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Additive Jordan isomorphisms of nest algebras on normed spaces

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Abstract

Let X be a real or complex Banach space. Let $\operatorname{alg}\mathcal{N}$ and $\operatorname{alg}\mathcal{M}$ be two nest algebras on X. Suppose that ϕ is an additive bijective mapping from $\operatorname{alg}\mathcal{N}$ onto $\operatorname{alg}\mathcal{M}$ such that $\phi(A^2) = \phi(A)^2$ for every $A \in \operatorname{alg}\mathcal{N}$. Then ϕ is either a ring isomorphism or a ring anti-isomorphism. Moreover, if X is a real space or an infinite dimensional complex space, then there exists a continuous (conjugate) linear bijective mapping T such that either $\phi(A) = TAT^{-1}$ for every $A \in \operatorname{alg}\mathcal{N}$ or $\phi(A) = TA^*T^{-1}$ for every $A \in \operatorname{alg}\mathcal{N}$.

Keywords: Jordan isomorphism; Nest algebra; Nilpotent Jordan ideal

1. Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be rings. An additive mapping $\phi : \mathcal{A} \to \mathcal{B}$ is called a Jordan homomorphism if $\phi(a^2) = \phi(a)^2$ holds for all $a \in \mathcal{A}$. In addition, if ϕ is bijective then ϕ is called a Jordan isomorphism. The study of Jordan homomorphisms between rings was initiated by Anococha [2] in connection with problems arising in projective geometry. Since then, Jordan homomorphisms between rings has been investigated in a series of papers (see [4] and references therein). Some results will be made of use in the present paper.

The utility of the study of Jordan isomorphisms of Banach algebras was noted by Kadison [7] in the study of isometries of C^*-algebras. In fact, it is often found that an isometry Φ between Banach algebras can be written as in the form $\Phi = U\phi$, where ϕ is a Jordan homomorphism and U is a suitable unitary element [3,10,12]. In [7], Kadison proved that

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a Jordan $*$-isomorphism from a von Neumann algebra onto another can be decomposed into the sum of a $*$-isomorphism and a $*$-anti-isomorphism by a central projection. This result was extended by Palacios [10]. In [12], Solel showed that every Jordan isomorphism of CSL algebras (on a Hilbert space), whose restriction to the diagonal of the algebra is a selfadjoint map, is the sum of an isomorphism and an anti-isomorphism. It follows that such a Jordan isomorphism of nest algebras on a Hilbert space is either an isomorphism or an anti-isomorphism. In [8], we extended this result. More precisely, we proved that every Jordan isomorphism between nest algebras on a Hilbert space is either an isomorphism or an anti-isomorphism. In these discussions, a Jordan homomorphism of Banach algebras is usually assumed to be linear. A more general approach would be to consider these algebras only as rings. Let us recall that a ring (anti-)isomorphism of algebras is a bijective additive and (anti-)multiplicative mapping. It is clear that a ring (anti-)isomorphism of algebras is a Jordan isomorphism.

In the present paper, we study additive Jordan isomorphisms between nest algebras on a Banach algebra. We define again: an additive Jordan isomorphism of Banach algebras is an additive bijective mapping which preserves squares, so it is not assumed to be linear. In Section 3, we shall prove that an additive mapping between nest algebras on a Banach space is an additive Jordan isomorphism if and only if it is a ring isomorphism or a ring anti-isomorphism. To prove this result, in Section 2 we improve the concept of nilpotent Jordan ideals in a nest algebra introduced in [8], where we characterized linear Jordan isomorphisms of nest algebras on Hilbert spaces. In Section 4, the general spatial form of additive Jordan isomorphisms between nest algebras on a Banach space will be obtained. That is, if the Banach space under considering is real or complex infinite dimensional, then all such mapping are linear or conjugate linear, and hence they are spatially implemented.

Now we recall some definitions and notations.

Throughout, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, X is a Banach space over $\mathbb{F}$, $\dim X > 1$, $B(X)$ is the set of all linear bounded operators on X, and X^* is the dual Banach space of X. For a subspace L of X and two subsets $\mathcal{T}$ and $\mathcal{S}$ of $B(X)$, we write $\mathcal{T}L = \{Tx\colon\ T \in \mathcal{T},\ x \in L\}$, $L^\perp = \{f \in X^*\colon f(x) = 0,\ \forall x \in L\}$, and $\mathcal{T}\mathcal{S} = \{TS\colon\ T \in \mathcal{T},\ S \in \mathcal{S}\}$. A chain $\mathcal{N}$ of closed subspaces of X is called a nest if it contains the trivial subspaces $\{0\}$ and X and if it is closed under intersection and closed span. We set $\mathcal{N}_0 = \mathcal{N} \setminus \{0, X\}$. For $E \in \mathcal{N}$, we define $E_- = \bigvee\{F \in \mathcal{N}\colon F < E\}$ and $E_+ = \bigwedge\{F \in \mathcal{N}\colon\ F > E\}$. We also define $0_- = 0$ and $X_+ = X$. If $0_+ = 0$ and $X_- = X$, we say that $\mathcal{N}$ is sub-continuous. The nest algebra $\operatorname{alg}\mathcal{N}$ corresponding to the nest $\mathcal{N}$ is defined by $\operatorname{alg}\mathcal{N} = \{T \in B(X)\colon\ TE \subseteq E,\ \forall E \in \mathcal{N}\}$. For non-zero vectors $x \in X$ and $f \in X^*$, a rank one operator $x \otimes f$ is defined by $(x \otimes f)y = f(y)x$ for every $y \in X$.

We need the following elementary facts about nest algebras and Jordan isomorphisms.

Lemma 1.1 [13, Lemma 1]. *Let $\mathcal{N}$ be a nest on X. Then $x \otimes f$ belongs to $\operatorname{alg}\mathcal{N}$ if and only if there exists an element $E \in \mathcal{N}$ such that $x \in E$ and $f \in (E_-)^\perp$.*

It should be mentioned that if $x \in E$ and $f \in E^\perp$ for $E \in \mathcal{N}_0$ then $x \otimes f \in \operatorname{alg}\mathcal{N}$.

Lemma 1.2. *Let $\mathcal{N}$ be a nests on X and T be in $B(X)$. If $TA = AT$ for every A in $\operatorname{alg}\mathcal{N}$, then $T = \lambda I$ for $\lambda \in \mathbb{F}$, where I is the identity operator on X.*

Proof. It is certainly well known. Here we give an elementary proof which will play a demonstrative role in the proof of the related results below. Let E be in $\mathcal{N}$ with $E_- < X$ and non-zero functional f be in $(E_-)^\perp$. Then $x \otimes f \in \operatorname{alg}\mathcal{N}$ for all $x \in E$. Thus $Tx \otimes f = x \otimes fT$, and hence there exists a scalar $\lambda(E) \in \mathbb{F}$ such that $Tx = \lambda(E)x$. Since $E_1 \leqslant E_2$ or $E_2 \leqslant E_1$ for all $E_1, E_2 \in \mathcal{N}$, we have that $\lambda(E_1) = \lambda(E_2)$. Consequently, there is a scalar λ in $\mathbb{F}$ such that $Tx = \lambda x$ for all $x \in \bigcup\{E \in \mathcal{N}\colon E_- < X\}$. Since $\bigvee\{E \in \mathcal{N}\colon E_- < X\} = X$, we obtain that $T = \lambda I$. □

Theorem 1.3. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then for any $A, B, C \in \operatorname{alg}\mathcal{N}$, we have*

(i) $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$;
(ii) $\phi(ABA) = \phi(A)\phi(B)\phi(A)$;
(iii) $\phi(ABC + CBA) = \phi(A)\phi(B)\phi(C) + \phi(C)\phi(B)\phi(A)$;
(iv) $\phi(I) = I$;
(v) $AB = 0$ *and* $BA = 0$ *if and only if* $\phi(A)\phi(B) = 0$ *and* $\phi(B)\phi(A) = 0$.

Proof. Parts (i)–(iii) can be found in [6]. Parts (iv) and (v) can be obtained using Lemma 1.2 and the same argument in proofs of Propositions 3 and 4 in [8]. □

2. Nilpotent Jordan ideals

We begin with a definition.

Definition 2.1. A subset $\mathcal{J}$ of a Banach algebra $\mathcal{A}$ is called a nilpotent Jordan ideal if $\mathcal{J}^2 = \{0\}$ and $AB + BA \in \mathcal{J}$ for every $A \in \mathcal{J}$ and for every $B \in \mathcal{A}$. $\mathcal{J}$ is a maximal nilpotent Jordan ideal if there are no nilpotent Jordan ideals properly containing it.

It follows from Theorem 1.3(i) and (v) that an additive isomorphism maps a nilpotent Jordan ideal to a nilpotent Jordan ideal. Now we shall give a model for maximal nilpotent Jordan ideals of a nest algebra. Let $\mathcal{N}$ be a nest on X. Recall that $\mathcal{N}_0 = \{E \in \mathcal{N}\colon 0 < E < X\}$. For $E \in \mathcal{N}_0$, we define

$$\mathcal{I}(\mathcal{N}, E) = \bigl\{T \in \operatorname{alg}\mathcal{N}\colon TE = \{0\} \text{ and } TX \subseteq E\bigr\}.$$

By the Hahn–Banach extension theorem, it is easy to see that

$$\mathcal{I}(\mathcal{N}, E) = \bigl\{T \in \operatorname{alg}\mathcal{N}\colon TE = \{0\} \text{ and } T^*E^\perp = \{0\}\bigr\}.$$

Remark 2.2. Let $\mathcal{N}$ be a nest on X. Then

(1) If $x \in E$ and $f \in E^\perp$ for $E \in \mathcal{N}_0$, then $x \otimes f$ is in $\mathcal{I}(\mathcal{N}, E)$. We will see that those rank one operators are useful elements in $\mathcal{I}(\mathcal{N}, E)$.
(2) Let E and F be in $\mathcal{N}_0$ such that $E \leqslant F$. For every $B \in \mathcal{I}(\mathcal{N}, E)$, since its range is contained in E, we have that $AB = 0$ for every $A \in \mathcal{I}(\mathcal{N}, F)$ and hence $\mathcal{I}(\mathcal{N}, F)\mathcal{I}(\mathcal{N}, E) = 0$. This simple observation will play an important role.

(3) Let E and F be in $\mathcal{N}_0$ such that $E \neq F$. We may assume that $E < F$. Then by the Hahn–Banach extension theorem, there exist $f \in E^{\perp}$ while $f \notin F^{\perp}$. Thus for a non-zero vector $x \in E$, $x \otimes f$ is in $\mathcal{I}(\mathcal{N}, E)$ but not in $\mathcal{I}(\mathcal{N}, F)$, which implies that $\mathcal{I}(\mathcal{N}, E) \neq \mathcal{I}(\mathcal{N}, F)$.

It is easy to verify that $\mathcal{I}(\mathcal{N}, E)$ is a nilpotent Jordan ideal of $\operatorname{alg}\mathcal{N}$. Moreover, we have

Lemma 2.3. *Let $\mathcal{N}$ be a nest on X. If $\mathcal{J}$ is a nilpotent Jordan ideal of* $\operatorname{alg}\mathcal{N}$ *such that $\mathcal{J} \supseteq \mathcal{I}(\mathcal{N}, E)$ for some $E \in \mathcal{N}_0$, then $\mathcal{J} = \mathcal{I}(\mathcal{N}, E)$.*

Proof. Suppose that $S \in \mathcal{J}$. Let x be in E and f be in $E^{\perp}$. Since $x \otimes f \in \mathcal{I}(\mathcal{N}, E)$, we have

$$Sx \otimes f = 0 \quad \text{and} \quad x \otimes fS = 0,$$

and hence $Ax = 0$ and $A^* f = 0$. Since x and f are arbitrary, $SE = \{0\}$ and $S^* E = \{0\}$. Namely, $S \in \mathcal{I}(\mathcal{N}, E)$. □

Theorem 2.4. *Let $\mathcal{N}$ be a nest on X. Suppose that $\mathcal{J}$ is a maximal nilpotent Jordan ideal of* $\operatorname{alg}\mathcal{N}$ *and $\mathcal{J} \neq \{0\}$. Then there exists an element E in $\mathcal{N}_0$ such that $\mathcal{J} = \mathcal{I}(\mathcal{N}, E)$.*

Proof. Define

$$E = \bigwedge\{L \in \mathcal{N}\colon \mathcal{J}X \subseteq L\}, \qquad F = \bigvee\bigl\{L \in \mathcal{N}\colon \mathcal{J}L = \{0\}\bigr\}.$$

Then $\mathcal{J}X \subseteq E$ and $\mathcal{J}F = \{0\}$. Since $\mathcal{J} \neq \{0\}$, we have that $E > 0$ and $F < X$.

We claim that $E \leqslant F$. Otherwise $E > F$. Then we can take $T, S \in \mathcal{J}$ and vectors e, f such that $e \otimes f \in \operatorname{alg}\mathcal{N}$ and $Te \otimes fS \neq 0$ as follows. Note that $E = \bigwedge\{L \in \mathcal{N}\colon \mathcal{J}^* L^{\perp} = \{0\}\}$. If $F = E_-$, by the definition of E and F, there exist $e \in E \setminus F$, $f \in F^{\perp}$, and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^* f$. If $F \neq E_-$, then there is an element P in $\mathcal{N}$ such that $F < P < E$. By the definition of E and F, there exit $e \in P \setminus F$, $f \in P^{\perp}$, and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^* f$.

Since $\mathcal{J}$ is a Jordan ideal, $A = Te \otimes f + e \otimes fT \in \mathcal{J}$. Thus $AS = 0$. But

$$AS = Te \otimes fS + e \otimes fTS = Te \otimes fS \neq 0.$$

Therefore $E \leqslant F$, and then $E \in \mathcal{N}_0$ and $\mathcal{J} \subseteq \mathcal{I}(\mathcal{N}, E)$. By the maximality, we have that $\mathcal{J} = \mathcal{I}(\mathcal{N}, E)$. □

Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from $\operatorname{alg}\mathcal{N}$ onto $\operatorname{alg}\mathcal{M}$. For every $E \in \mathcal{N}_0$, by Lemma 2.3, $\mathcal{I}(\mathcal{N}, E)$ is a maximal nilpotent Jordan ideal in $\operatorname{alg}\mathcal{N}$. It follows from Theorem 1.3(i) and (v) that $\phi(\mathcal{I}(\mathcal{N}, E))$ is also a maximal nilpotent Jordan ideal in $\operatorname{alg}\mathcal{M}$. By Theorem 2.4 and Remark 2.2(3), there is only one element $\hat{E} \in \mathcal{M}_0$ such that $\phi(\mathcal{I}(\mathcal{N}, E)) = \mathcal{I}(\mathcal{M}, \hat{E})$. Define a map $\hat{\phi}$ from $\mathcal{N}_0$ to $\mathcal{M}_0$ by $\hat{\phi}(E) = \hat{E}$ if E is in N_0 such that $\phi(\mathcal{I}(\mathcal{N}, E) = \mathcal{I}(\mathcal{M}, \hat{E})$. We will call $\hat{\phi}$ the induced map of ϕ.

Proposition 2.5. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then $\hat{\phi}$, the induced map of ϕ, is bijective.*

Proof. First we show that $\hat{\phi}$ is injective. Indeed, if $\hat{\phi}(E) = \hat{\phi}(F)$ for some E, F in $\mathcal{N}_0$, then $\phi(\mathcal{I}(\mathcal{N}, E)) = \phi(\mathcal{I}(\mathcal{N}, F))$. Applying ϕ^{-1} to this equality, we get $\mathcal{I}(\mathcal{N}, E) = \mathcal{I}(\mathcal{N}, F)$. By Remark 2.2(3), $E = F$.

Considering ϕ^{-1} instead of ϕ. For every element $F \in \mathcal{M}_0$, $\phi^{-1}(\mathcal{I}(\mathcal{M}, F))$ is a maximal nilpotent Jordan ideal of $\operatorname{alg}\mathcal{N}$. Hence there is an element E in $\mathcal{N}_0$ such that $\mathcal{I}(\mathcal{N}, E) = \phi^{-1}(\mathcal{I}(\mathcal{M}, F))$. Thus $\phi(\mathcal{I}(\mathcal{N}, E)) = \mathcal{I}(\mathcal{M}, F)$ and hence $F = \hat{\phi}(E)$. That is to say, $\hat{\phi}$ is surjective. □

The next goal is to prove that $\hat{\phi}$ is either order-preserving (i.e., $P < Q$ implies $\hat{\phi}(P) < \hat{\phi}(Q)$) or anti-order preserving (i.e., $P < Q$ implies $\hat{\phi}(P) > \hat{\phi}(Q)$). For convenience, in the rest of this section and in Section 3, we shall use $\hat{E}$ to denote the image of E under $\hat{\phi}$.

Lemma 2.6. *Let $\mathcal{N}$, $\mathcal{M}$, ϕ, and $\hat{\phi}$ be as in Proposition* 2.5. *Suppose that P and Q are in $\mathcal{N}_0$ such that $P < Q$ and $\hat{P} < \hat{Q}$. Then $\hat{Q} < \hat{E}$ for every E in $\mathcal{N}_0$ satisfying $Q < E$.*

Proof. Let E be in $\mathcal{N}_0$ such that $Q < E$.

If $\hat{P} < \hat{E} < \hat{Q}$, take $x \in P$, $y \in E$, $f \in Q^{\perp}$, $g \in E^{\perp}$ such that $f(y) = 1$. Then $y \otimes g$ is in $\mathcal{I}(\mathcal{N}, E)$ and $x \otimes f$ is in $\mathcal{I}(\mathcal{N}, P) \cap \mathcal{I}(\mathcal{N}, Q)$. Thus $\phi(y \otimes g)$ is in $\mathcal{I}(\mathcal{M}, \hat{E})$ and $\phi(x \otimes f)$ is in $\mathcal{I}(\mathcal{M}, \hat{P}) \cap \mathcal{I}(\mathcal{M}, \hat{Q})$. Since $\hat{P} < \hat{E} < \hat{Q}$, by Remark 2.2(2), we have that $\phi(y \otimes g)\phi(x \otimes f) = 0$ and $\phi(x \otimes f)\phi(y \otimes g) = 0$. It follows from Theorem 1.3(v) that $(x \otimes f)(y \otimes g) = 0$. But this is impossible, since $(x \otimes f)(y \otimes g) = x \otimes g \neq 0$.

If $\hat{E} < \hat{P} < \hat{Q}$, take $x \in P$, $y \in Q$, $f \in P^{\perp}$, $g \in E^{\perp}$ such that $f(y) = 1$. Then $x \otimes f$ is in $\mathcal{I}(\mathcal{N}, P)$ and $y \otimes g$ is in $\mathcal{I}(\mathcal{N}, Q) \cap \mathcal{I}(\mathcal{N}, E)$. Thus $\phi(x \otimes f)$ is in $\mathcal{I}(\mathcal{M}, \hat{P})$ and $\phi(y \otimes g)$ is in $\mathcal{I}(\mathcal{M}, \hat{Q}) \cap \mathcal{I}(\mathcal{M}, \hat{E})$. Since $\hat{E} < \hat{P} < \hat{Q}$, we have that $\phi(y \otimes g)\phi(x \otimes f) = 0$ and $\phi(x \otimes f)\phi(y \otimes g) = 0$. Hence, $(x \otimes f)(y \otimes g) = 0$. But this is impossible, since $(x \otimes f)(y \otimes g) = x \otimes g \neq 0$.

Consequently, $\hat{Q} < \hat{E}$. □

Lemma 2.7. *Let $\mathcal{N}$, $\mathcal{M}$, ϕ, and $\hat{\phi}$ be as in Proposition* 2.5. *Suppose that P and Q are in $\mathcal{N}_0$ such that $P < Q$ and $\hat{P} < \hat{Q}$. Then $\hat{E} < \hat{P}$ for every E in $\mathcal{N}_0$ satisfying $E < P$.*

Proof. Applying Lemma 2.6 to $\hat{\phi}^{-1}$, $\hat{P} < \hat{Q} < \hat{E}$ is impossible. If $\hat{P} < \hat{E} < \hat{Q}$, let non-zero vectors $x \in E$, $f \in E^{\perp}$, $y \in P$, $g \in Q^{\perp}$ such that $f(y) = 1$. Then $x \otimes f \in \mathcal{I}(\mathcal{N}, E)$ and $y \otimes g$ is in $\mathcal{I}(\mathcal{N}, P) \cap \mathcal{I}(\mathcal{N}, Q)$. Thus $\phi(y \otimes g)\phi(x \otimes f) = 0$ and $\phi(x \otimes f)\phi(y \otimes g) = 0$. It follows from Theorem 1.3(v) that $(x \otimes f)(y \otimes g) = 0$. But this is impossible. Consequently, $\hat{E} < \hat{P}$. □

Theorem 2.8. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then $\hat{\phi}$, the induced map of ϕ, is either order-preserving or anti-order preserving.*

Proof. Suppose that there are $P < Q$ in $\mathcal{N}_0$ such that $\hat{P} < \hat{Q}$. Let E and F be two arbitrary elements in $\mathcal{N}_0$ such that $E < F$. We will prove that $\hat{E} < \hat{F}$.

Case 1: $E < F < P < Q$. Then by Lemma 2.7, $\hat{F} < \hat{P}$. Hence by Lemma 2.7 again, $\hat{E} < \hat{F}$.

Case 2: $P < Q < E < F$. Then by Lemma 2.6, $\hat{Q} < \hat{E}$. By Lemma 2.6 again, $\hat{E} < \hat{F}$.

Case 3: $P < E < Q < F$. Then by Lemma 2.6, $\hat{P} < \hat{Q} < \hat{F}$. And hence, by Lemma 2.7, $\hat{E} < \hat{Q} < \hat{F}$.

Case 4: $E < P < Q < F$. Then by Lemmas 2.6 and 2.7, $\hat{E} < \hat{P} < \hat{Q} < \hat{F}$.

Case 5: $E < P < F < Q$. Then by Lemma 2.7, $\hat{E} < \hat{P}$. By Lemma 2.6, $\hat{E} < \hat{P} < \hat{F}$.

Case 6: $P < E < F < Q$. If $\hat{E} > \hat{F}$, by Lemma 2.6, $\hat{P} > \hat{E}$ and by Lemma 2.7, $\hat{F} > \hat{Q}$. Therefore $\hat{Q} < \hat{F} < \hat{E} < \hat{P}$, which conflicts with the hypothesis $\hat{P} < \hat{Q}$. Thus $\hat{E} < \hat{F}$.

Since Cases 1–6 exhaust all possibilities, we get $\hat{E} < \hat{F}$. □

In the next section, we shall refer to ϕ itself as being order preserving or anti-order preserving as $\hat{\phi}$ is order preserving or anti-order preserving, respectively.

Corollary 2.9. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that* $\operatorname{alg}\mathcal{N}$ *is additive Jordan isomorphic to* $\operatorname{alg}\mathcal{M}$. *Then if one of $\mathcal{N}$ and $\mathcal{M}$ is sub-continuous, so is the other.*

Proof. It is immediate from Theorem 2.8. □

3. Algebraic results

The main result in this section is the following.

Theorem 3.1. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then ϕ is either a ring isomorphism or a ring anti-isomorphism.*

We will prove Theorem 3.1 distinguishing two cases: $\mathcal{N}$ is sub-continuous and $\mathcal{N}$ is not sub-continuous.

First we consider the first case. Recall that for a nest $\mathcal{N}$ on X and $E \in \mathcal{N}_0$, $\mathcal{I}(\mathcal{N}, E) = \{T \in \operatorname{alg}\mathcal{N}\colon\ TE = 0 \text{ and } TX \subseteq E\}$.

Lemma 3.2. *Let $\mathcal{N}$ be a nest on X such that $0_+ = 0$ and A be an operator in $B(X)$. If $\mathcal{I}(\mathcal{N}, E)A = 0$ for every $E \in \mathcal{N}_0$, then $A = 0$.*

Proof. Suppose that $A \neq 0$. Then there is a vector x in X such that $Ax \neq 0$. Since $\bigcap\{E\colon\ E \in \mathcal{N}_0\} = \{0\}$, there is E in $\mathcal{N}_0$ such that $Ax \notin E$. By the Hahn–Banach extension theorem, there is $f \in E^\perp$ such that $f(Ax) \neq 0$, which conflicts with the hypothesis $A^* f = 0$. □

Lemma 3.3. *Let $\mathcal{N}$ be a nest on X such that $X_- = X$ and A be an operator in $B(X)$. If $A\mathcal{I}(\mathcal{N}, E) = 0$ for every $E \in \mathcal{N}_0$, then $A = 0$.*

Proof. Let E be in $\mathcal{N}_0$ and fix a non-zero functional f in $E^{\perp}$. Then for every $x \in E$, we have that $Ax \otimes f = 0$. Hence $Ax = 0$ for every $x \in E$. Since E is arbitrary and $\bigvee\{E\colon E \in \mathcal{N}_0\} = X$, we have that $A = 0$. □

Lemma 3.4. *Let $\mathcal{N}$ and M be nests on X and E_i $(i = 1, 2)$ be in $\mathcal{N}_0$. Suppose that ϕ is an additive Jordan isomorphism from* alg$\mathcal{N}$ *onto* alg$\mathcal{M}$. *Then*

(1) *If ϕ is order preserving, then $\phi(B_1AB_2) = \phi(B_1)\phi(A)\phi(B_2)$ for every $B_i \in \mathcal{I}(\mathcal{N}, E_i)$ $(i = 1, 2)$ and for every $A \in \mathrm{alg}\mathcal{N}$. In particular, $\phi(B_1B_2) = \phi(B_1)\phi(B_2)$.*
(2) *If ϕ is anti-order preserving, then $\phi(B_1AB_2) = \phi(B_2)\phi(A)\phi(B_1)$ for every $B_i \in \mathcal{I}(\mathcal{N}, E_i)$ $(i = 1, 2)$ and for every $A \in \mathrm{alg}\mathcal{N}$. In particular, $\phi(B_1B_2) = \phi(B_2)\phi(B_1)$.*

Proof. We only prove part (1). The proof of part (2) is similar.

Note that ϕ maps $\mathcal{I}(\mathcal{N}, E_i)$ onto $\mathcal{I}(\mathcal{M}, \hat{\phi}(E_i))$, where $\hat{\phi}$ is the induced map of ϕ. If $E_2 \leqslant E_1$, then by Remark 2.2(2), $B_1AB_2 = \phi(B_1)\phi(A)\phi(B_2) = 0$ and the desired equation holds.

Assume now that $E_1 \leqslant E_2$. Then $B_2AB_1 = \phi(B_2)\phi(A)\phi(B_1) = 0$. From Theorem 1.3(iii), we obtain

$$\begin{aligned}\phi(B_1AB_2) &= \phi(B_1AB_2 + B_2AB_1)\\ &= \phi(B_1)\phi(A)\phi(B_2) + \phi(B_2)\phi(A)\phi(B_1) = \phi(B_1)\phi(A)\phi(B_2).\end{aligned}$$

Putting $A = I$ in this equation, from Theorem 1.3(iv) we obtain $\phi(B_1B_2) = \phi(B_1)\phi(B_2)$. □

Proof of Theorem 3.1 (The case where $\mathcal{N}$ is sub-continuous). By Theorem 2.8, we only need to consider two cases.

Case 1. ϕ is anti-order preserving. We will show that ϕ is anti-multiplicative in this case.

Let E be in $\mathcal{N}_0$. First we show that $\phi(AD) = \phi(D)\phi(A)$ for A in alg$\mathcal{N}$ and D in $\mathcal{I}(\mathcal{N}, E)$. Let F be an arbitrary element in $\mathcal{N}_0$. Then by Lemma 3.4(2), we have that

$$\phi(AD)\phi\big(\mathcal{I}(\mathcal{N}, F)\big) = \phi\big(\mathcal{I}(\mathcal{N}, F)AD\big) = \phi(D)\phi(A)\phi\big(\mathcal{I}(\mathcal{N}, F)\big).$$

Since ϕ maps $\mathcal{I}(\mathcal{N}, F)$ onto $\mathcal{I}(\mathcal{M}, \hat{\phi}(F))$ and $\mathcal{M}$ is sub-continuous by Corollary 2.9, we have that $\phi(AD) = \phi(D)\phi(A)$ by Lemma 3.3.

Now let A and B be in alg$\mathcal{N}$. For every $E \in \mathcal{N}_0$, from the preceding result, we have that

$$\begin{aligned}\phi\big(\mathcal{I}(\mathcal{N}, E)\big)\phi(AB) &= \phi\big(AB\mathcal{I}(\mathcal{N}, E)\big)\\ &= \phi\big(B\mathcal{I}(\mathcal{N}, E)\big)\phi(A) = \phi\big(\mathcal{I}(\mathcal{N}, E)\big)\phi(B)\phi(A).\end{aligned}$$

Hence by Lemma 3.2 we have that $\phi(AB) = \phi(B)\phi(A)$.

Case 2. ϕ is order preserving. Similarly, we can show that ϕ is multiplicative in this case. □

Now we turn to consider the case in which $\mathcal{N}$ is not sub-continuous. In the following, when confusions may occur we write $0_+^{\mathcal{N}}$ and $X_-^{\mathcal{N}}$ instead of 0_+ and X_-, respectively, for a nest $\mathcal{N}$ on X.

Lemma 3.5. *Let $\mathcal{N}$ be a nest on a Banach space X and suppose that $X_- \neq X$. Let P be an idempotent in* $\operatorname{alg}\mathcal{N}$ *such that $PX \not\subseteq X_-$. If P has rank greater than one, then there are two non-zero idempotents P_1 and P_2 in* $\operatorname{alg}\mathcal{N}$ *such that $P = P_1 + P_2$.*

Proof. Suppose that $Px \notin X_-$ for some $x \in X$. Then we can take $f \in X_-^{\perp}$ such that $f(Px) = 1$. Set $P_1 = Px \otimes P^* f$ and $P_2 = P - Px \otimes P^* f$, as desired. □

Lemma 3.6. *Let $\mathcal{N}$ be a nest on a Banach space X and suppose that $0_+ \neq 0$. Let P be an idempotent in* $\operatorname{alg}\mathcal{N}$ *such that $P0_+ \neq \{0\}$. If P has rank greater than one, then there are two non-zero idempotents P_1 and P_2 such that $P = P_1 + P_2$.*

Proof. Suppose that $0 \neq Px \in 0_+$ for some $x \in 0_+$. Take $f \in X^*$ such that $f(Px) = 1$. Set $P_1 = Px \otimes P^* f$ and $P_2 = P - Px \otimes P^* f$, as desired. □

Lemma 3.7. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X and let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$*. Suppose that P is an idempotent of rank one in* $\operatorname{alg}\mathcal{N}$*. If $\phi(P)X \not\subseteq X_-^{\mathcal{M}}$ or $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$ then $\phi(P)$ is of rank one.*

Proof. Assume to the contrary that $\phi(P)$ has at least rank two. Then by Lemmas 3.5 and 3.6 there are Q_1 and Q_2 of non-zero idempotents in $\operatorname{alg}\mathcal{M}$ such that $\phi(P) = Q_1 + Q_2$. Hence P is a sum of two non-zero idempotents $\phi^{-1}(Q_1)$ and $\phi^{-1}(Q_2)$. This is impossible. □

Let $\mathcal{N}$ be a nest and suppose that N is in $\mathcal{N}$ such that $N_- \neq N$. In what follows, by $\operatorname{Idem}(\mathcal{N}, N)$ we denote the set $\{x \otimes f\colon\ x \in N,\ f \in N_-^{\perp},\ f(x) = 1\}$.

Lemma 3.8. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X and let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$*. Suppose that N is in $\mathcal{N}$ such that $N_- \neq N$. If for each $P \in$* $\operatorname{Idem}(\mathcal{N}, N)$ *either $\phi(P)X \not\subseteq X_-^{\mathcal{M}}$ or $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$, then for all $x \in N$ and $f \in N_-^{\perp}$, the operator $\phi(x \otimes f)$ is of rank one.*

Proof. Let $x \in N$ and $f \in N_-^{\perp}$. First suppose that $t = f(x) \neq 0$. Then $f(x/t) = 1$ and hence by Lemma 3.7, $\phi(x/t \otimes f)$ is of rank one. A direct computation shows that

$$(x \otimes f)\left(I - \frac{1}{t}x \otimes f\right) = \left(I - \frac{1}{t}x \otimes f\right)(x \otimes f) = 0.$$

By Theorem 1.3(v), we have that $\phi(x \otimes f)\phi(I - x/t \otimes f) = 0$. Hence $\phi(x \otimes f) = \phi(x \otimes f)\phi(x/t \otimes f)$, and so $\phi(x \otimes f)$ is of rank one.

Now suppose that $f(x) = 0$. If $f \notin N^{\perp}$, then we can pick $x_0 \in N$ such that $f(x_0) = 1$. Suppose that $A = \phi(x \otimes f)$ and $\phi(x_0 \otimes f) = y \otimes g$. By Theorem 1.3(ii), we have that

$$0 = \phi\big((x \otimes f)(x_0 \otimes f)(x \otimes f)\big) = Ay \otimes A^* g.$$

Therefore, one of Ay and A^*g is zero. Hence from the equations

$$A = \phi(x \otimes f) = \phi\big((x \otimes f)(x_0 \otimes f) + (x_0 \otimes f)(x \otimes f)\big) = Ay \otimes g + y \otimes A^*g$$

we see that A is of rank one. Now let f_0 be in $N_-^\perp$ but not in $N^\perp$. Then by the preceding result, $\phi(x \otimes f_0)$ is of rank one. Suppose that $\phi(x \otimes f_0) = z \otimes h$. Let $x_1 \in N$ be such that $f_0(x_1) = 1$ and let $B = \phi(x_1 \otimes f)$. Then

$$0 = \phi\big((x_1 \otimes f)(x \otimes f_0)(x_1 \otimes f)\big) = Bz \otimes B^*h.$$

Therefore, one of Bz and B^*h is zero. Hence from

$$\phi(x \otimes f) = \phi\big((x \otimes f_0)(x_1 \otimes f) + (x_1 \otimes f)(x \otimes f_0)\big) = z \otimes hB + Bz \otimes h$$

we see that $\phi(x \otimes f)$ is of rank one. $\square$

Lemma 3.9. *Let $\mathcal{N}$ and $\mathcal{M}$ be non-trivial nests on X and let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Suppose that $X_-^{\mathcal{N}} \neq X$ and that either $\phi(P)X \not\subseteq X_-^{\mathcal{M}}$ or $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$ for every $P \in \operatorname{Idem}(\mathcal{N}, X)$. Let $x_0 \in X$ and $f_0 \in (X_-^{\mathcal{N}})^\perp$ be such that $f_0(x_0) = 1$. Then the following hold*:

(i) *If $\phi(x_0 \otimes f_0)X \not\subseteq X_-^{\mathcal{M}}$, then there is $g_0 \in (X_-^{\mathcal{M}})^\perp$ with the property that for each $x \in X$ there exists a vector $y \in X$ such that $\phi(x \otimes f_0) = y \otimes g_0$*;

(ii) *If $\phi(x_0 \otimes f_0)0_+^{\mathcal{M}} \neq \{0\}$, then there is $y_0 \in 0_+^{\mathcal{M}}$ with the property that for each $x \in X$ there exists a functional $g \in X^*$ such that $\phi(x \otimes f_0) = y_0 \otimes g$.*

Proof. Suppose that $\phi(x_0 \otimes f_0) = y_0 \otimes g_0$. Take $x_1 \in X_-^{\mathcal{N}}$. Then $\phi(x_1 \otimes f_0) \in \mathcal{I}(\mathcal{M}, \hat{X}_-^{\mathcal{N}})$. Note that $0_+^{\mathcal{M}} \leqslant \hat{X}_-^{\mathcal{N}} \leqslant X_-^{\mathcal{M}}$.

If $\phi(x_0 \otimes f_0)X \not\subseteq X_-^{\mathcal{M}}$, then $g_0 \in (X_-^{\mathcal{M}})^\perp$ and hence $\phi(x_0 \otimes f_0)\phi(x_1 \otimes f_0) = 0$. Therefore,

$$\begin{aligned}\phi(x_1 \otimes f_0) &= \phi\big((x_1 \otimes f_0)(x_0 \otimes f_0) + (x_0 \otimes f_0)(x_1 \otimes f_0)\big)\\ &= \phi(x_1 \otimes f_0)\phi(x_0 \otimes f_0) = y_1 \otimes g_0,\end{aligned}$$

where $y_1 = \phi(x_1 \otimes f_0)y_0 \in \hat{X}_-^{\mathcal{N}} \subseteq X_-^{\mathcal{M}}$. For $x \in X$, by Lemma 3.8 we can suppose that $\phi(x \otimes f_0) = y \otimes g$. It suffices to prove that g is linearly dependent of g_0. Indeed, since $x_0 \otimes f_0 + x \otimes f_0$ and $x_1 \otimes f_0 + x \otimes f_0$ are both of rank one, so are $y_0 \otimes g_0 + y \otimes g$ and $y_1 \otimes g_0 + y \otimes g$ by Lemma 3.8. If g is not linearly dependent of g_0, then y and y_0 as well as y and y_1 are linearly dependent. Consequently, y_0 and y_1 are linearly dependent. But this is impossible since $y_1 \in X_-^{\mathcal{M}}$ while $y_0 \notin X_-^{\mathcal{M}}$.

If $\phi(x_0 \otimes f_0)0_+^{\mathcal{M}} \neq \{0\}$, then $y_0 \in 0_+^{\mathcal{M}}$ and hence $\phi(x_1 \otimes f_0)\phi(x_0 \otimes f_0) = 0$. Therefore,

$$\begin{aligned}\phi(x_1 \otimes f_0) &= \phi\big((x_1 \otimes f_0)(x_0 \otimes f_0) + (x_0 \otimes f_0)(x_1 \otimes f_0)\big)\\ &= \phi(x_0 \otimes f_0)\phi(x_1 \otimes f_0) = y_0 \otimes g_1,\end{aligned}$$

where $g_1 = \phi(x_1 \otimes f_0)^* g_0 \in (0_+^{\mathcal{M}})^\perp$. For $x \in X$, suppose that $\phi(x \otimes f_0) = y \otimes g$. Since $x_0 \otimes f_0 + x \otimes f_0$ and $x_1 \otimes f_0 + x \otimes f_0$ are both of rank one, so are $y_0 \otimes g_0 + y \otimes g$ and $y_1 \otimes g_0 + x \otimes g$. If y is not linearly dependent of y_0, then g and g_0 as well as g and g_1 are linearly dependent. Consequently, g_0 and g_1 are linearly dependent. But this is impossible since $g_1 \in (0_+^{\mathcal{M}})^\perp$ while $g_0 \notin (0_+^{\mathcal{M}})^\perp$. So y must be linearly dependent of y_0. $\square$

Lemma 3.10. *Let $\mathcal{N} = \{0, N, X\}$ and $\mathcal{M} = \{0, M, X\}$ be non-trivial nests on X. Let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then one of the following holds*:

(i) $\phi(x \otimes f)X \nsubseteq M$ *for all* $x \in X$ *and* $f \in N^\perp$ *with* $f(x) = 1$;
(ii) $\phi(x \otimes f)M \neq \{0\}$ *for all* $x \in X$ *and* $f \in N^\perp$ *with* $f(x) = 1$.

Proof. It is easy to see that either $\phi(x \otimes f)X \nsubseteq M$ or $\phi(x \otimes f)M \neq \{0\}$ for all $x \in X$ and $f \in N^\perp$ with $f(x) = 1$. Assume to the contrary that there are $x_1, x_2 \in X$ and $f_1, f_2 \in N^\perp$ with $f_1(x_1) = f_2(x_2) = 1$ such that $\phi(x_1 \otimes f_1)X \nsubseteq M$ and $\phi(x_2 \otimes f_2)M \neq \{0\}$. Suppose that

$$\phi(x_1 \otimes f_1) = y_1 \otimes g_1 \quad \text{and} \quad \phi(x_2 \otimes f_2) = y_2 \otimes g_2.$$

Then $g_1 \in M^\perp$, $y_2 \in M$, and $g_2 \notin M^\perp$. Moreover, by Lemma 3.9, $\phi(x_2 \otimes f_1) = y_3 \otimes g_1$ for some $y_3 \in X$. Since $x_2 \otimes f_2 + x_2 \otimes f_1$ is of rank one, it follows from Lemma 3.8 that $y_2 \otimes g_2 + y_3 \otimes g_1 = \phi((x_2 \otimes (f_1 + f_2))$ is also of rank one. Since g_1 and g_2 are linearly independent, it follows that y_3 is linearly dependent of y_2. Consequently, $y_3 \otimes g_1 \in \mathcal{I}(\mathcal{M}, M)$. But this is impossible since $x_2 \otimes f_1 \notin \mathcal{I}(\mathcal{N}, N)$. $\square$

Lemma 3.11. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X and let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Suppose that $\mathcal{N}$ has at least four elements and $X_-^{\mathcal{N}} \neq X$. If ϕ is order-preserving, then $\phi(P)X \nsubseteq X_-^{\mathcal{M}}$ for all $P \in \operatorname{Idem}(\mathcal{N}, X)$. If ϕ is anti-order-preserving, then $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$ for all $P \in \operatorname{Idem}(\mathcal{N}, X)$.*

Proof. Let $N \in \mathcal{N}$ with $0 < N < X_-^{\mathcal{N}}$. Take $x_2 \in N$, $x_1 \in X_-^{\mathcal{N}}$, and $f_2 \in N^\perp$ such that $f_2(x_1) = 1$. Let $x_0 \otimes f_0 \in \operatorname{Idem}(\mathcal{N}, X)$ and $f_1 = f_0$. Set

$$A_i = x_i \otimes f_i, \quad i = 0, 1, 2.$$

Then $\phi(A_1) \in \mathcal{I}(\mathcal{M}, \hat{X}_-^{\mathcal{N}})$ and $\phi(A_2) \in \mathcal{I}(\mathcal{M}, \hat{N})$. Since $A_0 A_1 = 0$, we have that

$$\begin{aligned} \phi(x_2 \otimes f_0) &= \phi(A_2 A_1 A_0) = \phi(A_2 A_1 A_0 + A_0 A_1 A_2) \\ &= \phi(A_2)\phi(A_1)\phi(A_0) + \phi(A_0)\phi(A_1)\phi(A_2). \end{aligned} \tag{3.1}$$

If ϕ is order-preserving and $\phi(x_0 \otimes f_0)X \subseteq X_-^{\mathcal{M}}$, then $\phi(A_1)\phi(A_0) = 0$ and $\phi(A_1) \times \phi(A_2) = 0$. Thus, from (3.1), $\phi(x_2 \otimes f_0) = 0$, which conflicts with the injectivity of ϕ. If ϕ is anti-order-preserving and $\phi(x_0 \otimes f_0)0_+^{\mathcal{M}} = \{0\}$, then $\phi(A_2)\phi(A_1) = 0$ and $\phi(A_0)\phi(A_1) = 0$. We therefore also get a contradiction. $\square$

Combining Lemmas 3.10 and 3.11 immediately yields

Proposition 3.12. *Let $\mathcal{N}$ and $\mathcal{M}$ be non-trivial nests on X and suppose that $X_-^{\mathcal{N}} \neq X$. Let ϕ be an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then one of the following holds*:

(i) $\phi(P)X \nsubseteq X_-^{\mathcal{M}}$ *for all* $P \in \mathrm{Idem}(\mathcal{N}, X)$;
(ii) $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$ *for all* $P \in \mathrm{Idem}(\mathcal{N}, X)$.

Similarly, we can prove

Proposition 3.13. *Let* $\mathcal{N}$ *and* $\mathcal{M}$ *be non-trivial nests on* X *and suppose that* $0_+^{\mathcal{N}} \neq 0$. *Let* ϕ *be an additive Jordan isomorphism from* $\mathrm{alg}\,\mathcal{N}$ *onto* $\mathrm{alg}\,\mathcal{M}$. *Then one of the following holds*:

(i) $\phi(P)0_+^{\mathcal{M}} \neq \{0\}$ *for all* $P \in \mathrm{Idem}(\mathcal{N}, 0_+^{\mathcal{N}})$;
(ii) $\phi(P)X \nsubseteq X_-^{\mathcal{M}}$ *for all* $P \in \mathrm{Idem}(\mathcal{N}, 0_+^{\mathcal{N}})$.

Lemma 3.14. *Let* $\mathcal{N}$ *and* $\mathcal{M}$ *be nests on* X *and suppose that* $X_-^{\mathcal{N}} \neq X$. *Let* ϕ *be an additive Jordan isomorphism from* $\mathrm{alg}\,\mathcal{N}$ *onto* $\mathrm{alg}\,\mathcal{M}$. *Let* ψ *be the restriction of* ϕ *to* $\mathrm{span}\{x \otimes f\colon\ x \in X,\ \ f \in (X_-^{\mathcal{N}})^\perp\}$. *If Proposition* 3.12(i) *holds, then* ψ *is an additive Jordan isomorphism onto* $\mathrm{span}\{y \otimes g\colon\ y \in X,\ \ g \in (X_-^{\mathcal{M}})^\perp\}$ *which preserves rank one operators in both directions. If Proposition* 3.12(ii) *holds, then* ψ *is an additive Jordan isomorphism onto* $\mathrm{span}\{y \otimes g\colon\ y \in 0_+^{\mathcal{M}},\ \ g \in X^*\}$ *which preserves rank one operators in both directions.*

Proof. First suppose that Proposition 3.12(i) holds. Then by Lemma 3.9,

$$\psi\bigl(\mathrm{span}\bigl\{x \otimes f\colon\ x \in X,\ \ f \in \bigl(X_-^{\mathcal{N}}\bigr)^\perp\bigr\}\bigr) \subseteq \mathrm{span}\bigl\{y \otimes g\colon\ y \in X,\ \ g \in \bigl(X_-^{\mathcal{M}}\bigr)^\perp\bigr\}.$$

To establish another inclusion, consider ϕ^{-1}. From the above relation, it is easy to see that there is $Q \in \mathrm{Idem}(\mathcal{M}, X)$ such that $\phi^{-1}(Q)X \nsubseteq X_-^{\mathcal{N}}$. Applying Proposition 3.12 to ϕ^{-1}, we get that $\phi^{-1}(Q)X \nsubseteq X_-^{\mathcal{N}}$ for all $Q \in \mathrm{Idem}(\mathcal{M}, X)$. It follows that the another inclusion is true.

Now suppose that Proposition 3.12(ii) holds. Then by Lemma 3.9,

$$\psi\bigl(\mathrm{span}\bigl\{x \otimes f\colon\ x \in X,\ \ f \in \bigl(X_-^{\mathcal{N}}\bigr)^\perp\bigr\}\bigr) \subseteq \mathrm{span}\bigl\{y \otimes g\colon\ y \in 0_+^{\mathcal{M}},\ \ g \in X^*\bigr\}.$$

The another inclusion can be obtained by applying Proposition 3.13 and Lemma 3.8 to ϕ^{-1}. $\square$

Proof of Theorem 3.1 (The case where $\mathcal{N}$ is not sub-continuous). We only consider the case $X_-^{\mathcal{N}} \neq X$. The proof for the case that $0_+^{\mathcal{N}} \neq 0$ is similar.

First we consider the exceptional case where $\mathcal{N}$ is the trivial nest $\{0, X\}$. By Theorem 2.8, $\mathcal{M}$ is also trivial. Thus ϕ is an additive Jordan isomorphism from $B(X)$ onto $B(X)$. Since $B(X)$ is a prime ring, it follows from [11] that ϕ is either a ring isomorphism or a ring anti-isomorphism.

In the following, we assume that $\mathcal{N}$ is not trivial. We distinguish two cases.

Case 1. Proposition 3.12(i) holds. Then the restriction of ϕ to $\mathrm{span}\{x \otimes f\colon\ x \in X,$ $f \in (X_-^{\mathcal{N}})^\perp\}$ is an additive Jordan isomorphism onto $\mathrm{span}\{y \otimes g\colon\ y \in X,\ \ g \in (X_-^{\mathcal{M}})^\perp\}$ which preserves rank one operators in both directions. Moreover, by Lemma 3.9, for every

$f \in X_-^{\mathcal{N}}$ there is $g \in (X_-^{\mathcal{M}})^\perp$ with the property that for each $x \in X$ there exists a vector $y \in X$ such that $\phi(x \otimes f) = y \otimes g$.

Now arguing as in [9], we have that there exist two additive maps $T : X \to X$ and $S : (X_-^{\mathcal{N}})^\perp \to (X_-^{\mathcal{M}})^\perp$ such that

$$\phi(x \otimes f) = Tx \otimes Sf \quad \text{for all } x \in X,\ f \in X_-^\perp.$$

Let $A \in \operatorname{alg}\mathcal{N}$ be arbitrary. Fix a non-zero functional $f \in (X_-^{\mathcal{N}})^\perp$. Then for all $x \in X$,

$$\begin{aligned} &TAx \otimes Sf + Tx \otimes SA^* f = \phi(Ax \otimes f + x \otimes fA) \\ &\quad = \phi(A)\phi(x \otimes f) + \phi(x \otimes f)\phi(A) = \phi(A)Tx \otimes Sf + Tx \otimes Sf\phi(A). \end{aligned}$$

Therefore

$$\phi(A)Tx \otimes Sf = TAx \otimes Sf + Tx \otimes \big(SA^* f - \phi(A)^* Sf\big), \quad \forall x \in X.$$

Applying both sides of the above equation to $y_0 \in X$ satisfying $(Sf)(y_0) = 1$, we get that

$$\phi(A)Tx = TAx + \mu_A Tx, \quad \forall x \in X, \tag{3.2}$$

where $\mu_A \in \mathbb{F}$ is a constant not depending on x. Let $y \in X_-^{\mathcal{N}}$ and $g \in (X_-^{\mathcal{N}})^\perp$. Set $B = y \otimes g$. Then

$$0 = \phi(B^2) = \phi(B)\phi(B) = \phi(B)Ty \otimes Sg,$$

which implies that $\phi(B)Ty = 0$. Thus from (3.2) we have that

$$0 = \phi(B)Ty = TBy + \mu_B Ty = \mu_B Ty.$$

It follows that $\phi(B)Tx = TBx$ for all $x \in X$. Hence we have that

$$\phi(A)\phi(B)Tx = \phi(A)TBx = TABx + \mu_A TBx$$

and

$$\phi(B)\phi(A)Tx = \phi(B)(TAx + \mu_A Tx) = TBAx + \mu_A TBx.$$

On the other hand

$$\phi(AB + BA)Tx = TABx + TBAx.$$

It follows from Theorem 1.3(i) that $2\mu_A TBx = 0$ for all $x \in X$. Taking $x \in X$ such that $g(x) = 1$, we get that $\mu_A Ty = 0$ and hence $\mu_A = 0$, and so $\phi(A)Tx = TAx$ for all $x \in X$. Consequently, $\phi(A) = TAT^{-1}$ for all $A \in \operatorname{alg}\mathcal{N}$, from which it is easy to see that ϕ is multiplicative.

Case 2. Proposition 3.12(ii) holds. Then the restriction of ϕ to $\operatorname{span}\{x \otimes f\colon x \in X, f \in (X_-^{\mathcal{N}})^\perp\}$ is an additive Jordan isomorphism onto $\operatorname{span}\{y \otimes g\colon\ y \in 0_+^{\mathcal{M}},\ g \in X^*\}$ which preserves rank one operators in both directions. Moreover, by Lemma 3.9, for every $f \in X_-^{\mathcal{N}}$ there is $y \in 0_+^{\mathcal{M}}$ with the property that for each $x \in X$ there exists a vector $g \in X^*$ such that $\phi(x \otimes f) = y \otimes g$.

Arguing as in [9] again, we have that there are two additive bijections $T : X_-^\perp \to 0_+^{\mathcal{M}}$ and $S : X \to X^*$ such that

$$\phi(x \otimes f) = Tf \otimes Sx \quad \text{for all } x \in X,\ f \in X_-^\perp,$$

from which we get that $\phi(A)^* = SAS^{-1}$ and hence ϕ is anti-multiplicative. $\square$

Corollary 3.15. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Let k be a fixed integer greater than one. Let ϕ is an additive bijective mapping from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$ *such that $\phi(A^k) = \phi(A)^k$ for all A in* $\operatorname{alg}\mathcal{N}$. *Then $\phi = \mu\Phi$, where $\mu \in \mathbb{F}$ such that $\mu^{k-1} = 1$ and Φ is either a ring isomorphism or a ring anti-isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$.

Proof. It suffices to prove that there exist a scalar $\mu \in \mathbb{F}$ satisfying $\mu^{k-1} = 1$ and an additive Jordan isomorphism Φ from $\operatorname{alg}\mathcal{N}$ onto $\operatorname{alg}\mathcal{M}$ such that $\phi = \mu\Phi$.

Let A be in $\operatorname{alg}\mathcal{N}$ and t be positive integer. Consider

$$\phi\bigl((I + tA)^k\bigr) = \phi(I) + k\phi(A)t + \frac{1}{2}k(k-1)\phi(A^2)t^2 + \text{sum of other terms} \tag{3.3}$$

and

$$\begin{aligned}\phi(I + tA)^k &= \bigl(\phi(I) + t\phi(A)\bigr)^k \\ &= \phi(I)^k + t\sum_{j=0}^{k-1}\phi(I)^j\phi(A)\phi(I)^{k-1-j} + \text{sum of other terms.}\end{aligned} \tag{3.4}$$

Comparing the coefficients of t, we get

$$k\phi(A) = \sum_{j=0}^{k-1}\phi(I)^j\phi(A)\phi(I)^{k-1-j}. \tag{3.5}$$

Suppose that $\phi(B) = I$. Putting $A = B$ in (3.5), we get

$$\phi(I)^{k-1} = I. \tag{3.6}$$

Thus (3.5) becomes

$$(k-2)\phi(A) = \sum_{j=1}^{k-2}\phi(I)^j\phi(A)\phi(I)^{k-1-j}.$$

Further, applying (3.6) we have that

$$\begin{aligned}(k-2)\phi(I)\phi(A)\phi(I)^{k-2} &= \sum_{j=1}^{k-2}\phi(I)^{j+1}\phi(A)\phi(I)^{2k-3-j} \\ = \sum_{j=1}^{k-2}\phi(I)^{j+1}\phi(A)\phi(I)^{k-1-(j+1)} &= \sum_{j=2}^{k-2}\phi(I)^j\phi(A)\phi(I)^{k-1-j} + \phi(A) \\ &= (k-2)\phi(A) - \phi(I)\phi(A)\phi(I)^{k-2} + \phi(A).\end{aligned}$$

Thus $\phi(A) = \phi(I)\phi(A)\phi(I)^{k-2}$. Multiplying this equation by $\phi(I)$ from the right and applying (3.6), we get $\phi(I)\phi(A) = \phi(A)\phi(I)$. Since ϕ is surjective, by Lemma 1.2, $\phi(I) = \mu I$ for some $\mu \in \mathbb{F}$. Moreover, by (3.6), $\mu^{k-1} = 1$.

Now we can rewrite (3.4) as

$$\phi(I+tA)^k = \phi(I)^k + k\phi(A)t + \frac{1}{2}k(k-1)\mu^{k-2}\phi(A)^2t^2 + \text{sum of other terms.} \tag{3.7}$$

Comparing the coefficients of t^2 in (3.3) and (3.7), we have $\phi(A) = \mu^{k-2}\phi(A)^2$.

Define a mapping Φ from $\operatorname{alg}\mathcal{N}$ onto $\operatorname{alg}\mathcal{M}$ by $\Phi(A) = \mu^{-1}\phi(A)$. Then Φ is additive and bijective. Moreover $\phi = \mu\Phi$ and $\Phi(A^2) = \mu^{-1}\phi(A^2) = \mu^{k-3}\phi(A)^2 = \mu^{-2}\phi(A)^2 = \Phi(A)^2$. $\square$

4. Spatial results

In this section, we shall give the spatial structure of additive Jordan isomorphisms of nest algebras. Recall that a (conjugate) algebraic isomorphism of algebras is an additive, (conjugate) linear, multiplicative, bijective mapping. By Theorem 3.5 in [5] and its proof (or the related result in [13]), we have

Lemma 4.1. *Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is a mapping from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then ϕ is an* (*conjugate*) *algebraic isomorphism if and only if ϕ is of the form*

$$\phi(A) = TAT^{-1},$$

where T is a continuous (*respectively, conjugate*) *linear invertible operator from X onto X. ϕ is a* (*conjugate*) *algebraic anti-isomorphism if and only if ϕ is of the form*

$$\phi(A) = SA^*S^{-1},$$

where S is a continuous (*respectively, conjugate*) *linear invertible operator from $\bigvee\{f \in E_-^\perp\colon E \in \mathcal{N}\}$ onto X.*

Remark 4.2. Let $\mathcal{N}$ be a nest on X. If $0_+ \neq 0$ or X is reflexive, it is easy to see $\bigvee\{E^\perp\colon E \in \mathcal{N}_0\} = X^*$. But this equality may not hold if $0_+ = 0$ and X is not reflexive. For instance, set $X = L^1[0,1]$. For $t \in [0,1]$, let $E_t = \{f \in L^1\colon f([1, 1-t]) = 0\}$. Then $\mathcal{N} = \{E_t\colon t \in [0,1]\}$ is a nest with $0_+ = 0$ and $X_- = X$. It is obvious that $X^* = L^\infty$ and $E_t^\perp = \{g \in L^\infty\colon g([0,t)) = 0\}$ for every $t \in (0,1)$. Let $g_0 = 1 \in X^*$, then $\|g - g_0\|_\infty \geqslant 1$ for every $g \in E_t^\perp$ and every $t \in (0,1)$, which implies that $\bigvee\{E_t^\perp\colon t \in (0,1)\} \neq X^*$.

Lemma 4.3. *Let $\mathcal{N}$ and $\mathcal{M}$ be nest on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then there is a ring automorphism $h:\mathbb{F}\to\mathbb{F}$ such that $\phi(\alpha A) = h(\alpha)\phi(A)$ for every $\alpha \in \mathbb{F}$ and for every $A \in \operatorname{alg}\mathcal{N}$.*

Proof. By Theorem 3.1, we may assume that ϕ is a ring anti-isomorphism.

Let α be in $\mathbb{F}$. Then $\phi(\alpha I)\phi(A) = \phi(A(\alpha I)) = \phi((\alpha I)A) = \phi(A)\phi(\alpha I)$ for every A in $\operatorname{alg}\mathcal{N}$. Since ϕ is surjective, by Lemma 1.2 there exists a scalar $h(\alpha) \in \mathbb{F}$ such that $\phi(\alpha I) = h(\alpha)I$. It is easy to verify that $h(\alpha)$ is a ring automorphism of $\mathbb{F}$. Now for $\alpha \in \mathbb{F}$ and $A \in \operatorname{alg}\mathcal{N}$, we have that

$$\phi(\alpha A) = \phi(A)\phi(\alpha I) = \phi(A)h(\alpha)I = h(\alpha)\phi(A). \qquad \square$$

Lemma 4.4. *Let X be either a real Banach space,* $\dim X > 1$, *or an infinite dimensional complex Banach space. Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that ϕ is an additive Jordan isomorphism from* $\operatorname{alg}\mathcal{N}$ *onto* $\operatorname{alg}\mathcal{M}$. *Then in the real case ϕ is linear, and in the complex case ϕ is either linear or conjugate linear.*

Proof. By Lemma 4.3, there is a ring autoisomorphism $h:\mathbb{F}\to\mathbb{F}$ such that $\phi(\alpha A) = h(\alpha)\phi(A)$ for every $\alpha\in\mathbb{F}$ and for every $A\in\operatorname{alg}\mathcal{N}$.

Case 1: $\mathbb{F}=\mathbb{R}$. By a result of [1, p. 57], $h(\alpha)=\alpha$ for every $\alpha\in\mathbb{F}$. Consequently, ϕ is linear.

Case 2: $\mathbb{F}=\mathbb{C}$ and X is of infinite dimensions. By additivity and multiplicativity of h, it suffices to prove that h is continuous.

We take a sequence $\{x_k\otimes f_k\}$ of rank one operators in $\operatorname{alg}\mathcal{N}$ such that

(a) $f_k(x_{k+1})=1$ for each k,
(b) $f_k(x_j)=0$ for $j\leqslant k$,
(c) $x_k\otimes f_{k+1}$ is also in $\operatorname{alg}\mathcal{N}$ for each k,

as follows. If $\mathcal{N}$ is an infinite set, then there is a sequence $\{N_k\}$ in $\mathcal{N}$ such that $0 < N_k < N_{k+1} < X$ for every k. Take non-zero vectors x_k in N_k and f_k in $N_k^{\perp}$ such that $f_k(x_{k+1})=1$. If $\mathcal{N}$ is a finite set, there must be an element E in $\mathcal{N}$ such that $E\setminus E_-$ has an infinite subset of linearly independent vectors since $\dim X=\infty$. Let $\{x_k\}$ be a subset of linearly independent vectors in $E\setminus E_-$. Then by Hahn–Banach extension theorem, there is a sequence $\{f_k\}$ of functionals in $E_-^{\perp}$ satisfying conditions (a)–(c).

If h is not continuous, then by [1], h is unbounded on every neighborhood of 0. We pick $\alpha_k\in\mathbb{C}$ such that $\|\alpha_k(x_k\otimes f_k)\| < 1/2^k$ and

$$\left|h(\alpha_k)\right| > \frac{k\|\phi(x_{k+1}\otimes f_{k+1})\| + \|\phi(\sum_{i=1}^{k-1}\alpha_i(x_i\otimes f_i)(x_{k+1}\otimes f_{k+1})\|}{\|\phi(x_k\otimes f_{k+1})\|}.$$

Define $A=\sum_{k=1}^{\infty}\alpha_k x_k\otimes f_k$. Then A is in $\operatorname{alg}\mathcal{N}$ and

$$Ax_{k+1}\otimes f_{k+1} = \sum_{i=1}^{k-1}\alpha_i(x_i\otimes f_i)(x_{k+1}\otimes f_{k+1}) + \alpha_k x_k\otimes f_{k+1}.$$

Since ϕ is a ring isomorphism or a ring anti-isomorphism, we have that

$$\begin{aligned}
&\left\|\phi(A)\right\|\left\|\phi(x_{k+1}\otimes f_{k+1})\right\|\\
&\quad\geqslant \left\|\phi\left(\sum_{i=1}^{k-1}\alpha_i(x_i\otimes f_i)(x_{k+1}\otimes f_{k+1}) + \alpha_k x_k\otimes f_{k+1}\right)\right\|\\
&\quad= \left\|\phi\left(\sum_{i=1}^{k-1}\alpha_i(x_i\otimes f_i)(x_{k+1}\otimes f_{k+1})\right) + h(\alpha_k)\phi(x_k\otimes f_{k+1})\right\|\\
&\quad\geqslant \left|h(\alpha_k)\right|\left\|\phi(x_k\otimes f_{k+1})\right\| - \left\|\phi\left(\sum_{i=1}^{k-1}\alpha_i(x_i\otimes f_i)(x_{k+1}\otimes f_{k+1})\right)\right\|.
\end{aligned}$$

Therefore $\|\phi(A)\|\geqslant k$ $(k=1,2,\ldots)$, which contradicts that fact that $\phi(A)$ is bounded. □

Combining Theorem 3.1 and Lemmas 4.1 and 4.4, we obtain

Theorem 4.5. *Let X be either a real Banach space,* $\dim X > 1$, *or an infinite dimensional complex Banach space. Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X. Suppose that $\phi : \operatorname{alg}\mathcal{N} \to \operatorname{alg}\mathcal{M}$ is an additive Jordan isomorphism. Then we have in the real case either*

$$\phi(A) = TAT^{-1}$$

for all A in $\operatorname{alg}\mathcal{N}$, where $T : X \to X$ is a continuous linear bijective mapping, or

$$\phi(A) = SA^*S^{-1}$$

for all A in $\operatorname{alg}\mathcal{N}$, where $S : \bigvee\{f \in E_-^\perp\colon\ E \in \mathcal{N}\} \to X$ is a continuous linear bijective mapping. In the complex case ϕ is either of one of the above forms, or of one of the following:

$$\phi(A) = TAT^{-1}$$

for all A in $\operatorname{alg}\mathcal{N}$, where $T : X \to X$ is a continuous conjugate linear bijective mapping, or

$$\phi(A) = SA^*S^{-1}$$

for all A in $\operatorname{alg}\mathcal{N}$, where $S : \bigvee\{f \in E_-^\perp\colon\ E \in \mathcal{N}\} \to X$ is a continuous conjugate linear bijective mapping.

Now we treat an additive Jordan isomorphism of nest algebras on a finite dimensional complex Banach space. By $M_n(\mathbb{C})$ we denote the algebra of all $n \times n$ matrices over $\mathbb{C}$. For every finite sequence of positive integers $n_1, n_2, \ldots, n_k$, satisfying $n_1 + n_2 + \cdots + n_k = n$, we associate an algebra consisting of all $n \times n$ matrices of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ 0 & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{kk} \end{bmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix. We will call such an algebra a block upper triangular algebra in $M_n(\mathbb{C})$. Given two nest algebras on $\mathbb{C}^n$, one of them can be assumed to be a block upper triangular algebra in $M_n(\mathbb{C})$ and another can be assumed to be a subalgebra of $M_n(\mathbb{C})$ which is called a nest algebra in $M_n(\mathbb{C})$.

Theorem 4.6. *Let $\mathcal{A}$ be a block upper triangular algebra in $M_n(\mathbb{C})$ and $\mathcal{B}$ be a nest algebra in $M_n(\mathbb{C})$. Let ϕ is an additive Jordan isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then there is a ring automorphism $h : \mathbb{C} \to \mathbb{C}$ and an invertible matrix T such that ϕ is of the form $\phi(A) = TH(A)T^{-1}$ or $\phi(A) = TH(A)^tT^{-1}$ for all $A \in \mathcal{A}$, where $H[\alpha_{ij}] = [h(\alpha_{ij})]$ and t stands for the transpose.*

Proof. By Lemma 4.3, there is a ring autoisomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $\phi(\alpha A) = h(\alpha)\phi(A)$ for every $\alpha \in \mathbb{C}$ and for every $A \in \mathcal{A}$. For a matrix $[\alpha_{ij}]$, we define

$$H[\alpha_{ij}] = \bigl[h(\alpha_{ij})\bigr] \quad \text{and} \quad H^{-1}[\alpha_{ij}] = \bigl[h^{-1}(\alpha_{ij})\bigr].$$

Then both H and H^{-1} are ring isomorphisms.

By Theorem 3.1, we distinguish two cases.

Case 1. ϕ is a ring isomorphism. We introduce a new mapping ψ on $\mathcal{A}$ by

$$\psi(A) = \phi\big(H^{-1}(A)\big), \quad A \in \mathcal{A}.$$

It is not difficult to see that ψ is a linear isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. By Lemma 4.1, there is an invertible matrix T such that $\psi(A) = TAT^{-1}$. Consequently, $\phi(A) = TH(A)T^{-1}$ for every $A \in \mathcal{A}$.

Case 2. ϕ is a ring automorphism. Let J be a particular permutation matrix in $M_n(\mathbb{C})$ given by $J = [\delta_{i,n+1-i}]$, where δ_{ij} is the Kronecker delta symbol. Then $J\mathcal{A}^tJ$ is also a block upper triangular algebra. We introduce a new mapping ψ on $J\mathcal{A}^tJ$ by

$$\psi(JA^tJ) = \phi\big(H^{-1}(A)\big), \quad A \in \mathcal{A}.$$

It is not difficult to see that ψ is a linear isomorphism from $J\mathcal{A}^tJ$ onto $\mathcal{B}$. By Lemma 4.1, there is an invertible matrix T such that $\psi(JA^tJ) = TJA^tJT^{-1}$. Consequently, $\phi(A) = TJH(A)^tJT^{-1} = (TJ)H(A)^t(TJ)^{-1}$ for every $A \in \mathcal{A}$. □

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