\right) \tag{2.16}$$

by means of (1.8), it is not difficult to show that the finite summation formula (2.9) is equivalent to (2.1) and (2.2); the generating function (2.10) is equivalent to (2.3) and (2.4); and (2.12), (2.13), and the generating functions (2.17) to (2.20) below are *all* equivalent to one another:

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} F_n^{(\alpha-k,\,\beta)}(x;a,b,c)t^k$$
$$= (1-t)^{\alpha}\left(1+\frac{(x-a)t}{a-b}\right)^n$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)x+b(x-a)t}{a-b+(x-a)t};a,b,c\right) \qquad (|t|<1); \quad (2.17)$$

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} F_n^{(\alpha,\,\beta-k)}(x;a,b,c)t^k$$
$$= (1-t)^{\beta}\left(1-\frac{(x-b)t}{a-b}\right)^n$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)x-a(x-b)t}{a-b-(x-b)t};a,b,c\right) \qquad (|t|<1); \quad (2.18)$$

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} F_n^{(\alpha+k,\,\beta)}(x;a,b,c)t^k$$
$$= (1-t)^{-\alpha-\beta-n-1} F_n^{(\alpha,\,\beta)}\left(\frac{x-bt}{1-t};a,b,c\right) \qquad (|t|<1); \quad (2.19)$$

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} F_n^{(\alpha,\,\beta+k)}(x;a,b,c)t^k$$
$$= (1-t)^{-\alpha-\beta-n-1} F_n^{(\alpha,\,\beta)}\left(\frac{x-at}{1-t};a,b,c\right) \qquad (|t|<1). \quad (2.20)$$

Each of the double-series generating functions (2.5), (2.6), (2.11), and (2.14) would follow immediately when we appropriately combine two of the single-series generating functions (2.1), (2.2), (2.3), (2.4), (2.9), (2.10), (2.12), and (2.13). In fact, by similarly combining two or more of the above-listed single-series generating functions, one can easily derive numerous other double-, triple-, and multiple-series generating functions involving the extended Jacobi polynomials $F_n^{(\alpha,\,\beta)}(x;a,b,c)$. For example, we thus obtain the following analogues and variants of the double-series generating functions (2.5), (2.6), (2.11), and (2.14):

$$\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n+k-j}{k}\binom{j-\beta-n-1}{j}F_{n+k-j}^{(\alpha-k+j,\,\beta)}(x;a,b,c)\,t^k\tau^j$$
$$=(1-\lambda t)^{\alpha}\left(1-\frac{\tau}{\lambda}+t\tau\right)^{n}\{1-c(x-a)t\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{\lambda\{x-bc(x-a)t\}-b\tau(1-\lambda t)\{1-c(x-a)t\}}{\{\lambda(1+t\tau)-\tau\}\{1-c(x-a)t\}};a,b,c\right)$$
$$(|t|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}),\quad (2.21)$$

which follows immediately from (2.1) and (2.3) (and which appeared erroneously in the work of Chongdar *et al*. [7, p. 380, Eq. (4.4)]);

$$\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n+k-j}{k}\binom{j-\alpha-n-1}{j}F_{n+k-j}^{(\alpha,\,\beta-k+j)}(x;a,b,c)\,t^k\tau^j$$
$$=(1+\lambda t)^{\beta}\left(1+\frac{\tau}{\lambda}+t\tau\right)^{n}\{1-c(x-b)t\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{\lambda\{x-ac(x-b)t\}+a\tau(1+\lambda t)\{1-c(x-b)t\}}{\{\lambda(1+t\tau)+\tau\}\{1-c(x-b)t\}};a,b,c\right)$$
$$(|t|<\min\{\lambda^{-1},|c(x-b)|^{-1}\}),\quad (2.22)$$

which follows immediately from (2.2) and (2.4) (and which appeared erroneously in the work of Mukherjee [14, p. 8, Eq. (3.6)]);

$$\sum_{k=0}^{\infty}\sum_{j=0}^{n}\binom{n+k-j}{k}\binom{\alpha+\beta+n+j}{j}F_{n+k-j}^{(\alpha-k+j,\,\beta-k+j)}(x;a,b,c)\,t^k\tau^j$$
$$=\{1+c(x-b)t\}^{\alpha}\{1+c(x-a)t\}^{\beta}$$
$$\cdot F_n^{(\alpha,\,\beta)}\Bigg(x+c(x-a)(x-b)t$$
$$+\frac{\tau}{c}\{1+c(x-a)t\}\{1+c(x-b)t\};a,b,c\Bigg)$$
$$(|t|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}),\quad (2.23)$$

which follows immediately from (2.9) and (2.10);

$$\sum_{k,\,j=0}^{\infty}\binom{\alpha+\beta+n+k-j}{k}\binom{j-\alpha-n-1}{j}F_n^{(\alpha+k-j,\,\beta)}(x;a,b,c)\,t^k\tau^j$$
$$=(1-t)^{-\alpha-\beta-n-1}\{1-(1-t)\tau\}^{\alpha}\left(1+t\tau+\frac{(x-a)\tau}{a-b}\right)^{n}$$

$$\cdot F_n^{(\alpha,\beta)}\left(\frac{(a-b)(x-bt)+b\tau(1-t)\{x-a+(a-b)t\}}{(1-t)\{(a-b)(1+t\tau)+(x-a)\tau\}};a,b,c\right)$$
$$(|t|<1;|\tau|<|1-t|^{-1}), \quad (2.24)$$

which follows immediately from (2.17) and (2.19);

$$\sum_{k,j=0}^{\infty}\binom{\alpha+\beta+n+k-j}{k}\binom{j-\beta-n-1}{j}F_n^{(\alpha,\beta+k-j)}(x;a,b,c)\,t^k\tau^j$$
$$=(1-t)^{-\alpha-\beta-n-1}\{1-(1-t)\tau\}^{\beta}\left(1+t\tau-\frac{(x-b)\tau}{a-b}\right)^n$$
$$\cdot F_n^{(\alpha,\beta)}\left(\frac{(a-b)(x-at)-a\tau(1-t)\{x-b-(a-b)t\}}{(1-t)\{(a-b)(1+t\tau)-(x-b)\tau\}};a,b,c\right)$$
$$(|t|<1;|\tau|<|1-t|^{-1}), \quad (2.25)$$

which follows immediately from (2.18) and (2.20);

$$\sum_{k=0}^{n}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{n-k}\binom{j-\alpha-n-1}{j}F_{n-k-j}^{(\alpha+k,\beta+j)}(x;a,b,c)\,\tau^j$$
$$=\left(1+\frac{\tau-t}{\lambda}\right)^n F_n^{(\alpha,\beta)}\left(\frac{\lambda x+a\tau-bt}{\lambda+\tau-t};a,b,c\right), \quad (2.26)$$

which follows immediately from (2.1) and (2.2);

$$\sum_{k=0}^{n}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{n-k}\binom{j-\beta-n-1}{j}F_{n-k-j}^{(\alpha+j,\beta+k)}(x;a,b,c)\,\tau^j$$
$$=\left(1+\frac{t-\tau}{\lambda}\right)^n F_n^{(\alpha,\beta)}\left(\frac{\lambda x+at-b\tau}{\lambda+t-\tau};a,b,c\right), \quad (2.27)$$

which also follows immediately from (2.1) and (2.2);

$$\sum_{k=0}^{n}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{j-\alpha-n-1}{j}F_{n-k}^{(\alpha+k-j,\beta+j)}(x;a,b,c)\,\tau^j$$
$$=(1-\tau)^{\alpha}\left(1-\frac{t(1-\tau)}{\lambda}\right)^n$$
$$\cdot F_n^{(\alpha,\beta)}\left(\frac{\lambda\{x-(x-a)\tau\}-bt(1-\tau)}{\lambda-t(1-\tau)};a,b,c\right) \qquad (|\tau|<1), \quad (2.28)$$

which follows immediately from (2.1) and (2.13);

$$\sum_{k=0}^{n}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{j-\beta-n-1}{j}F_{n-k}^{(\alpha+j,\,\beta+k-j)}(x;a,b,c)\,\tau^j$$
$$=(1-\tau)^{\beta}\left(1+\frac{t(1-\tau)}{\lambda}\right)^{n}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{\lambda\{x-(x-b)\tau\}+at(1-\tau)}{\lambda+t(1-\tau)};a,b,c\right)\qquad(|\tau|<1),\quad(2.29)$$

which follows immediately from (2.2) and (2.12);

$$\sum_{k=0}^{\infty}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha+k-j,\,\beta-k)}(x;a,b,c)\,\tau^j$$
$$=(1-\lambda\tau)^{\alpha}\{1-t(1-\lambda\tau)\}^{\beta}\{1-c(x-a)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-bc(x-a)\tau-(x-b)t(1-\lambda\tau)}{1-c(x-a)\tau};a,b,c\right)$$
$$(|t|<|1-\lambda\tau|^{-1};|\tau|<\lambda^{-1}),\quad(2.30)$$

which follows immediately from (2.3) and (2.12);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-k-j,\,\beta+k)}(x;a,b,c)\tau^j$$
$$=(1-t-\lambda\tau)^{\alpha}\{1-c(x-a)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-(x-a)(t+bc\tau)}{1-c(x-a)\tau};a,b,c\right)$$
$$(|t|<|1-\lambda\tau|;|\tau|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}),\quad(2.31)$$

which follows immediately from (2.3) and (2.13);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\left(\binom{n+j}{j}\right)F_{n+j}^{(\alpha-k-j,\,\beta)}(x;a,b,c)\,\tau^j$$
$$=(1-\lambda\tau-\{1-c(x-a)\tau\}t)^{\alpha}\{1-c(x-a)\tau\}^{-\alpha-\beta-n-1}\left(1+\frac{(x-a)t}{a-b}\right)^{n}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x-bc(x-a)\tau\}+b(x-a)t\{1-c(x-a)\tau\}}{\{a-b+(x-a)t\}\{1-c(x-a)\tau\}};a,b,c\right)$$
$$(|t|<|1-\lambda\tau|\cdot|1-c(x-a)\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}),\quad(2.32)$$

which follows immediately from (2.3) and (2.17);

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} t^k \sum_{j=0}^{\infty} \binom{n+j}{j} F_{n+j}^{(\alpha-j,\,\beta-k)}(x;a,b,c)\,\tau^j$$
$$= (1-\lambda\tau)^{\alpha}\{1-t+c(x-a)t\tau\}^{\beta}\{1-c(x-a)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot\left(1-\frac{(x-b)t}{a-b}\right)^n$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x-bc(x-a)\tau\}-a(x-b)t\{1-c(x-a)\tau\}}{\{a-b-(x-b)t\}\{1-c(x-a)\tau\}};a,b,c\right)$$
$$(|t|<|1-c(x-a)\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}), \quad (2.33)$$

which follows immediately from (2.3) and (2.18);

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} t^k \sum_{j=0}^{\infty} \binom{n+j}{j} F_{n+j}^{(\alpha+k-j,\,\beta)}(x;a,b,c)\,\tau^j$$
$$= (1-\lambda\tau)^{\alpha}\{1-c(x-a)\tau-t(1-\lambda\tau)\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-bc(x-a)\tau-bt(1-\lambda\tau)}{1-c(x-a)\tau-t(1-\lambda\tau)};a,b,c\right)$$
$$(|t|<|1-c(x-a)\tau|\cdot|1-\lambda\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}), \quad (2.34)$$

which follows immediately from (2.3) and (2.19);

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} t^k \sum_{j=0}^{\infty} \binom{n+j}{j} F_{n+j}^{(\alpha-j,\,\beta+k)}(x;a,b,c)\,\tau^j$$
$$= (1-\lambda\tau)^{\alpha}\{1-t-c(x-a)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-at-bc(x-a)\tau}{1-t-c(x-a)\tau};a,b,c\right)$$
$$(|t|<|1-c(x-a)\tau|;|\tau|<\min\{\lambda^{-1},|c(x-a)|^{-1}\}), \quad (2.35)$$

which follows immediately from (2.3) and (2.20);

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} t^k \sum_{j=0}^{\infty} \binom{n+j}{j} F_{n+j}^{(\alpha+k,\,\beta-k-j)}(x;a,b,c)\,\tau^j$$
$$= (1-t+\lambda\tau)^{\beta}\{1-c(x-b)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-(x-b)(t+ac\tau)}{1-c(x-b)\tau};a,b,c\right)$$
$$(|t|<1;|\tau|<\min\{\lambda^{-1}|1-t|,|c(x-b)|^{-1}\}), \quad (2.36)$$

which follows immediately from (2.4) and (2.12);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\left(\binom{n+j}{j}\right)F_{n+j}^{(\alpha-k,\,\beta+k-j)}(x;a,b,c)\,\tau^j$$
$$=(1+\lambda\tau)^{\beta}\,\{1-t(1+\lambda\tau)\}^{\alpha}\,\{1-c(x-b)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-ac(x-b)\tau-(x-a)t(1+\lambda\tau)}{1-c(x-b)\tau};a,b,c\right)$$
$$(|t|<|1+\lambda\tau|^{-1};|\tau|<\lambda^{-1}),\quad(2.37)$$

which follows immediately from (2.4) and (2.13);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-k,\,\beta-j)}(x;a,b,c)\,\tau^j$$
$$=(1+\lambda\tau)^{\beta}\{1-c(x-b)\tau\}^{-\alpha-\beta-n-1}\,(1-\{1-c(x-b)\tau\}t)^{\alpha}$$
$$\cdot\left(1+\frac{(x-a)t}{a-b}\right)^{n}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x-ac(x-b)\tau\}+b(x-a)t\{1-c(x-b)\tau\}}{\{a-b+(x-a)t\}\{1-c(x-b)\tau\}};a,b,c\right)$$
$$(|t|<|1-c(x-b)\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-b)|^{-1}\}),\quad(2.38)$$

which follows immediately from (2.4) and (2.17);

$$\sum_{k=0}^{\infty}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha,\,\beta-k-j)}(x;a,b,c)\,\tau^j$$
$$=(1+\lambda\tau-\{1-c(x-b)\tau\}t)^{\beta}\,\{1-c(x-b)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot\left(1-\frac{(x-b)t}{a-b}\right)^{n}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x-ac(x-b)\tau\}-a(x-b)t\{1-c(x-b)\tau\}}{\{a-b-(x-b)t\}\{1-c(x-b)\tau\}};a,b,c\right)$$
$$(|t|<|1+\lambda\tau|\cdot|1-c(x-b)\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-b)|^{-1}\}),\quad(2.39)$$

which follows immediately from (2.4) and (2.18);

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta+n+k}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha+k,\,\beta-j)}(x;a,b,c)\,\tau^j$$
$$=(1+\lambda\tau)^{\beta}\{1-t-c(x-b)\tau\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-bt-ac(x-b)\tau}{1-t-c(x-b)\tau};a,b,c\right)$$
$$(|t|<|1-c(x-b)\tau|;|\tau|<\min\{\lambda^{-1},|c(x-b)|^{-1}\}),\quad(2.40)$$

which follows immediately from (2.4) and (2.19);

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta+n+k}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha,\,\beta+k-j)}(x;a,b,c)\,\tau^j$$
$$=(1+\lambda\tau)^{\beta}\{1-c(x-b)\tau-t(1+\lambda\tau)\}^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x-ac(x-b)\tau-at(1+\lambda\tau)}{1-c(x-b)\tau-t(1+\lambda\tau)};a,b,c\right)$$
$$(|t|<|1-c(x-b)\tau|\cdot|1+\lambda\tau|^{-1};|\tau|<\min\{\lambda^{-1},|c(x-b)|^{-1}\}),\quad(2.41)$$

which follows immediately from (2.4) and (2.20);

$$\sum_{k=0}^{\infty}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha+k-j,\,\beta-k-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-b)\tau\}^{\alpha}(1+c(x-a)\tau-\{1+c(x-b)\tau\}t)^{\beta}$$
$$\cdot F_n^{(\alpha,\,\beta)}\Big(x+c(x-a)(x-b)\tau-(x-b)\{1+c(x-b)\tau\}t;a,b,c\Big)$$
$$\big(|t|<|1+c(x-a)\tau|\cdot|1+c(x-b)\tau|^{-1};$$
$$|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}\big),\quad(2.42)$$

which follows immediately from (2.10) and (2.12);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-k-j,\,\beta+k-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-a)\tau\}^{\beta}(1+c(x-b)\tau-\{1+c(x-a)\tau\}t)^{\alpha}$$
$$\cdot F_n^{(\alpha,\,\beta)}\Big(x+c(x-a)(x-b)\tau-(x-a)\{1+c(x-a)\tau\}t;a,b,c\Big)$$
$$\big(|t|<|1+c(x-b)\tau|\cdot|1+c(x-a)\tau|^{-1};$$
$$|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}\big),\quad(2.43)$$

which follows immediately from (2.10) and (2.13);

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-k-j,\,\beta-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-a)\tau\}^{\beta}\{1-t+c(x-b)\tau\}^{\alpha}\left(1+\frac{(x-a)t}{a-b}\right)^{n}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x+c(x-a)(x-b)\tau\}+b(x-a)t}{a-b+(x-a)t};a,b,c\right)$$
$$(|t|<|1+c(x-b)\tau|;|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}),\quad(2.44)$$

which follows immediately from (2.10) and (2.17);

$$\sum_{k=0}^{\infty}\binom{k-\beta-n-1}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-j,\,\beta-k-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-b)\tau\}^{\alpha}\{1-t+c(x-a)\tau\}^{\beta}\left(1-\frac{(x-b)t}{a-b}\right)^n$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{(a-b)\{x+c(x-a)(x-b)\tau\}-a(x-b)t}{a-b-(x-b)t};a,b,c\right)$$
$$(|t|<|1+c(x-a)\tau|^{-1};|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}),\tag{2.45}$$

which follows immediately from (2.10) and (2.18);

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta+n+k}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha+k-j,\,\beta-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-b)\tau\}^{\alpha}\{1+c(x-a)\tau\}^{\beta}(1-\{1+c(x-b)\tau\}t)^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x+c(x-a)(x-b)\tau-b\{1+c(x-b)\tau\}t}{1-\{1+c(x-b)\tau\}t};a,b,c\right)$$
$$(|t|<|1+c(x-b)\tau|^{-1};|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}),\tag{2.46}$$

which follows immediately from (2.10) and (2.19); and

$$\sum_{k=0}^{\infty}\binom{\alpha+\beta+n+k}{k}t^k\sum_{j=0}^{\infty}\binom{n+j}{j}F_{n+j}^{(\alpha-j,\,\beta+k-j)}(x;a,b,c)\,\tau^j$$
$$=\{1+c(x-b)\tau\}^{\alpha}\{1+c(x-a)\tau\}^{\beta}(1-\{1+c(x-a)\tau\}t)^{-\alpha-\beta-n-1}$$
$$\cdot F_n^{(\alpha,\,\beta)}\left(\frac{x+c(x-a)(x-b)\tau-a\{1+c(x-a)\tau\}t}{1-\{1+c(x-a)\tau\}t};a,b,c\right)$$
$$(|t|<|1+c(x-a)\tau|^{-1};|\tau|<\min\{|c(x-a)|^{-1},|c(x-b)|^{-1}\}),\tag{2.47}$$

which follows immediately from (2.10) and (2.20).

In view of the relationship (1.8), the single-series generating functions (2.1), (2.2), (2.9), (2.3), (2.4), (2.10), (2.12), (2.13), and (2.17) to (2.20), which readily imply each of the aforementioned multiple-series generating functions, are merely *disguised* forms of the following known generating functions for the classical Jacobi polynomials (cf., e.g., Hansen [9], Srivastava and Manocha [22], Chen and Srivastava [3], and the references cited therein):

$$\sum_{k=0}^{n}\binom{k-\beta-n-1}{k}P_{n-k}^{(\alpha+k,\,\beta)}(x)\,t^k=(1+t)^n\,P_n^{(\alpha,\,\beta)}\left(\frac{x-t}{1+t}\right),\tag{2.48}$$

$$\sum_{k=0}^{n} \binom{k-\alpha-n-1}{k} P_{n-k}^{(\alpha,\,\beta+k)}(x)\, t^k = (1-t)^n\, P_n^{(\alpha,\,\beta)}\left(\frac{x-t}{1-t}\right), \tag{2.49}$$

$$\sum_{k=0}^{n} \binom{\alpha+\beta+n+k}{k} P_{n-k}^{(\alpha+k,\,\beta+k)}(x)\, t^k = P_n^{(\alpha,\,\beta)}(x+2t), \tag{2.50}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha-k,\,\beta)}(x)\, t^k = (1+t)^{\alpha} \left\{1-\frac{1}{2}(x-1)t\right\}^{-\alpha-\beta-n-1} \cdot P_n^{(\alpha,\,\beta)}\left(\frac{x+\frac{1}{2}(x-1)t}{1-\frac{1}{2}(x-1)t}\right) \quad \left(|t| < \min\{1, 2|x-1|^{-1}\}\right), \tag{2.51}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha,\,\beta-k)}(x)\, t^k = (1-t)^{\beta} \left\{1-\tfrac{1}{2}(x+1)t\right\}^{-\alpha-\beta-n-1} \cdot P_n^{(\alpha,\,\beta)}\left(\frac{x-\frac{1}{2}(x+1)t}{1-\frac{1}{2}(x+1)t}\right) \quad \left(|t| < \min\{1, 2|x+1|^{-1}\}\right), \tag{2.52}$$

$$\sum_{k=0}^{\infty} \binom{n+k}{k} P_{n+k}^{(\alpha-k,\,\beta-k)}(x)\, t^k = \left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha} \left\{1+\frac{1}{2}(x-1)t\right\}^{\beta} \cdot P_n^{(\alpha,\,\beta)}\left(x+\frac{1}{2}(x^2-1)t\right) \quad \left(|t| < \min\{2|x+1|^{-1}, 2|x-1|^{-1}\}\right), \tag{2.53}$$

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} P_n^{(\alpha+k,\,\beta-k)}(x)\, t^k = (1-t)^{\beta}\, P_n^{(\alpha,\,\beta)}\Big(x-(x+1)t\Big) \qquad (|t|<1), \tag{2.54}$$

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\, \beta+k)}(x)\, t^k$$
$$= (1-t)^{\alpha} P_n^{(\alpha,\, \beta)}\bigl(x-(x-1)t\bigr) \qquad (|t|<1), \tag{2.55}$$

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\, \beta)}(x)\, t^k$$
$$= (1-t)^{\alpha} \left\{1+\frac{1}{2}(x-1)t\right\}^n$$
$$\cdot\, P_n^{(\alpha,\, \beta)}\left(\frac{x-\frac{1}{2}(x-1)t}{1+\frac{1}{2}(x-1)t}\right) \qquad (|t|<1), \tag{2.56}$$

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} P_n^{(\alpha,\, \beta-k)}(x)\, t^k$$
$$= (1-t)^{\beta} \left\{1-\frac{1}{2}(x+1)t\right\}^n$$
$$\cdot\, P_n^{(\alpha,\, \beta)}\left(\frac{x-\frac{1}{2}(x+1)t}{1-\frac{1}{2}(x+1)t}\right) \qquad (|t|<1), \tag{2.57}$$

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} P_n^{(\alpha+k,\, \beta)}(x)\, t^k$$
$$= (1-t)^{-\alpha-\beta-n-1}\, P_n^{(\alpha,\, \beta)}\left(\frac{x+t}{1-t}\right) \qquad (|t|<1), \tag{2.58}$$

and

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} P_n^{(\alpha,\, \beta+k)}(x)\, t^k$$
$$= (1-t)^{-\alpha-\beta-n-1} P_n^{(\alpha,\, \beta)}\left(\frac{x-t}{1-t}\right) \qquad (|t|<1), \tag{2.59}$$

respectively.

In the fairly vast (and widely scattered) literature on generating functions, much more general results than those that are mentioned above can be found for the Jacobi polynomials $P_n^{(\alpha,\, \beta)}(x)$ as well as for their numerous genaralizations. For example, in terms of the Appell function F_1 defined by

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) := \sum_{l, m=0}^{\infty} \frac{(\alpha)_{l+m}(\beta)_l(\beta')_m}{(\gamma)_{l+m}} \frac{x^l}{l!} \frac{y^m}{m!}$$
$$\left(\max\{|x|, |y|\} < 1; \gamma \neq 0, -1, -2, \dots\right), \quad (2.60)$$

it is known that (cf., e.g., [22, p. 114, Eq. 2.3(40)])

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \frac{(\gamma)_k}{(-\alpha-\beta-n)_k} P_{n+k}^{(\alpha-k, \beta-k)}(x) t^k$$
$$= \binom{\alpha+\beta+2n}{n} \left(\frac{x+1}{2}\right)^n \left\{1 + \frac{1}{2}(x+1)t\right\}^{-\gamma}$$
$$\cdot F_1\left(-\beta-n, -n, \gamma; -\alpha-\beta-2n; \frac{2}{x+1}, \frac{t}{1+\frac{1}{2}(x+1)t}\right)$$
$$\left(|t| < \min\{2|x+1|^{-1}, 2|x-1|^{-1}\}\right), \quad (2.61)$$

which reduces to (2.53) in the special case when

$$\gamma = -\alpha - \beta - n \qquad (n \in \mathbb{N}_0),$$

since [22, p. 105, Eq. 2.3(6)]

$$F_1(\alpha, \beta, \beta'; \beta+\beta'; x, y) = (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta; \beta+\beta'; \frac{x-y}{1-y}\right)$$
$$\left(|\arg(1-y)| \leq \pi - \epsilon \ (0 < \epsilon < \pi)\right). \quad (2.62)$$

In the F-notation, the generating function (2.61) can easily be rewritten in the following form by appealing to the relationship (1.8):

$$\sum_{k=0}^{\infty} \binom{n+k}{k} \frac{(\gamma)_k}{(-\alpha-\beta-n)_k} F_{n+k}^{(\alpha-k, \beta-k)}(x; a, b, c) t^k$$
$$= \binom{\alpha+\beta+2n}{n} \{c(x-b)\}^n \{1 + c(x-b)t\}^{-\gamma}$$
$$\cdot F_1\left(-\beta-n, -n, \gamma; -\alpha-\beta-2n; \frac{a-b}{x-b}, -\frac{\lambda t}{1+c(x-b)t}\right)$$
$$\left(|t| < \min\{|c(x-a)|^{-1}, |c(x-b)|^{-1}\}\right). \quad (2.63)$$

3. BILINEAR AND BILATERAL GENERATING FUNCTIONS

A familiar bilinear generating function for the classical Jacobi polynomials is the Bailey formula (cf. [2]; see also [22, p. 116, Eq. 2.3(47)]):

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) t^n$$
$$= (1+t)^{-\alpha-\beta-1} F_4\left[\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; \frac{(1-x)(1-y)t}{(1+t)^2}, \frac{(1+x)(1+y)t}{(1+t)^2}\right]$$
$$(|t|<1), \quad (3.1)$$

where the Appell function

$$F_4 := F_C^{(2)}$$

is the two-variable ($s=2$) case of the Lauricella function $F_C^{(s)}$ of s complex variables $z_1, \ldots, z_s$ ($s \in \mathbb{N}$), defined by [22, p. 60, Eq. 1.7(3)] (see also [1, p. 114, Eq. (3)])

$$F_C^{(s)}[\alpha, \beta; \gamma_1, \ldots, \gamma_s; z_1, \ldots, z_s]$$
$$:= \sum_{l_1,\ldots,l_s=0}^{\infty} \frac{(\alpha)_{l_1+\cdots+l_s}(\beta)_{l_1+\cdots+l_s}}{(\gamma_1)_{l_1}\cdots(\gamma_s)_{l_s}} \frac{z_1^{l_1}}{l_1!} \cdots \frac{z_s^{l_s}}{l_s!}$$
$$\left(|z_1|^{1/2} + \cdots + |z_s|^{1/2} < 1; \gamma_j \neq 0, -1, -2, \ldots (j=1,\ldots,s)\right). \quad (3.2)$$

In view of the relationship (1.8), Bailey's formula (3.1) immediately yields the following bilinear generating function for the extended Jacobi polynomials:

$$\sum_{n=0}^{\infty} \frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} F_n^{(\alpha,\beta)}(x; a, b, c) F_n^{(\alpha,\beta)}(y; A, B, C) t^n$$
$$= (1+\lambda\Lambda t)^{-\alpha-\beta-1} F_4\left[\frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; \frac{4cC(x-a)(y-A)t}{(1+\lambda\Lambda t)^2}, \frac{4cC(x-b)(y-B)t}{(1+\lambda\Lambda t)^2}\right]$$
$$\left(|t| < (\lambda\Lambda)^{-1}; \lambda := c(b-a); \Lambda := C(B-A)\right). \quad (3.3)$$

In fact, by merely applying the relationship (1.8) to various known generalizations of the Bailey formula (3.1), one can easily derive much more general results than the bilinear generating function (3.3). Just as an illustration, we recall the following generating function for the classical Jacobi polynomials (cf. [22, p. 115, Eq. 2.3(45)]):

$$\sum_{n=0}^{\infty} \frac{(m+n)!(\alpha+\beta+m+1)_n}{(\gamma+1)_n(\delta+1)_n} P_{m+n}^{(\alpha,\beta)}(x)\, P_n^{(\gamma,\delta)}(y)\, t^n$$
$$= (\alpha+1)_m \left(\frac{x+1}{2}\right)^{-\alpha-\beta-m-1}$$
$$\cdot F_C^{(3)}\Bigg[\alpha+\beta+m+1, \alpha+m+1; \alpha+1, \gamma+1, \delta+1;$$
$$\frac{x-1}{x+1}, \frac{(y-1)t}{x+1}, \frac{(y+1)t}{x+1}\Bigg]$$
$$\left(|t|^{1/2} < (|x+1|^{1/2} - |x-1|^{1/2})(|y+1|^{1/2} + |y-1|^{1/2})^{-1}; m \in \mathbb{N}_0\right), \quad (3.4)$$

which, in the special case when

$$m = 0, \qquad \gamma = \alpha, \qquad \text{and} \qquad \delta = \beta$$

yields the Bailey formula (3.1), since [22, p. 117, Eq. 2.3(50)]

$$F_C^{(3)}[\alpha+\beta+1, \beta+1; \alpha+1, \beta+1, \beta+1; x, y, z]$$
$$= (1+x-y-z)^{-\alpha-\beta-1}$$
$$\cdot F_4\left[\tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha+\beta+2); \alpha+1, \beta+1; X, Y\right], \quad (3.5)$$

where, for convenience,

$$X := \frac{4x}{(1+x-y-z)^2} \quad \text{and} \quad Y := \frac{4yz}{(1+x-y-z)^2}. \quad (3.6)$$

Thus, by applying the relationship (1.8), we can easily obtain the following *disguised* form of (3.4):

$$\sum_{n=0}^{\infty} \frac{(m+n)!(\alpha+\beta+m+1)_n}{(\gamma+1)_n(\delta+1)_n} F_{m+n}^{(\alpha,\beta)}(x; a, b, c) F_n^{(\gamma,\delta)}(y; A, B, C) t^n$$
$$= (\alpha+1)_m \{c(a-b)\}^m \left(\frac{x-b}{a-b}\right)^{-\alpha-\beta-m-1}$$
$$\cdot F_C^{(3)}\Bigg[\alpha+\beta+m+1, \alpha+m+1; \alpha+1, \gamma+1, \delta+1;$$
$$\frac{x-a}{x-b}, \frac{cC(a-b)^2(y-A)t}{x-b}, \frac{cC(a-b)^2(y-B)t}{x-b}\Bigg]$$
$$\left(|t|^{1/2} < \left|\frac{A-B}{a-b}\right|^{1/2} \frac{|x-b|^{1/2} - |x-a|^{1/2}}{|y-B|^{1/2} + |y-A|^{1/2}}; m \in \mathbb{N}_0\right), \quad (3.7)$$

which, in view of the reduction formula (3.6), would reduce to the bilinear generating function (3.3) in the special case when

$$m = 0, \qquad \gamma = \alpha, \qquad \text{and} \qquad \delta = \beta.$$

With a view to obtaining numerous families of bilinear, bilateral, or mixed multilateral generating functions for the extended Jacobi polynomials, we first observe that each of the generating functions (2.3) [with α replaced trivially by $\alpha - n$ ($n \in \mathbb{N}_0$)], (2.4) [with β replaced trivially by $\beta - n$ ($n \in \mathbb{N}_0$)], (2.10) [with α and β replaced trivially by $\alpha - n$ and $\beta - n$, respectively ($n \in \mathbb{N}_0$)], (2.12) [with α and β replaced trivially by $\alpha + m$ and $\beta - m$, respectively ($m \in \mathbb{N}_0$)], (2.13) [with α and β replaced trivially by $\alpha - m$ and $\beta + m$, respectively ($m \in \mathbb{N}_0$)], (2.17) [with α replaced trivially by $\alpha - m$ ($m \in \mathbb{N}_0$)], (2.18) [with β replaced trivially by $\beta - m$ ($m \in \mathbb{N}_0$)], (2.19) [with α replaced trivially by $\alpha + m$ ($m \in \mathbb{N}_0$)], and (2.20) [with β replaced trivially by $\beta + m$ ($m \in \mathbb{N}_0$)] fits easily into the Singhal–Srivastava definition [19, p. 755, Eq. (1)]:

$$\sum_{k=0}^{\infty} A_{m,k} S_{m+k}(x) t^k = f(x,t)\bigl\{g(x,t)\bigr\}^{-m} S_m\bigl(h(x,t)\bigr) \qquad (m \in \mathbb{N}_0). \tag{3.8}$$

Thus, by comparing the Singhal–Srivastava generating function (3.8) with the aforementioned (trivially modified) versions of the generating functions (2.3), (2.4), (2.10), (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20), respectively, we obtain the following special cases of (3.8):

$$A_{m,k} = \binom{m+k}{k}, \qquad f = (1-\lambda t)^{\alpha}\{1 - c(x-a)t\}^{-\alpha-\beta-1},$$
$$g = 1 - \lambda t, \qquad h = \frac{x - bc(x-a)t}{1 - c(x-a)t}, \qquad \text{and}$$
$$S_k(x) = F_k^{(\alpha-k,\,\beta)}(x; a, b, c); \tag{3.9}$$

$$A_{m,k} = \binom{m+k}{k}, \qquad f = (1+\lambda t)^{\beta}\{1 - c(x-b)t\}^{-\alpha-\beta-1},$$
$$g = 1 + \lambda t, \qquad h = \frac{x - ac(x-b)t}{1 - c(x-b)t}, \qquad \text{and}$$
$$S_k(x) = F_k^{(\alpha,\,\beta-k)}(x; a, b, c); \tag{3.10}$$

$$\begin{aligned} A_{m,k} &= \binom{m+k}{k}, \qquad f = \{1+c(x-b)t\}^{\alpha}\{1+c(x-a)t\}^{\beta}, \\ g &= \{1+c(x-b)t\}\{1+c(x-a)t\}, \\ h &= x + c(x-a)(x-b)t, \qquad \text{and} \qquad S_k(x) = F_k^{(\alpha-k,\,\beta-k)}(x;a,b,c); \end{aligned} \tag{3.11}$$

$$\begin{aligned} A_{m,k} &= \binom{k-\beta+m-n-1}{k}, \qquad f = (1-t)^{\beta}, \qquad g = 1-t, \\ h &= x-(x-b)t, \qquad \text{and} \qquad S_k(x) = F_n^{(\alpha+k,\,\beta-k)}(x;a,b,c); \end{aligned} \tag{3.12}$$

$$\begin{aligned} A_{m,k} &= \binom{k-\alpha+m-n-1}{k}, \qquad f = (1-t)^{\alpha}, \\ g &= 1-t, \quad h = x-(x-a)t, \quad \text{and} \quad S_k(x) = F_n^{(\alpha-k,\,\beta+k)}(x;a,b,c); \end{aligned} \tag{3.13}$$

$$\begin{aligned} A_{m,k} &= \binom{k-\alpha+m-n-1}{k}, \\ f &= (1-t)^{\alpha}\left(1+\frac{(x-a)t}{a-b}\right)^{n}, \qquad g = 1-t, \\ h &= \frac{(a-b)x+b(x-a)t}{a-b+(x-a)t}, \qquad \text{and} \qquad S_k(x) = F_n^{(\alpha-k,\,\beta)}(x;a,b,c); \end{aligned} \tag{3.14}$$

$$\begin{aligned} A_{m,k} &= \binom{k-\beta+m-n-1}{k}, \\ f &= (1-t)^{\beta}\left(1-\frac{(x-b)t}{a-b}\right)^{n}, \qquad g = 1-t, \\ h &= \frac{(a-b)x-a(x-b)t}{a-b-(x-b)t}, \qquad \text{and} \qquad S_k(x) = F_n^{(\alpha,\,\beta-k)}(x;a,b,c) \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
A_{m,k} &= \binom{\alpha+\beta+m+n+k}{k}, \\
f &= (1-t)^{-\alpha-\beta-n-1}, \qquad g = 1-t, \\
h &= \frac{x-bt}{1-t}, \qquad \text{and} \qquad S_k(x) = F_n^{(\alpha+k,\,\beta)}(x;a,b,c);
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
A_{m,k} &= \binom{\alpha+\beta+m+n+k}{k}, \\
f &= (1-t)^{-\alpha-\beta-n-1}, \qquad g = 1-t, \\
h &= \frac{x-at}{1-t}, \qquad \text{and} \qquad S_k(x) = F_n^{(\alpha,\,\beta+k)}(x;a,b,c).
\end{aligned}
\tag{3.17}
$$

In view of the connections exibited by (3.9) to (3.17), the *entire* development stemming from the Singhal–Srivastava generating function (3.8) would readily apply also to the generating functions (2.3), (2.4), (2.10), (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20). Alternatively, however, by appealing *directly* to each of the generating functions (2.3), (2.4), (2.10), (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20), we can derive a set of nine families of bilinear, bilateral, or mixed multilateral generating functions for the extended Jacobi polynomials, which are given by Theorems 1 to 9 below:

THEOREM 1. *Corresponding to a non-vanishing function* $\Omega_\mu(y_1, \ldots, y_s)$ *of s complex variables* $y_1, \ldots, y_s$ $(s \in \mathbb{N})$ *and of* (*complex*) *order* μ, *let*

$$
\begin{aligned}
&\Lambda^{(1)}_{n,\rho,\sigma}\left[x; y_1, \ldots, y_s; z\right] \\
&\quad := \sum_{k=0}^{\infty} a_k\, F_{n+qk}^{(\alpha-\rho qk,\,\beta+\sigma qk)}(x;a,b,c) \\
&\qquad \cdot \Omega_{\mu+pk}(y_1, \ldots, y_s)\, z^k \\
&\qquad\qquad \left(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}\right)
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
&\Theta^{(1)}_{k,\rho,\sigma}\left(x; y_1, \ldots, y_s; z\right) \\
&\quad := \sum_{j=0}^{[k/q]} \binom{n+k}{k-qj} a_j F_{n+k}^{(\alpha-k+\rho qj,\,\beta+\sigma qj)}(x;a,b,c) \\
&\qquad \cdot \Omega_{\mu+pj}(y_1, \ldots, y_s) z^j,
\end{aligned}
\tag{3.19}
$$

where ρ and σ are suitable complex parameters and (as usual) $[\lambda]$ represents the greatest integer in $\lambda \in \mathbb{R}$.

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(1)}(x; y_1, \ldots, y_s; z)t^k$$
$$= \{1 - c(b-a)t\}^{\alpha}\{1 - c(x-a)t\}^{-\alpha-\beta-n-1}$$
$$\cdot \Lambda_{n,1-\rho,\sigma}^{(1)}\left[\frac{x - bc(x-a)t}{1 - c(x-a)t}; y_1, \ldots, y_s; \frac{z\,t^q\{1 - c(b-a)t\}^{(\rho-1)q}}{\{1 - c(x-a)t\}^{(\rho+\sigma)q}}\right], \tag{3.20}$$

provided that each member of (3.20) *exists.*

THEOREM 2. *Under the hypotheses of Theorem 1, let*

$$\Lambda_{n,\rho,\sigma}^{(2)}[x; y_1, \ldots, y_s; z]$$
$$:= \sum_{k=0}^{\infty} a_k F_{n+qk}^{(\alpha+\rho qk,\, \beta-\sigma qk)}(x; a, b, c)$$
$$\cdot \Omega_{\mu+pk}(y_1, \ldots, y_s)z^k$$
$$(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}) \tag{3.21}$$

and

$$\Theta_{k,\rho,\sigma}^{(2)}(x; y_1, \ldots, y_s; z)$$
$$:= \sum_{j=0}^{[k/q]} \binom{n+k}{k-qj} a_j\, F_{n+k}^{(\alpha+\rho qj,\, \beta-k+\sigma qj)}(x; a, b, c)$$
$$\cdot \Omega_{\mu+pj}(y_1, \ldots, y_s)z^j, \tag{3.22}$$

where ρ and σ are suitable complex parameters.

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(2)}(x; y_1, \ldots, y_s; z)t^k$$
$$= \{1 + c(b-a)t\}^{\beta}\{1 - c(x-b)t\}^{-\alpha-\beta-n-1}$$
$$\cdot \Lambda_{n,\rho,1-\sigma}^{(2)}\left[\frac{x - ac(x-b)t}{1 - c(x-b)t}; y_1, \ldots, y_s; \frac{zt^q\{1 + c(b-a)t\}^{(\sigma-1)q}}{\{1 - c(x-b)t\}^{(\rho+\sigma)q}}\right], \tag{3.23}$$

provided that each member of (3.23) *exists.*

THEOREM 3. *Under the hypotheses of Theorem* 1, *let*

$$\begin{aligned}\Lambda^{(3)}_{n,\rho,\sigma}&[x;y_1,\ldots,y_s;z]\\&:=\sum_{k=0}^{\infty}a_kF^{(\alpha-\rho qk,\,\beta-\sigma qk)}_{n+qk}(x;a,b,c)\\&\quad\cdot\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k\\&\qquad\qquad\big(a_k\neq 0;n\in\mathbb{N}_0;\,p,q\in\mathbb{N}\big)\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}\Theta^{(3)}_{k,\rho,\sigma}&\big(x;\,y_1,\ldots,y_s;z\big)\\&:=\sum_{j=0}^{[k/q]}\binom{n+k}{k-qj}a_jF^{(\alpha-k+\rho qj,\,\beta-k+\sigma qj)}_{n+k}(x;a,b,c)\\&\quad\cdot\Omega_{\mu+pj}(y_1,\ldots,y_s)z^j,\end{aligned}\tag{3.25}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\begin{aligned}\sum_{k=0}^{\infty}&\Theta^{(3)}_{k,\rho,\sigma}\big(x;y_1,\ldots,y_s;z\big),\,t^k\\&=\{1+c(x-b)t\}^{\alpha}\{1+c(x-a)t\}^{\beta}\\&\quad\cdot\Lambda^{(3)}_{n,1-\rho,1-\sigma}\big[x+c(x-a)(x-b)t;y_1,\ldots,y_s;\\&\qquad zt^q\{1+c(x-b)t\}^{(\rho-1)q}\{1+c(x-a)t\}^{(\sigma-1)q}\big],\end{aligned}\tag{3.26}$$

provided that each member of (3.26) *exists.*

THEOREM 4. *Under the hypotheses of Theorem* 1, *let*

$$\begin{aligned}\Lambda^{(4)}_{n,\rho,\sigma}&[x;y_1,\ldots,y_s;z]\\&:=\sum_{k=0}^{\infty}a_kF^{(\alpha+\rho qk,\,\beta-\sigma qk)}_{n}(x;a,b,c)\\&\quad\cdot\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k\\&\qquad\qquad\big(a_k\neq 0;n\in\mathbb{N}_0;\,p,q\in\mathbb{N}\big)\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}\Theta^{(4)}_{k,\rho,\sigma}&\big(x;y_1,\ldots,y_s;z\big)\\&:=\sum_{j=0}^{[k/q]}\binom{k-\beta+\sigma qj-n-1}{k-qj}a_j\Omega_{\mu+pj}(y_1,\ldots,y_s)\\&\quad\cdot F^{(\alpha+k+\rho qj,\,\beta-k-\sigma qj)}_{n}(x;a,b,c)\,z^j,\end{aligned}\tag{3.28}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(4)}\big(x; y_1, \ldots, y_s; z\big) t^k$$
$$= (1-t)^{\beta}$$
$$\cdot \Lambda_{n,\rho+1,\sigma+1}^{(4)}\left[x-(x-b)t; y_1, \ldots, y_s; \frac{z\, t^q}{(1-t)^{(\sigma+1)q}}\right], \quad (3.29)$$

provided that each member of (3.29) *exists.*

THEOREM 5. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda_{n,\rho,\sigma}^{(5)}\big[x; y_1, \ldots, y_s; z\big]$$
$$:= \sum_{k=0}^{\infty} a_k F_n^{(\alpha-\rho q k,\, \beta+\sigma q k)}(x; a, b, c)$$
$$\cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k$$
$$\big(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}\big) \quad (3.30)$$

and

$$\Theta_{k,\rho,\sigma}^{(5)}\big(x; y_1, \ldots, y_s; z\big)$$
$$:= \sum_{j=0}^{[k/q]} \binom{k-\alpha+\rho q j-n-1}{k-qj} a_j \Omega_{\mu+pj}(y_1, \ldots, y_s)$$
$$\cdot F_n^{(\alpha-k-\rho q j,\, \beta+k+\sigma q j)}(x; a, b, c) z^j, \quad (3.31)$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(5)}\big(x; y_1, \ldots, y_s; z\big) t^k$$
$$= (1-t)^{\alpha}$$
$$\cdot \Lambda_{n,\rho+1,\sigma+1}^{(5)}\left[x-(x-a)t; y_1, \ldots, y_s; \frac{z t^q}{(1-t)^{(\rho+1)q}}\right], \quad (3.32)$$

provided that each member of (3.32) *exists.*

THEOREM 6. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda^{(6)}_{n,\rho,\sigma}[x;y_1,\ldots,y_s;z] := \sum_{k=0}^{\infty} a_k F_n^{(\alpha-\rho qk,\,\beta-\sigma qk)}(x;a,b,c) \cdot \Omega_{\mu+pk}(y_1,\ldots,y_s)z^k \quad (a_k \neq 0;\, n \in \mathbb{N}_0;\, p,q \in \mathbb{N}) \tag{3.33}$$

and

$$\Theta^{(6)}_{k,\rho,\sigma}(x;y_1,\ldots,y_s;z) := \sum_{j=0}^{[k/q]} \binom{k-\alpha+\rho qj-n-1}{k-qj} a_j \Omega_{\mu+pj}(y_1,\ldots,y_s) \cdot F_n^{(\alpha-k-\rho qj,\,\beta-\sigma qj)}(x;a,b,c)z^j, \tag{3.34}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta^{(6)}_{k,\rho,\sigma}(x;y_1,\ldots,y_s;z)t^k = (1-t)^{\alpha}\left(1+\frac{(x-a)t}{a-b}\right)^n \cdot \Lambda^{(6)}_{n,\rho+1,\sigma}\left[\frac{(a-b)x+b(x-a)t}{a-b+(x-a)t};y_1,\ldots,y_s;\frac{z\,t^q}{(1-t)^{(\rho+1)q}}\right], \tag{3.35}$$

provided that each member of (3.35) *exists.*

THEOREM 7. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda^{(7)}_{n,\rho,\sigma}[x;y_1,\ldots,y_s;z] := \sum_{k=0}^{\infty} a_k F_n^{(\alpha-\rho qk,\,\beta-\sigma qk)}(x;a,b,c) \cdot \Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k \quad (a_k \neq 0;\, n \in \mathbb{N}_0;\, p,q \in \mathbb{N}) \tag{3.36}$$

and

$$\Theta^{(7)}_{k,\rho,\sigma}(x;\,y_1,\ldots,y_s;z) := \sum_{j=0}^{[k/q]} \binom{k-\beta+\sigma qj-n-1}{k-qj} a_j \Omega_{\mu+pj}(y_1,\ldots,y_s) \cdot F_n^{(\alpha-\rho qj,\,\beta-k-\sigma qj)}(x;a,b,c)z^j, \tag{3.37}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(7)}\big(x;\, y_1, \ldots, y_s; z\big) t^k$$
$$= (1-t)^{\beta}\left(1 - \frac{(x-b)t}{a-b}\right)^n$$
$$\cdot \Lambda_{n,\rho,\sigma+1}^{(7)}\left[\frac{(a-b)x - a(x-b)t}{a-b-(x-b)t}; y_1, \ldots, y_s; \frac{zt^q}{(1-t)^{(\sigma+1)q}}\right], \tag{3.38}$$

provided that each member of (3.38) *exists.*

THEOREM 8. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda_{n,\rho,\sigma}^{(8)}\big[x; y_1, \ldots, y_s; z\big]$$
$$:= \sum_{k=0}^{\infty} a_k F_n^{(\alpha+\rho qk,\, \beta+\sigma qk)}(x; a, b, c)$$
$$\cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k$$
$$\big(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}\big) \tag{3.39}$$

and

$$\Theta_{k,\rho,\sigma}^{(8)}\big(x;\, y_1, \ldots, y_s; z\big)$$
$$:= \sum_{j=0}^{[k/q]} \binom{\alpha+\beta-(\rho+\sigma)qj+n+k}{k-qj} a_j \Omega_{\mu+pj}(y_1, \ldots, y_s)$$
$$\cdot F_n^{(\alpha+k-\rho qj,\, \beta-\sigma qj)}(x; a, b, c)\, z^j, \tag{3.40}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta_{k,\rho,\sigma}^{(8)}\big(x; y_1, \ldots, y_s; z\big) t^k = (1-t)^{-\alpha-\beta-n-1}$$
$$\cdot \Lambda_{n,1-\rho,\sigma}^{(8)}\left[\frac{x-bt}{1-t}; y_1, \ldots, y_s; \frac{z\,t^q}{(1-t)^{(1-\rho-\sigma)q}}\right], \tag{3.41}$$

provided that each member of (3.41) *exists.*

THEOREM 9. *Under the hypotheses of Theorem* 1, *let*

$$\Lambda_{n,\rho,\sigma}^{(9)}\big[x; y_1, \ldots, y_s; z\big]$$
$$:= \sum_{k=0}^{\infty} a_k F_n^{(\alpha-\rho qk,\, \beta+\sigma qk)}(x; a, b, c)$$
$$\cdot \Omega_{\mu+pk}(y_1, \ldots, y_s) z^k$$
$$\big(a_k \neq 0; n \in \mathbb{N}_0; p, q \in \mathbb{N}\big) \tag{3.42}$$

and

$$\Theta^{(9)}_{k,\,\rho,\,\sigma}\big(x;\, y_1, \ldots, y_s; z\big) := \sum_{j=0}^{[k/q]} \binom{\alpha+\beta-(\rho+\sigma)qj+n+k}{k-qj} a_j \Omega_{\mu+pj}(y_1, \ldots, y_s) \cdot F_n^{(\alpha-\rho qj,\, \beta+k-\sigma qj)}(x; a, b, c) z^j, \tag{3.43}$$

where ρ *and* σ *are suitable complex parameters.*

Then

$$\sum_{k=0}^{\infty} \Theta^{(9)}_{k,\,\rho,\,\sigma}\big(x; y_1, \ldots, y_s; z\big) t^k = (1-t)^{-\alpha-\beta-n-1} \cdot \Lambda^{(9)}_{n,\,\rho,\,1-\sigma}\left[\frac{x-at}{1-t}; y_1, \ldots, y_s; \frac{z\,t^q}{(1-t)^{(1-\rho-\sigma)q}}\right], \tag{3.44}$$

provided that each member of (3.44) *exists.*

Proofs of Theorems 1 *to* 9. We give a *direct* proof of Theorem 1 only; each of the other Theorems 2 to 9 can indeed be proven in a similar manner.

Denote, for convenience, the left-hand side of the assertion (3.20) of Theorem 1 by $\mathcal{S}$. Then, upon substituting for the polynomials

$$\Theta^{(1)}_{k,\,\rho,\,\sigma}\big(x; y_1, \ldots, y_s; z\big)$$

from (3.19) into the left-hand side of (3.20), we obtain

$$\begin{aligned}\mathcal{S} &= \sum_{k=0}^{\infty} t^k \sum_{j=0}^{[k/q]} \binom{n+k}{k-qj} a_j F_{n+k}^{(\alpha-k+\rho qj,\, \beta+\sigma qj)}(x; a, b, c) \cdot \Omega_{\mu+pj}(y_1, \ldots, y_s) z^j \\ &= \sum_{j=0}^{\infty} a_j \Omega_{\mu+pj}(y_1, \ldots, y_s)(zt^q)^j \cdot \sum_{k=0}^{\infty} \binom{n+k+qj}{k} F_{n+k+qj}^{(\alpha-k-(1-\rho)qj,\, \beta+\sigma qj)}(x; a, b, c) t^k,\end{aligned} \tag{3.45}$$

by inverting the order of the double summation involved.

The inner series in (3.45) can be summed by applying the generating function (2.3) [with α, β, and n replaced by

$$\alpha-(1-\rho)qj, \qquad \beta+\sigma qj, \qquad \text{and} \qquad n+qj,$$

respectively $(q \in \mathbb{N}; j \in \mathbb{N}_0; \rho, \sigma \in \mathbb{C})$], and we thus find from (3.45) and (2.3) that

$$\begin{aligned}\mathscr{S} &= \{1 - c(b-a)t\}^{\alpha}\{1 - c(x-a)t\}^{-\alpha-\beta-n-1} \\ &\quad \cdot \sum_{j=0}^{\infty} a_j F_{n+qj}^{(\alpha-(1-\rho)qj,\, \beta+\sigma qj)}\left(\frac{x - bc(x-a)t}{1 - c(x-a)t}; a, b, c\right) \\ &\quad \cdot \Omega_{\mu+pj}(y_1, \ldots, y_s)\left(\frac{zt^q\{1 - c(b-a)t\}^{(\rho-1)q}}{\{1 - c(x-a)t\}^{(\rho+\sigma)q}}\right)^j \\ &\qquad \left(|t| < \min\left\{[c(b-a)]^{-1}, |c(x-a)|^{-1}\right\}\right). \qquad (3.46)\end{aligned}$$

Now, upon interpreting this last infinite series in (3.46) by means of the definition (3.18), we arrive immediately at the right-hand side of the assertion (3.20) of Theorem 1.

This evidently completes the direct proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true (at least as a relation between formal power series) for those values of the various parameters and variables involved for which each member of the assertion (3.20) exists.

The direct proof of each of Theorems 2 to 9 is much akin to that of Theorem 1, which we already have detailed here fairly adequately. In place of the generating function (2.3) used in proving Theorem 1, we shall require the generating functions (2.4), (2.10), (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20) in proving Theorems 2 to 9, respectively. The details are being omitted here.

For each suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function

$$\Omega_{\mu}(y_1, \ldots, y_s) \qquad \left(s \in \mathbb{N} \setminus \{1\}\right)$$

is expressed as an appropriate product of several simpler functions, each of our results (Theorems 1 to 9 above) can be applied to derive various families of mixed multilateral generating functions for the extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x; a, b, c)$. We choose to leave the details involved in these applications of Theorems 1 to 9 as an exercise for the interested reader.

In terms of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, Theorem 3 (and hence also its essentially equivalent forms asserted by Theorems 1 and 2) was given, over one decade ago, by Srivastava and Popov [23] (see also Srivastava and Handa [21] for further extensions of Theorems 1, 2, and 3 involving a general sequence of functions defined by a Rodrigues formula). Furthermore, the special cases of Theorem 1, 2, and 3 above when

$\rho = \sigma = 0$ correspond to the known families of mixed multilateral generating functions for $P_n^{(\alpha-n,\,\beta)}(x)$, $P_n^{(\alpha,\,\beta-n)}(x)$, and $P_n^{(\alpha-n,\,\beta-n)}(x)$, which were given, almost two decades ago, by Srivastava [20, p. 230, Corollaries 5, 6, and 7] and which were subsequently reproduced in the treatise on the subject of generating functions by Srivastava and Manocha [22, pp. 423–424, Corollaries 5, 6, and 7].

Each of our Theorems 4 to 9, on the other hand, can be deduced *alternatively* from a general family of mixed multilateral generating functions, which was given recently by Chen and Srivastava [3, p. 180, Theorem 1] (see also [3, p. 182, Theorem 2] for a general multivariable extension) for the sequence $\{\zeta_k^{(\lambda,\,r)}(z)\}_{k=0}^{\infty}$ defined by (cf. [3, p. 171, Eq. (5.14)])

$$\begin{aligned}\zeta_k^{(\lambda,\,r)}(z) &= \zeta_k^{(\lambda,\,r)}\big[\alpha_1,\ldots,\alpha_u;\beta_1,\ldots,\beta_v : z\big]\\ &:= {}_uF_{v+r}\big(\alpha_1,\ldots,\alpha_u;\Delta(r;1-\lambda-k),\ \beta_1,\ldots,\beta_v;z\big),\end{aligned}\tag{3.47}$$

where, for convenience, $\Delta(r;\lambda)$ abbreviates the array of r parameters

$$\frac{\lambda}{r}, \frac{\lambda+1}{r}, \ldots, \frac{\lambda+r-1}{r} \qquad (r \in \mathbb{N}).$$

Indeed the sequence $\{\zeta_k^{(\lambda,\,r)}(z)\}_{k=0}^{\infty}$ possesses the following generating function (cf. [3, p. 171, Eq. (5.15)]):

$$\begin{aligned}&\sum_{k=0}^{\infty}\binom{\lambda+m+k-1}{k}\zeta_{m+k}^{(\lambda,\,r)}(z)t^k\\ &\quad= (1-t)^{-\lambda-m}\zeta_m^{(\lambda,\,r)}\big(z(1-t)^r\big) \qquad \big(m \in \mathbb{N}_0; |t| < 1\big),\end{aligned}\tag{3.48}$$

which obviously is a special case of the Singhal–Srivastava generating function (3.8) when

$$\begin{aligned}A_{m,\,k} &= \binom{\lambda+m+k-1}{k}, \qquad f = (1-t)^{-\lambda}, \qquad g = 1-t,\\ h &= x(1-t)^r, \qquad \text{and} \qquad S_k(x) = \zeta_k^{(\lambda,\,r)}(x).\end{aligned}\tag{3.49}$$

It is not difficult to verify that each of the generating functions (2.12), (2.13), (2.17), (2.18), (2.19), and (2.20), upon which the assertions of Theorems 4 to 9 are based rather heavily, would follow from (3.48) in its special cases when

$$r = 1, \qquad u-2 = v = 0, \qquad \text{and} \quad \alpha_1 = -n\ (n \in \mathbb{N}_0)\tag{3.50}$$

with appropriate choices for α_2, λ, and z.

We conclude this paper by remarking further that various very specialized versions of the many families of bilinear, bilateral, or mixed multilateral generating functions for the Jacobi (or the extended Jacobi) and related polynomials, which we have considered in this section rather systematically, continue to be rederived by one method or the other in the literature on bilateral generating functions since the publication of the monograph by Srivastava and Manocha [22] (especially see [22, Chap. 8]). To the numerous references cited by Chen and Srivastava [3] for rederivations of *obvious* special cases of readily accessible *known* results on bilateral generating functions, we should add the *main results* in the works of (among others) Hazra [10], Chongdar [4, 5], Majumdar and Chongdar [11], and Mukherjee [14, 15]. In particular, Chongdar [4, 5] gave two very specialized cases of Theorem 1 when

$$n = 0, \qquad q = 1, \qquad \rho = \sigma = 0, \qquad \text{and} \qquad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{3.51}$$

and

$$n = 0, \qquad q = 1, \qquad \rho = \sigma = 0, \qquad \text{and} \qquad s = 1, \tag{3.52}$$

respectively, while (much more recently) Mukherjee [14, 15] gave two very specialized cases of the equivalent results (Theorem 2 and Theorem 1) when

$$n = 0, \qquad q = 1, \qquad \rho = \sigma - 1 = 0, \qquad \text{and} \qquad \Omega_\mu(y_1, \ldots, y_s) \equiv 1 \tag{3.53}$$

and

$$q = 1, \qquad \rho - 1 = \sigma = 0, \qquad \text{and} \qquad \Omega_\mu(y_1, \ldots, y_s) \equiv 1, \tag{3.54}$$

respectively. The *main result* of Hazra [10], on the other hand, happens to be a very specialized case of Theorem 3.

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