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Existence and Multiplicity of Solutions of Semilinear Elliptic Equations[1]

Chun-Lei Tang and Xing-Ping Wu

Department of Mathematics, Southwest Normal University, Chongqing 400715, People's Republic of China

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The existence and multiplicity results of solutions are obtained by the reduction method and the minimax methods for nonautonomous semilinear elliptic Dirichlet boundary value problem. Some well-known results are generalized.

Key Words: semilinear elliptic equation; Dirichlet boundary value problem; nontrivial solution; reduction method; critical point; (PS) condition; Sobolev's embedding theorem.

1. INTRODUCTION AND MAIN RESULTS

Consider the semilinear elliptic Dirichlet boundary problem,

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^N$ $(N \geq 1)$ is a bounded smooth domain and f: $\overline{\Omega} \times R \to R$ is a Carathéodory function. Throughout this paper we assume that there are a positive constant C_1 and a real function $f_0 \in L^q(\Omega)$ such that

$$|f(x,t)| \leq C_1 |t|^{p-1} + f_0(x) \qquad \text{and} \qquad f(\cdot, 0) \in L^\infty(\Omega) \tag{2}$$

for all $t \in R$ and a.e. $x \in \Omega$, where $p \in]2, \frac{2N}{N-2}[$ for $N \geq 3$, $p \in]2, +\infty[$ for $N = 1, 2$, and $1/q + 1/p = 1$. Let

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

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be the sequence of the distinct eigenvalues of the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

and let k be a fixed positive integer.

With the reduction method, the existence and multiplicity results of solutions are obtained for the nonresonant or resonant elliptic problem (1) (see [1, 2, 4–6, 8] and their references). Motivated by the results in [8], Cac [4] proved the following theorem under the condition that there are positive constants c_1, c_2 such that

$$|f(x,t)| \le c_1 + c_2|t| \tag{i}$$

for all $t \in R$ and a.e. $x \in \Omega$.

THEOREM A [4]. *Suppose that*

(P1) *There exists a real function* $a \in L^\infty(\Omega)$ *with* $a(x) \le \lambda_{k+1}$ *for a.e.* $x \in \Omega$,

$$\operatorname{meas}\{x \in \Omega \mid a(x) < \lambda_{k+1}\} > 0 \tag{ii}$$

and

$$\frac{f(x,s) - f(x,t)}{s - t} \le a(x) \tag{iii}$$

for all $s, t \in R$, $s \neq t$, *and a.e.* $x \in \Omega$.

(P2) *There exist real functions* $h_1, h_2 \in L^\infty(\Omega \times R)$ *and a constant* $M > 0$ *such that*

$$\frac{\lambda_k t + h_1(x,t)}{t} \le \frac{f(x,t)}{t} \le \frac{\lambda_{k+1} t - h_2(x,t)}{t} \tag{iv}$$

for all $|t| \ge M$ *and a.e.* $x \in \Omega$. *Moreover, we assume that*

$$\int_\Omega H_1(x,u)\,dx \to +\infty \tag{v}$$

as $\|u\| \to \infty$ *in* $E(\lambda_k)$, *and that*

$$\int_\Omega H_2(x,u)\,dx \to +\infty \tag{vi}$$

as $\|u\| \to \infty$ *in* $E(\lambda_{k+1})$, *where* $H_i(x,t) = \int_0^t h_i(x,s)\,ds$, $i = 1, 2$.

(I) *Then under the assumptions* (P1) *and* (P2) *the problem in* (1) *has at least one solution in* $H_0^1(\Omega)$.

(II) *Assume in addition to* (P1) *and* (P2) *that*

(P3) $f(x,0) = 0$ *for a.e.* $x \in \Omega$ *and there exists* $\delta > 0$ *such that*

$$\frac{f(x,t)}{t} < \lambda_k \tag{vii}$$

for all $0 < |t| \le \delta$ *and a.e.* $x \in \Omega$.

Then the problem in (1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

(III) *In the case* $k > 1$ *assume in addition to* (P1), (P2), *and* (P3) *that*

(P4)

$$\frac{f(x,t)}{t} \ge \lambda_{k-1} \tag{viii}$$

for all $t \ne 0$ *and a.e.* $x \in \Omega$.

Then the problem in (1) *has at least two nontrivial solutions if the eigenvalue* λ_k *is simple.*

Motivated by [4], *we obtain some existence and multiplicity results which generalize the results mentioned above. Our approach is based on the reduction method and the minimax methods. The main results are the following theorems.*

THEOREM 1. *Suppose that* (2) *holds and that there exists a real function* $a \in L^\infty(\Omega)$ *with* $a(x) \le \lambda_{k+1}$ *for a.e.* $x \in \Omega$,

$$\text{meas}\{x \in \Omega \mid a(x) < \lambda_{k+1}\} > 0 \tag{3}$$

and

$$\frac{f(x,s) - f(x,t)}{s-t} \le a(x) \tag{4}$$

for all $s, t \in R$, $s \ne t$, *and a.e.* $x \in \Omega$. *Assume that there exist* $M > 0$ *and a measurable function* $h\colon \Omega \times R \to R$ *with*

$$|h(x,t)| \le g(x) \tag{5}$$

for all $t \in R$, *a.e.* $x \in \Omega$, *and some real function* $g \in L^1(\Omega)$, *and*

$$\int_\Omega H(x, u)\, dx \to +\infty \tag{6}$$

as $\|u\| \to \infty$ *in* $E(\lambda_k)$, *such that*

$$\frac{f(x,t)}{t} \geq \frac{\lambda_k t + h(x,t)}{t} \tag{7}$$

for all $|t| \geq M$ *and a.e.* $x \in \Omega$, *where* $H(x,t) = \int_0^t h(x,s)\, ds$. *Then the problem in* (1) *has at least one solution in* $H_0^1(\Omega)$.

Remark 1. Theorem 1 generalizes (I) of Theorem A. In fact, Theorem 1 does not need condition (vi) and the right inequality of (iv) of Theorem A. Moreover, (5) is weaker than $h_1 \in L^\infty(\Omega \times R)$ and (2) is weaker than (i). There are functions f satisfying our Theorem 1 and not satisfying Theorem A. For example, let

$$f(x,t) = \lambda_k t + (\lambda_{k+1} - \lambda_k)\alpha(t)\beta(x),$$

where

$$\alpha(t) = \begin{cases} t/(1+t^2) & |t| \leq 1 \\ t/(2|t) & |t| \geq 1 \end{cases}$$

and

$$\beta(x) = \begin{cases} -|x - x_0|^{-(2N-1)/4} & |x - x_0| \leq r \\ 1 & |x - x_0| \geq r. \end{cases}$$

Choose $x_0 \in \Omega$, r small enough that the ball $B(x_0, r) \subset \Omega$, and $\int_\Omega \beta(x)\, dx > 0$. Then f satisfies our Theorem 1 and does not satisfy (iv) and (v) in Theorem A.

THEOREM 2. *Suppose that* (2)–(7) *hold and that there exist a real function* $b(\cdot) \in L^\infty(\Omega)$ *with* $b(x) \leq \lambda_k$ *for a.e.* $x \in \Omega$,

$$\operatorname{meas}\{x \in \Omega \mid b(x) < \lambda_k\} > 0, \tag{8}$$

and $\delta > 0$ *such that*

$$\frac{f(x,t)}{t} \leq b(x) \tag{9}$$

for all $0 < |t| \leq \delta$ *and a.e.* $x \in \Omega$. *Then the problem in* (1) *has at least one nontrivial solution in* $H_0^1(\Omega)$.

Remark 2. Theorem 2 generalizes (II) of Theorem A in [4]. In fact, besides the reason in Remark 1, (9) is weaker than (vii) of Theorem A. There are functions f satisfying our Theorem 2 and not satisfying Theorem A. For example, let

$$f(x,t) = \begin{cases} b(x)t & |t| \leq 1/2 \\ (2b(x) - \lambda)t + (\lambda - b(x))|t|t + (\lambda - b(x))t/(4|t|) & \\ & 1/2 \leq |t| \leq 1 \\ \lambda t - (\lambda - b(x))3t/(4|t|) & |t| \geq 1, \end{cases}$$

where λ is a constant in $]\lambda_k, \lambda_{k+1}[$ and $b(x)$ is a real function in $L^\infty(\Omega)$ with $b(x) \leq \lambda_k$ for a.e. $x \in \Omega$,

$$\operatorname{meas}\{x \in \Omega \mid b(x) < \lambda_k\} > 0.$$

THEOREM 3. *Suppose that* (2)–(7) *hold and that there exist* $0 < m \leq k$ *and a real function* $b(\cdot) \in L^\infty(\Omega)$ *with* $b(x) \leq \lambda_m$ *for a.e.* $x \in \Omega$,

$$\operatorname{meas}\{x \in \Omega \mid b(x) < \lambda_m\} > 0, \tag{10}$$

and $\delta > 0$ *such that*

$$\lambda_{m-1} \leq \frac{f(x,t)}{t} \leq b(x) \tag{11}$$

for all $0 < |t| \leq \delta$ *and a.e.* $x \in \Omega$. *Then the problem in* (1) *has at least two nontrivial solutions in* $H_0^1(\Omega)$.

Remark 3. Theorem 3 generalizes (III) of Theorem A. In fact, besides the reason in Remarks 1 and 2, Theorem 3 does not need the condition that the eigenvalue λ_k is simple, and the left inequality of (11) is weaker than (viii) of Theorem A. There are functions f satisfying our Theorem 3 and not satisfying the corresponding results in [1, 2, 4–6, 8]. For example, let

$$f(x,t) = \begin{cases} b(x)t & |t| \leq 1/2 \\ (2b(x) - \lambda)t + (\lambda - b(x))|t|t + (\lambda - b(x))t/(4|t|) & \\ & 1/2 \leq |t| \leq 1 \\ \lambda t - (\lambda - b(x))3t/(4|t|) & |t| \geq 1, \end{cases}$$

where λ is a constant in $]\lambda_k, \lambda_{k+1}[$ and $b(x)$ is a real function in $L^\infty(\Omega)$ with $\lambda_{m-1} \le b(x) \le \lambda_m$ $(0 < m \le k)$ for a.e. $x \in \Omega$, and

$$\operatorname{meas}\{x \in \Omega \mid b(x) < \lambda_m\} > 0.$$

2. PROOF OF THEOREMS

Define the functional φ on the Sobolev space $H_0^1(\Omega)$ by

$$\varphi(u) = -\tfrac{1}{2}\|u\|^2 + \int_\Omega F(x,u)\,dx, \qquad u \in H_0^1(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)\,ds$, and $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ is the usual norm in $H_0^1(\Omega)$. Then φ is continuously differentiable and

$$\langle \varphi'(u), v\rangle = -\int_\Omega \nabla u\,\nabla v\,dx + \int_\Omega f(x,u)v\,dx$$

for $u, v \in H_0^1(\Omega)$. It is well known that $u \in H_0^1(\Omega)$ is a solution of problem (1) if and only if u is a critical point of φ. Let

$$H_k = E(\lambda_1) \oplus \cdots \oplus E(\lambda_k),$$

$V = H_k$, and $W = V^\perp$, where $E(\lambda_i)$ stands for the eigenspace corresponding to λ_i, i.e., the finite-dimensional space spanned by the eigenfunctions corresponding to λ_i. Define the functional ψ,

$$\psi(v) = \sup_{w \in W} \varphi(v + w), \qquad v \in V.$$

LEMMA 1. *Suppose that $a(\cdot) \in L^\infty(\Omega)$ with $a(x) \le \lambda_l$ for a.e. $x \in \Omega$, and*

$$\operatorname{meas}\{x \in \Omega \mid a(x) < \lambda_l\} > 0.$$

Then there exists a constant $a_0 < 1$ such that

$$\int_\Omega a(x)w^2\,dx \le a_0 \int_\Omega |\nabla w|^2\,dx$$

for all $w \in H_{l-1}^\perp$.

Proof. If not, there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset H_{l-1}^{\perp}$ such that

$$\int_{\Omega} a(x) w_n^2 \, dx > \left(1 - \frac{1}{n}\right) \int_{\Omega} |\nabla w_n|^2 \, dx$$

for all n, which implies that $w_n \neq 0$ for all n. By the homogeneity of the above inequality we may assume that $\|w_n\| = 1$ and

$$\int_{\Omega} a(x) w_n^2 \, dx > 1 - \frac{1}{n} \tag{12}$$

for all n. It follows from the weak compactness of the unit ball of $H_{l-1}^{\perp}$ that there exists a subsequence, say $\{w_n\}$, such that w_n weakly converges to w in $H_{l-1}^{\perp}$. Now Sobolev's embedding theorem suggests that w_n converges to w in $L^2(\Omega)$. From (12) we obtain

$$\int_{\Omega} a(x) w^2 \, dx \geq 1.$$

Moreover, one has

$$1 \geq \int_{\Omega} |\nabla w|^2 \, dx \geq \lambda_l \int_{\Omega} |w|^2 \, dx \geq \int_{\Omega} a(x) |w|^2 \, dx \geq 1.$$

Hence we have

$$1 = \int_{\Omega} |\nabla w|^2 \, dx = \lambda_l \int_{\Omega} |w|^2 \, dx$$

and

$$\int_{\Omega} (\lambda_l - a(x)) |w|^2 \, dx = 0,$$

which implies that $w \in E(\lambda_l) \backslash \{0\}$ and $w = 0$ on a positive measure subset. It contradicts the unique continuation property of the eigenfunction.

LEMMA 2. *Suppose that* (2) *and* (4) *hold. Then* $\psi: V \to R$ *is continuously differentiable and*

$$\psi'(v) = P_V \varphi'(v + \theta(v)), \qquad v \in V,$$

where $P_V: H_0^1(\Omega) \to V$ *is the corresponding projection onto* V *along* W, *and* $\theta: V \to W$ *is a continuous mapping satisfying*

$$\psi(v) = \varphi(v + \theta(v))$$

for every $v \in V$.

It is a simple corollary that $v + \theta(v)$ is a critical point of φ if v is a critical point of ψ.

Proof. For $v \in V$, $w \in W$, it follows from (4) and Lemma 1 that

$$\begin{aligned}
&\left\langle -\phi'_w(v + w_1) - (-\phi'_w(v + w_2)), w_1 - w_2 \right\rangle \\
&\quad = \int_\Omega |\nabla(w_1 - w_2)|^2\, dx \\
&\qquad - \int_\Omega (f(x, v + w_1) - f(x, v + w_2))(w_1 - w_2)\, dx \\
&\quad \geq \int_\Omega |\nabla(w_1 - w_2)|^2\, dx - \int_\Omega a(x)(w_1 - w_2)^2\, dx \\
&\quad \geq (1 - a_0)\int_\Omega |\nabla(w_1 - w_2)|^2\, dx.
\end{aligned}$$

It follows easily from Theorem 2.3 of Amann [1] that the lemma holds.

LEMMA 3. *Suppose that* (2) *and* (5)–(7) *hold. Then φ is coercive on V, i.e.,*

$$\varphi(v) \to +\infty \qquad \text{as } \|v\| \to \infty \text{ in } V.$$

Thus ψ is coercive.

Proof. Let $Q(x,t) = f(x,t) - \lambda_k t - h(x,t)$ and

$$C(x) = \min\left\{ \inf_{0 \leq t < M} Q(x,t), \sup_{-M < t \leq 0} Q(x,t) \right\}.$$

Then one has

$$Q(x,t)t \geq \begin{cases} 0 & |t| \geq M \\ C(x)t & |t| < M \end{cases}$$

for a.e. $x \in \Omega$. Hence we have

$$Q(x,st)t \geq \begin{cases} 0 & s \geq \dfrac{M}{|t|} \\[2ex] C(x)t & 0 < s < \dfrac{M}{|t|} \end{cases}$$

for a.e. $x \in \Omega$. Integrating the two sides in s over $[0, 1]$, we obtain

$$\int_0^1 Q(x, st)t\,ds \geq \int_0^{M/|t|} C(x)t\,ds = C(x)M\frac{t}{|t|}$$

for all $t \neq 0$ and a.e. $x \in \Omega$, which implies that

$$F(x,t) \geq \tfrac{1}{2}\lambda_k t^2 + H(x,t) - |C(x)|M$$

for all $t \in R$ and a.e. $x \in \Omega$. Hence for $u \in H_{k-1}$, $v \in E(\lambda_k)$, we have

$$\begin{aligned}
\psi(u+v) &\geq \varphi(u+v) \\
&= -\tfrac{1}{2}\int_\Omega |\nabla(u+v)|^2\,dx + \int_\Omega F(x, u+v)\,dx \\
&\geq -\tfrac{1}{2}\int_\Omega |\nabla(u+v)|^2\,dx + \tfrac{1}{2}\lambda_k\int_\Omega (u+v)^2\,dx \\
&\quad + \int_\Omega H(x, u+v)\,dx - \int_\Omega M|C(x)|\,dx \\
&= -\tfrac{1}{2}\int_\Omega |\nabla(u+v)|^2\,dx + \tfrac{1}{2}\lambda_k\int_\Omega (u+v)^2 dx + \int_\Omega H(x,v)\,dx \\
&\quad + \int_\Omega \int_0^1 h(x, su+v)u\,dt\,dx - \int_\Omega M|C(x)|\,dx \\
&\geq \tfrac{1}{2}(\lambda_k - \lambda_{k-1})\int_\Omega u^2\,dx - \int_\Omega g(x)\,dx\|u\|_\infty \\
&\quad + \int_\Omega H(x,v)\,dx - \int_\Omega M|C(x)|\,dx,
\end{aligned}$$

which implies that φ is coercive on V by (6) and the equivalence of the norms $\|u\|_{L^2}$ and $\|u\|_\infty$ in the finite-dimensional space H_{k-1}.

LEMMA 4. *Let* $V_2 = E(\lambda_m) + \cdots + E(\lambda_k)$. *Suppose that* (3), (4), *and* (11) *hold. Then there exists* $\delta_0 > 0$ *such that*

$$\psi(v) \leq 0 \qquad \text{for } v \in V_2 \text{ with } \|v\| \leq \delta_0.$$

Proof. It follows from (4) that

$$f(x,t)t \leq f(x,0)t + a(x)t^2$$

for all $t \in R$ and a.e. $x \in \Omega$. Hence one has

$$f(x, ts)t \leq f(x,0)t + a(x)st^2$$

for all $t \in R$, $0 < s \leq 1$, and a.e. $x \in \Omega$, which implies that

$$\begin{aligned} F(x,t) &\leq f(x,0)t + \tfrac{1}{2}a(x)t^2 \\ &\leq \| f(x,0) \|_\infty |t| + \tfrac{1}{2}\lambda_{k+1}t^2 \\ &\leq C_2|t|^p \end{aligned}$$

for all $|t| \geq \delta$ and a.e. $x \in \Omega$, where $C_2 = \|f(x,0)\|_\infty \delta^{1-p} + \frac{1}{2}\lambda_{k+1}\delta^{2-p}$. Moreover, by (11) we have

$$F(x,t) \leq \tfrac{1}{2}b(x)t^2 + C_2|t|^p \tag{13}$$

for all $t \in R$ and a.e. $x \in \Omega$. By Sobolev's embedding theorem, there exists $C > 0$ such that

$$\|u\|_{L^p} \leq C\|u\|$$

for all $u \in H_0^1(\Omega)$. From Lemma 1 there exists $b_0 < 1$ such that

$$\int_\Omega b(x)u^2\,dx \leq b_0 \int_\Omega |\nabla u|^2\,dx \tag{14}$$

for all $u \in H_{m-1}^\perp$. It follows from the continuity of θ that there exists $\delta_0 \in]0, \delta[$ such that

$$\| v + \theta(v) \| \leq \left(\frac{1 - b_0}{2C_2C^p} \right)^{1/(p-2)}$$

for all $v \in V_2$ with $\|v\| \leq \delta_0$. Thus (13) and (14) imply that

$$\begin{aligned} \psi(v) &= \varphi(v + \theta(v)) \\ &\leq -\tfrac{1}{2}\| v + \theta(v) \|^2 + \tfrac{1}{2}b_0\| v + \theta(v) \|^2 + C_2\| v + \theta(v) \|_{L^p}^p \\ &\leq -\frac{1 - b_0}{2}\| v + \theta(v) \|^2 + C_2C^p\| v + \theta(v) \|^p \\ &\leq 0 \end{aligned}$$

for all $v \in V_2$ with $\|v\| \leq \delta_0$.

Proof of Theorem 1. It follows from Lemma 3 that ψ is coercive, which implies that ψ satisfies the (PS) condition. By the proof of Lemma 3, to prove that φ is bounded from below we only need to prove that $\int_\Omega H(x,v)\,dx$ is bounded from below for v in $E(\lambda_k)$. From (6) there exists $M > 0$ such that $\int_\Omega H(x,v)\,dx \geq 0$ for all $v \in E(\lambda_k)$ with $\|v\| \geq M$. By the finite dimensionality of $E(\lambda_k)$, there exists $C_3 > 0$ such that

$$\sup\{|v(x)|\,|\,x \in \Omega\} \leq C_3\|v\|$$

for all $v \in E(\lambda_k)$. Moreover, it follows from (5) that

$$\begin{aligned}\left|\int_\Omega H(x,v)\,dx\right| &\leq \int_\Omega g(x)|v|\,dx \\ &\leq C_3\|v\|\int_\Omega g(x)\,dx \\ &\leq C_4\|v\|,\end{aligned}$$

which implies that

$$\int_\Omega H(x,v)\,dx \geq -C_4 M$$

for all $v \in E(\lambda_k)$. Hence ψ is bounded from below. It follows from Theorem 4.4 in [7] that ψ has a minimum, which leads to Theorem 1.

Proof of Theorem 2. It follows from the proof of Theorem 1 that ψ has a minimum v_0. We only need to prove $v_0 \neq 0$. $\varphi(0) = 0$ implies that $\psi(0) \geq 0$. By Lemma 4, one has

$$\psi(v) \leq 0$$

for all $v \in E(\lambda_k)$ with $\|v\| \leq \delta_0$. In the case where $\inf \psi < 0$, it is obvious that $v_0 \neq 0$. In the case where $\inf \psi = 0$, all $v \in E(\lambda_k)$ with $\|v\| \leq \delta_0$ are minimum of ψ. Hence ψ has at least one nontrivial minimum, which implies Theorem 2.

Proof of Theorem 3. By the finite dimensionality of V and Lemma 3 we know that ψ satisfies the (PS) condition.

Let $V_1 = E(\lambda_1) + \cdots + E(\lambda_{m-1})$, $V_2 = E(\lambda_m) + \cdots + E(\lambda_k)$. By the finite dimensionality of V_1, there exists $C_5 > 0$ such that

$$\sup\{|v(x)|\,|\,x \in \Omega\} \leq C_5\|v\|$$

for all $v \in V_1$. Let $\delta_1 = \min\{\delta_0, \delta/C_5\}$. Then for $v \in V_1$, $\|v\| \leq \delta_1$, one has

$$\psi(v) \geq \varphi(v) \geq -\tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\lambda_{m-1}\|v\|_{L^2}^2 \geq 0.$$

In the case where $\inf_{v \in V}\psi(v) = 0$, all $v \in V_2$ with $\|v\| \leq \delta_0$ are infimum of ψ by Lemma 4, which implies that ψ has infinite critical points.

In the case where $\inf_{v \in V}\psi(v) < 0$, from the proof of Theorem 1 we know that ψ is bounded from below. It follows from Theorem 4 in [3] that ψ has at least two nonzero critical points by Lemma 4. Hence φ has at least two nonzero critical points. Thus problem (1) has at least two nontrivial solutions in $H_0^1(\Omega)$.

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