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Optimal birth control of age-dependent competitive species ☆

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Abstract

This work is concerned with optimal policies for two age-structured biological populations in a competing system, which is controlled by fertilities. The maximum principles for problems with free terminal, infinite horizon and target sets are obtained respectively via Dubovitskii–Milyutin's general extremal theory.

Keywords: Population model; Age-dependence; Optimal control; Maximum principle

1. Introduction

There have been many studies for the control problems of multi-species by the help of mathematical modelling, see Refs. [1–9]. However all of these investigations are concentrated on the systems without age-dependence. It is well known that age-composition is one of the key factors in population dynamics, since the fertility and mortality of an individual depend heavily on its age. To check the age effects on the control problems of multi-species, we in the sequel examine several optimal control problems for systems composed of two age-dependent populations competing each other.

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The purpose of this article is to establish the necessary optimality conditions for the optimal control problems. This task is finished via a powerful functional approach first suggested by Dubovitskii and Milyutin for general extremal problems (Ref. [10]). We will follow the spirits in Refs. [11,12]. Nevertheless, because of the presence of interactions, additional difficulties have to be overcome for going through our investigations. The work generalizes the corresponding results in Ref. [11].

The remainder of this paper is organized as follows. In Section 2, we formulate the basic model and treat its well-posedness. From Section 3 to 5, we study, respectively, optimal control problems with free terminal state, infinite horizon and target sets. Section 6 is composed of some comments and a topic for future research.

2. The model and its well-posedness

In [13], Webb formulated the following model for two age-dependent species with interaction of competition or predator–prey:

$$\begin{cases} \frac{\partial l_i}{\partial t} + \frac{\partial l_i}{\partial a} = -[\mu_{i1}(\mathrm{P}l_1(\cdot,t)) + \mu_{i2}(\mathrm{P}l_2(\cdot,t))]l_i(a,t), \quad i=1,2, \\ l_i(0,t) = \int_0^\infty \beta_i(1-e^{\alpha_i a})l_i(a,t)\,da, \quad i=1,2, \\ l_i(a,0) = \varphi_i(a), \quad i=1,2, \\ \mathrm{P}l_i(\cdot,t) = \int_0^\infty l_i(a,t)\,da, \quad i=1,2,\ (a,t)\in(0,\infty)\times(0,\infty), \end{cases}$$

where $l_i(a,t)$ $(i=1,2)$ are the density with respect to age a of ith population at time t; $\alpha_1,\alpha_2,\beta_1,\beta_2$ are all positive constants; mortality modula μ_{ij} $(i,j=1,2)$ are all bounded and twice continuously differentiable functions from R to $(0,\infty)$.

Under certain conditions he considered first the extinction of one of the species, then the local stability of nontrivial equilibrium solution.

Motivated by the idea of Webb, we introduce the following model:

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)P_2(t)p_1, \\ \frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} = -\mu_2(a,t)p_2 - \lambda_2(a,t)P_1(t)p_2, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\,da, \\ p_i(a,0) = p_{i0}(a), \\ P_i(t) = \int_0^A p_i(a,t)\,da, \quad i=1,2,\ (a,t)\in Q, \end{cases} \tag{1}$$

where $Q=(0,A)\times(0,+\infty)$, $[a_1,a_2]$ is the fertility interval, and the other parameters mean as follows $(i=1,2)$:

- $p_i(a,t)$: the density of population i of age a at time t;
- $\mu_i(a,t)$: the average mortality of population p_i;
- $\beta_i(t)$: the average fertility of population p_i;
- $\lambda_i(a,t)$: the interaction coefficients;
- $m_i(a,t)$: the ratio of females of population p_i;
- $p_{i0}(a)$: the initial age distribution of population p_i;
- A: the life expectancy, $0<A<+\infty$. Here, without loss of generality, we assume that the two populations have the same life expectancy.

Throughout this paper, we suppose the following conditions hold:

(H_1) $\mu_i \in L^1_{\mathrm{loc}}(Q)$, $\mu_i(a,t) \geqslant 0$, $\int_0^A \mu_i(a, t+a)\,da = +\infty$, $(a,t) \in Q$.
(H_2) $0 \leqslant \lambda_i(a,t) \leqslant A_i$, A_i are constants.
(H_3) $0 \leqslant m_i(a,t) \leqslant M_i$, M_i are constants; $m_i(a,t) \equiv 0$ when $a < a_1$ or $a > a_2$.
(H_4) $\beta_i \in U_i := \{h_i \in L^\infty(0,\infty)\colon 0 \leqslant \beta_0 \leqslant h_i(t) \leqslant \beta^0,\ \forall t > 0\}$, β_0 and β^0 are constants, $U = U_1 \times U_2$.
(H_5) $p_{i0} \in L^\infty(0,A)$, $p_{i0}(a) \geqslant 0$, $\forall a \in (0,A)$.

For any given $T > 0$ and $v = (v_1, v_2) \in L^2(Q_T, R^2)$, $Q_T = (0,A) \times (0,T)$, $v \geqslant 0$, define

$$V_i(t) = \int_0^A v_i(a,t)\,da, \quad i = 1,2.$$

Consider the system

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)V_2(t)p_1, \\ \frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} = -\mu_2(a,t)p_2 - \lambda_2(a,t)V_1(t)p_2, \\ p_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)p_i(a,t)\,da, \\ p_i(a,0) = p_{i0}(a), \quad i = 1,2,\ (a,t) \in Q_T. \end{cases} \tag{2}$$

The above system has a unique nonnegative solution (Refs. [14,15])

$$p^v = \left(p_1^v, p_2^v\right) \in C\left(0,T; L^2(0,A; R^2)\right) \cap L^\infty(Q_T; R^2)$$

and

$$p_i^v(A,t) = 0, \quad \forall t \in [0,T],\ i = 1,2.$$

Note that, from the comparison principle of linear system (Ref. [15]), it follows that $p_i^v(a,t) \leqslant \bar{p}_i(a,t)$, $(a,t) \in Q_T$, $i = 1,2$, where $\bar{p}_i$ is the unique nonnegative and bounded solution of the following system:

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} = -\mu_i(a,t)y, \\ y(0,t) = \beta^0 \int_{a_1}^{a_2} m_i(a,t)y(a,t)\,da, \\ y(a,0) = p_{i0}(a), \quad (a,t) \in Q_T. \end{cases}$$

For any $v_{(i)} = (v_{i1}, v_{i2}) \in L^2(Q_T; R^2)$, $0 \leqslant v_{ij} \leqslant \bar{p}_j$, let the corresponding state be $p_{(i)} = (p_{i1}, p_{i2})$, $i = 1,2$, $x = (x_1, x_2) := p_{(1)} - p_{(2)}$.

It follows from (2) that

$$\begin{cases} \frac{\partial x_1}{\partial t} + \frac{\partial x_1}{\partial a} = -\mu_1 x_1 - \lambda_1 V_{12}(t)x_1 - (V_{12}(t) - V_{22}(t))\lambda_1 p_{21}, \\ \frac{\partial x_2}{\partial t} + \frac{\partial x_2}{\partial a} = -\mu_2 x_2 - \lambda_2 V_{11}(t)x_2 - (V_{11}(t) - V_{21}(t))\lambda_2 p_{22}, \\ x_i(0,t) = \beta_i(t)\int_{a_1}^{a_2} m_i(a,t)x_i(a,t)\,da, \\ x_i(a,0) = 0, \\ V_{ij}(t) = \int_0^A v_{ij}(a,t)\,da, \quad i,j = 1,2,\ (a,t) \in Q_T. \end{cases} \tag{3}$$

Multiplying $(3)_i$ (i.e., the ith equation in (3)) by x_i, $i = 1, 2$, and integrating on $(0, A) \times (0, t)$ yield

$$\left\|x_1(\cdot\,,t)\right\|^2 \leqslant c \int_0^t \left\|v_{12}(\cdot\,,s) - v_{22}(\cdot\,,s)\right\|^2 ds \tag{4}$$

and

$$\left\|x_2(\cdot\,,t)\right\|^2 \leqslant c \int_0^t \left\|v_{11}(\cdot\,,s) - v_{21}(\cdot\,,s)\right\|^2 ds, \tag{5}$$

where c is a constant independent of v_i, $i = 1, 2$, $\|\cdot\|$ is the ordinary norm in $L^2(0, A)$.

Consider the set

$$I = \left\{(v_1, v_2) \in L^2(Q_T; R^2)\colon\ 0 \leqslant v_i(a,t) \leqslant \bar{p}_i(a,t),\ i = 1, 2;\ \forall (a,t) \in Q_T\right\}.$$

Define the mapping $G : I \to I$,

$$(Gv)(a,t) = p^v(a,t), \quad \forall (a,t) \in Q_T,$$

and an equivalent norm $\|v\|_* = (\|v_1\|_*^2 + \|v_2\|_*^2)^{1/2}$,

$$\|v_i\|_*^2 = \int_0^T \left\|v_i(\cdot\,,t)\right\|^2 \exp(-2ct)\,dt, \quad i = 1, 2.$$

Using (4) and (5), we get that

$$\begin{aligned}
&\|Gv_{(1)} - Gv_{(2)}\|_* = \|p_{(1)} - p_{(2)}\|_* \\
&\quad = \left[\int_0^T \left(\left\|x_1(\cdot\,,t)\right\|^2 + \left\|x_2(\cdot\,,t)\right\|^2\right) \exp(-2ct)\,dt\right]^{1/2} \\
&\quad \leqslant \left[\int_0^T \int_0^t c\left(\left\|v_{11}(\cdot\,,s) - v_{21}(\cdot\,,s)\right\|^2 + \left\|v_{12}(\cdot\,,s) - v_{22}(\cdot\,,s)\right\|^2\right) ds \right. \\
&\qquad \left. \times \exp(-2ct)\,dt\right]^{1/2} \\
&\quad \leqslant \left[\int_0^T \left(\left\|v_{11}(\cdot\,,s) - v_{21}(\cdot\,,s)\right\|^2 + \left\|v_{12}(\cdot\,,s) - v_{22}(\cdot\,,s)\right\|^2\right) \right. \\
&\qquad \left. \times \int_s^T c \exp(-2ct)\,dt\,ds\right]^{1/2}
\end{aligned}$$

$$\leqslant \frac{1}{\sqrt{2}}\left[\int_0^T \left(\left\|v_{11}(\cdot,s)-v_{21}(\cdot,s)\right\|^2+\left\|v_{12}(\cdot,s)-v_{22}(\cdot,s)\right\|^2\right)\right.$$
$$\left.\times \exp(-2cs)\,ds\right]^{1/2}$$
$$=\frac{1}{\sqrt{2}}\|v_{(1)}-v_{(2)}\|_*.$$

So, G is a contraction on $(I, \|\cdot\|_*)$, which has a unique fixed point v^*. It is clear that v^* is the solution of the system (1). We have proved that

Theorem 1. *For any given $\beta \in U$, there is a unique solution p^β to system* (1)*, such that*

(i) $p^\beta \in C(0,\infty; L^2(0,A))$;
(ii) $0 \leqslant p_i^\beta(a,t) \leqslant \bar{p}_i(a,t)$, $\forall (a,t) \in Q$, $i=1,2$;
(iii) *It can be shown in a similar manner that p^β depends continuously on β.*

3. Free terminal problem

Consider the control problem: Determine (β^*, p^*), $\beta^* \in U$, such that

$$\begin{cases} J(\beta^*, p^*) = \min\{J(\beta,p)\colon\ \beta\in U,\ (\beta,p) \text{ is subject to (1)}\},\\ J(\beta,p) = \int_0^T\int_0^A L(\beta_1(t),\beta_2(t),p_1(a,t),p_2(a,t),a,t)\,da\,dt \\ \qquad\qquad + \frac{1}{2}\sum_{i=1}^2\int_0^A [p_i(a,t)-\bar{p}_i(a)]^2\,da, \end{cases} \tag{6}$$

where $T>0$ and $\bar{p}_i(a) \geqslant 0$ $(i=1,2)$ are prescribed. The functional L, defined on $[\beta_0,\beta^0]^2 \times [L^2(0,A)]^2 \times [0,A] \times [0,\infty)$, satisfies the following conditions:

(i) $\partial L/\partial\beta_i$ and $\partial L/\partial p_i$ $(i=1,2)$ are continuous in the first four arguments, and L is continuous with respect to its all variables.
(ii) $\int_0^A |\partial L(\beta_1,\beta_2,p_1(a),p_2(a),a,t)/\partial\beta_i|\,da$, $\int_0^A |\partial L(\beta_1,\beta_2,p_1(a),p_2(a),a,t)/\partial p_i|\,da$ $(i=1,2)$ are bounded for any $t \in [0,T]$ and any bounded subset of $[\beta_0,\beta^0]^2 \times [L^2(0,A)]^2 \times [0,A] \times [0,T]$.

In the sequel, by (β,p,a,t) we denote $(\beta_1(t),\beta_2(t),p_1(a,t),p_2(a,t),a,t)$.

Theorem 2. *Any solution (β^*, p^*) of problem* (6) *satisfies*

$$\beta_i^*(t)S_i(t) = \max\bigl\{\beta_i S_i(t)\colon\ \beta_0 \leqslant \beta_i \leqslant \beta^0\bigr\}, \quad \text{a.e. } t \in [0,T],\ i=1,2,$$

where

$$S_i(t) = \int_0^A \bigl[q_i(0,t)\bigl(m_i p_i^*\bigr)(a,t) - \partial L(\beta^*,p^*,a,t)/\partial\beta_i\bigr]\,da,$$

q_i, $i = 1, 2$, *is the solution of the adjoint system*

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1 \beta_1^* q_1(0,t) + \lambda_1 q_1 P_2^*(t) \\ \qquad\qquad + \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) + \int_0^A (\lambda_2 p_2^* q_2)(a,t)\,da, \\ \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} = \mu_2 q_2 - m_2 \beta_2^* q_2(0,t) + \lambda_2 q_2 P_1^*(t) \\ \qquad\qquad + \frac{\partial L}{\partial p_2}(\beta^*, p^*, a, t) + \int_0^A (\lambda_1 p_1^* q_1)(a,t)\,da, \\ q_i(a,T) = \bar{p}_i(a) - p_i^*(a,T), \\ q_i(A,t) = 0, \quad P_i^*(t) = \int_0^A p_i^*(a,t)\,da, \quad (a,t) \in Q_T. \end{cases} \tag{7}$$

Proof. For any given $h = (h_1, h_2) \in T_U(\beta^*)$ (the tangent cone to U at β^*) and $\varepsilon > 0$ small enough, we have $\beta^\varepsilon := \beta^* + \varepsilon h \in U$.

Denoting by p^ε the state corresponding to β^ε, we can write

$$J(\beta^\varepsilon, p^\varepsilon) \geqslant J(\beta^*, p^*),$$

i.e.,

$$\int_0^T \int_0^A L(\beta^\varepsilon, p^\varepsilon, a, t)\,da\,dt + \frac{1}{2}\sum_{i=1}^2 \int_0^A \big[p_i^\varepsilon(a,T) - \bar{p}_i(a)\big]^2\,da$$
$$\geqslant \int_0^T \int_0^A L(\beta^*, p^*, a, t)\,da\,dt + \frac{1}{2}\sum_{i=1}^2 \int_0^A \big[p_i^*(a,T) - \bar{p}_i(a)\big]^2\,da. \tag{8}$$

Dividing (8) by ε and passing to the limit as $\varepsilon \to 0^+$, we obtain that

$$\sum_{i=1}^2 \left\{ \int_0^T \int_0^A \left[h_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + z_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \right] da\,dt \right.$$
$$\left. + \int_0^A z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,da \right\} \geqslant 0, \tag{9}$$

where $z_i(a,t) := \lim_{\varepsilon \to 0^+} \varepsilon^{-1}[p_i^\varepsilon(a,t) - p_i^*(a,t)]$ satisfies

$$\begin{cases} \frac{\partial z_1}{\partial t} + \frac{\partial z_1}{\partial a} = -\mu_1 z_1 - \lambda_1 p_1^* Z_2(t) - \lambda_1 P_2^*(t) z_1, \\ \frac{\partial z_2}{\partial t} + \frac{\partial z_2}{\partial a} = -\mu_2 z_2 - \lambda_2 p_2^* Z_1(t) - \lambda_2 P_1^*(t) z_2, \\ z_i(0,t) = \beta_i^*(t) \int_{a_1}^{a_2} (m_i z_i)(a,t)\,da + h_i(t) \int_{a_1}^{a_2} (m_i p_i^*)(a,t)\,da, \\ z_i(a,0) = 0, \quad Z_i(t) = \int_0^A z_i(a,t)\,da, \quad (a,t) \in Q_T. \end{cases} \tag{10}$$

Multiplying $(10)_i$ by $q_i(a,t)$, integrating on Q_T and using the system (7), we derive out that

$$\sum_{i=1}^2 \left\{ \int_0^T \int_0^A z_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\,da\,dt + \int_0^A z_i(a,T)\big[p_i^*(a,T) - \bar{p}_i(a)\big]\,da \right\}$$

$$= -\sum_{i=1}^{2}\int_0^T q_i(0,t)\int_0^A \big(m_i p_i^*\big)(a,t)\,da\cdot h_i(t)\,dt. \tag{11}$$

Combining (11) with (9), we are led to that

$$\sum_{i=1}^{2}\left\{\int_0^T\int_0^A\left[q_i(0,t)m_i(a,t)p_i^*(a,t)-\frac{\partial L}{\partial\beta_i}(\beta^*,p^*,a,t)\right]da\cdot h_i(t)\,dt\right\}\leqslant 0$$

holds for any $h\in T_U(\beta^*)$, so $S_i\in N_U(\beta^*)$ (the normal cone to U at β^*). Consequently the conclusion of Theorem 2 follows immediately.

4. Infinite horizon problem

We consider further the optimal control problem: Find (β^*, p^*), $\beta_*\in U$, such that

$$\begin{cases} J(\beta^*,p^*)=\min\{J(\beta,p)\colon\ \beta\in U,\ (\beta,p)\text{ is subject to (1)}\},\\ J(\beta,p)=\int_0^\infty\int_0^A L(\beta_1(t),\beta_2(t),p_1(a,t),p_2(a,t),a,t)\,da\,dt,\end{cases} \tag{12}$$

with other conditions similar to problem (6). Moreover we suppose that for each admissible pair (β,p), the integral in (12) is convergent.

It is trivial to prove that

Lemma 1. *If* (β^*,p^*) *is a solution to the problem* (12), *then for any given* $T>0$, (β^*,p^*) *is a solution to the following problem*:

$$\begin{cases} J_T(\beta^*,p^*)=\min\{J_T(\beta,p)\colon\ \beta\in U\},\\ J_T(\beta,p)=\int_0^T\int_0^A L(\beta_1(t),\beta_2(t),p_1(a,t),p_2(a,t),a,t)\,da\,dt,\end{cases} \tag{13}$$

where (β,p) *is subject to the system*

$$\begin{cases} \frac{\partial p_1}{\partial t}+\frac{\partial p_1}{\partial a}=-\mu_1(a,t)p_1-\lambda_1(a,t)P_2(t)p_1,\\ \frac{\partial p_2}{\partial t}+\frac{\partial p_2}{\partial a}=-\mu_2(a,t)p_2-\lambda_2(a,t)P_1(t)p_2,\\ p_i(0,t)=\beta_i(t)\int_{a_1}^{a_2}m_i(a,t)p_i(a,t)\,da,\\ p_i(a,0)=p_{i0}(a),\quad p_i(a,T)=p_i^*(a,T),\quad (a,t)\in Q.\end{cases} \tag{14}$$

Let $X=L^\infty(0,T;R^2)\times C(0,T;L^2(0,A;R^2))$. We first investigate the necessary conditions which must be satisfied for the solution to the problem (13)–(14). Define

$$\Omega_1=\big\{(\beta,p)\in X\colon\ \beta_0\leqslant\beta_i(t)\leqslant\beta^0,\ \text{a.e. } t\in[0,T],\ i=1,2\big\},$$

$$\Omega_2=\big\{(\beta,p)\in X\colon\ (\beta,p)\text{ solves the system (14)}\big\}.$$

Then the problem (13)–(14) is equivalent to the problem: Find $(\beta^*,p^*)\in\Omega_1\cap\Omega_2$, such that

$$J_T(\beta^*,p^*)=\min\big\{J_T(\beta,p)\colon\ (\beta,p)\in\Omega_1\cap\Omega_2\big\}. \tag{15}$$

In what follows, we will use the general theory of Dubovitskii and Milyutin for extremal problems to deal with the problem (15), which needs to determine the corresponding cones.

Under the assumptions for $J(\beta, p)$, the functional J_T is differentiable at any point $(\tilde{\beta}, \tilde{p})$ and

$$J'_T(\tilde{\beta}, \tilde{p})(\beta, p) = \sum_{i=1}^{2} \int_0^T \int_0^A \left[\beta_i(t) \frac{\partial L}{\partial \beta_i}(\tilde{\beta}, \tilde{p}, a, t) + p_i(a, t) \frac{\partial L}{\partial p_i}(\tilde{\beta}, \tilde{p}, a, t)\right] da\, dt.$$

Since $J_T(\beta, p)$ is regularly decreasing at (β^*, p^*), its directions of decrease cone is

$$K_0 = \left\{(\beta, p) \in X\colon\ J'_T(\beta^*, p^*)(\beta, p) < 0\right\}.$$

If $K_0 \neq \emptyset$, then for any $f_0 \in K_0^*$ (the dual cone of K_0), there exists $\lambda_0 \geqslant 0$ such that

$$\begin{aligned} f_0(\beta, p) = -\lambda_0 \sum_{i=1}^{2} \int_0^T \int_0^A \Bigg[& \beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) \\ & + p_i(a, t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\Bigg] da\, dt. \end{aligned} \tag{16}$$

Note that $\Omega_1 = \hat{\Omega}_1 \times C(0, T; L^2(0, A; R^2))$ (where $\hat{\Omega}_1 = \{\beta \in L^\infty(0, T; R^2)\colon\ \beta_0 \leqslant \beta_i(t) \leqslant \beta^0\}$) is a closed convex subset of X. Thus

$$\operatorname{int}(\Omega_1) = \operatorname{int}(\hat{\Omega}_1) \times C\big(0, T; L^2(0, A; R_2)\big) \neq \emptyset,$$

where $\operatorname{int}(\Omega_1)$ denotes the interior of Ω_1. Hence the feasible directions cone for Ω_1 at (β^*, p^*) is

$$\begin{aligned} K_1 &= \left\{\lambda\big(\operatorname{int}(\Omega_1) - (\beta^*, p^*)\big)\colon\ \lambda > 0\right\} \\ &:= \left\{\lambda\big((\beta, p) - (\beta^*, p^*)\big)\colon\ (\beta, p) \in \operatorname{int}(\Omega_1),\ \lambda > 0\right\}. \end{aligned}$$

For any functional $f_1 \in K_1^*$, if there exists $a_i(t) \in L^1(0, T)$, $i = 1, 2$, such that

$$f_1(\beta, p) = \sum_{i=1}^{2} \int_0^T a_i(t) \beta_i(t)\, dt, \tag{17}$$

then [10, p. 76]

$$\sum_{i=1}^{2} a_i(t)\big[\beta_i - \beta_i^*(t)\big] \geqslant 0, \quad \forall \beta_i \in [\beta_0, \beta^0],\ \text{a.e. } t \in [0, T]. \tag{18}$$

Next we determine the tangent directions cone for Ω_2 at (β^*, p^*). As far as the mild solutions are concerned, system (14) is equivalent to the following system:

$$\begin{cases} u_1(a,t) := \int_0^a [p_1(\tau,t) - p_{10}(\tau)]\,d\tau + \int_0^t p_1(a,s)\,ds \\ \qquad - \int_0^t \int_{a_1}^{a_2} \beta_1(s) m_1(a,s) p_1(a,s)\,da\,ds \\ \qquad + \int_0^t \int_0^a p_1(\tau,s)[\mu_1(\tau,s) + \lambda_1(\tau,s) P_2(s)]\,d\tau\,ds = 0, \\ u_2(a,t) := \int_0^a [p_2(\tau,t) - p_{20}(\tau)]\,d\tau + \int_0^t p_2(a,s)\,ds \\ \qquad - \int_0^t \int_{a_1}^{a_2} \beta_2(s) m_2(a,s) p_2(a,s)\,da\,ds \\ \qquad + \int_0^t \int_0^a p_2(\tau,s)[\mu_2(\tau,s) + \lambda_2(\tau,s) P_1(s)]\,d\tau\,ds = 0, \\ p_i(a,T) = p_i^*(a,T), \quad i = 1,2. \end{cases} \tag{19}$$

Define the operator $G : X \to C(0,T; L^2(0,A; R^2))$,

$$\big[G(\beta,p)\big](a,t) = \big(u_1(a,t), u_2(a,t), p_1(a,T) - p_1^*(a,T), p_2(a,T) - p_2^*(a,T)\big).$$

So, $\Omega_2 = \{(\beta,p) \in X:\ G(\beta,p) = 0\}$, and

$$G'(\beta^*,p^*)(\beta,p) = \big(v_1(a,t), v_2(a,t), p_1(a,T), p_2(a,T)\big),$$

where

$$\begin{aligned} v_1(a,t) = {} & \int_0^a p_1(\tau,t)\,d\tau + \int_0^t p_1(a,s)\,ds + \int_0^t \int_0^a (\mu_1 p_1)(\tau,s)\,d\tau\,ds \\ & - \int_0^t \int_{a_1}^{a_2} m_1(a,s)\big[\beta_1^*(s) p_1(a,s) + \beta_1(s) p_1^*(a,s)\big]\,da\,ds \\ & + \int_0^t \int_0^a \lambda_1(\tau,s)\big[p_1^*(\tau,s) P_2(s) + p_1(\tau,s) P_2^*(s)\big]\,d\tau\,ds, \end{aligned} \tag{20}$$

$$\begin{aligned} v_2(a,t) = {} & \int_0^a p_2(\tau,t)\,d\tau + \int_0^t p_2(a,s)\,ds + \int_0^t \int_0^a (\mu_2 p_2)(\tau,s)\,d\tau\,ds \\ & - \int_0^t \int_{a_1}^{a_2} m_2(a,s)\big[\beta_2^*(s) p_2(a,s) + \beta_2(s) p_2^*(a,s)\big]\,da\,ds \\ & + \int_0^t \int_0^a \lambda_2(\tau,s)\big[p_2^*(\tau,s) P_1(s) + p_2(\tau,s) P_1^*(s)\big]\,d\tau\,ds. \end{aligned} \tag{21}$$

To show that $G'(\beta^*,p^*)$ is an onto mapping, we solve equation $G'(\beta^*,p^*)(\beta,p) = (w_1, w_2, w_3, w_4)$, i.e.,

$$\begin{cases} \int_0^a p_1(\tau,t)\,d\tau + \int_0^t p_1(a,s)\,ds + \int_0^t \int_0^a (\mu_1 p_1)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_1(a,s)[\beta_1^*(s)p_1(a,s) + \beta_1(s)p_1^*(a,s)]\,da\,ds \\ \quad + \int_0^t \int_0^a \lambda_1(\tau,s)[p_1^*(\tau,s)P_2(s) + p_1(\tau,s)P_2^*(s)]\,d\tau\,ds = w_1(a,t), \\ \int_0^a p_2(\tau,t)\,d\tau + \int_0^t p_2(a,s)\,ds + \int_0^t \int_0^a (\mu_2 p_2)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_2(a,s)[\beta_2^*(s)p_2(a,s) + \beta_2(s)p_2^*(a,s)]\,da\,ds \\ \quad + \int_0^t \int_0^a \lambda_2(\tau,s)[p_2^*(\tau,s)P_1(s) + p_2(\tau,s)P_1^*(s)]\,d\tau\,ds = w_2(a,t), \\ p_1(a,T) = w_3(a), \qquad p_2(a,T) = w_4(a), \end{cases} \tag{22}$$

where (w_1, w_2, w_3, w_4) is prescribed.

Note that the linearized system of (1) at (β^*, p^*) is

$$\begin{cases} \frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial a} = -\mu_1(a,t)p_1 - \lambda_1(a,t)[P_2^*(t)p_1 + P_2(t)p_1^*], \\ \frac{\partial p_2}{\partial t} + \frac{\partial p_2}{\partial a} = -\mu_2(a,t)p_2 - \lambda_2(a,t)[P_1^*(t)p_2 + P_1(t)p_2^*], \\ p_i(0,t) = \int_{a_1}^{a_2} m_i(a,t)[\beta_i^*(t)p_i(a,t) + \beta_i(t)p_i^*(a,t)]\,da, \\ p_i(a,0) = 0, \quad (a,t) \in Q. \end{cases} \tag{23}$$

It is easily seen that each mild solution of (23) satisfies Eqs. $(22)_1$ and $(22)_2$. So there is at least one solution to (22) if the system (23) is controllable. In fact, there exists $(\hat{\beta}_1, \hat{\beta}_2)$ such that the corresponding solution of the system (23) satisfies

$$\hat{p}_1(a,T) = w_3(a) - \gamma_1(a,T), \qquad \hat{p}_2(a,T) = w_4(a) - \gamma_2(a,T),$$

where $\gamma_i(a,t)$ is the solution to the following system:

$$\begin{cases} \int_0^a \gamma_1(\tau,t)\,d\tau + \int_0^t \gamma_1(a,s)\,ds + \int_0^t \int_0^a (\mu_1\gamma_1)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_1(a,s)\beta_1^*(s)\gamma_1(a,s)\,da\,ds \\ \quad + \int_0^t \int_0^a \lambda_1(\tau,s)[p_1^*(\tau,s)\Gamma_2(s) + \gamma_1(\tau,s)P_2^*(s)]\,d\tau\,ds = w_1(a,t), \\ \int_0^a \gamma_2(\tau,t)\,d\tau + \int_0^t \gamma_2(a,s)\,ds + \int_0^t \int_0^a (\mu_2\gamma_2)(\tau,s)\,d\tau\,ds \\ \quad - \int_0^t \int_{a_1}^{a_2} m_2(a,s)\beta_2^*(s)\gamma_2(a,s)\,da\,ds \\ \quad + \int_0^t \int_0^a \lambda_2(\tau,s)[p_2^*(\tau,s)\Gamma_1(s) + \gamma_2(\tau,s)P_1^*(s)]\,d\tau\,ds = w_2(a,t), \\ \Gamma_i(s) = \int_0^{\bar{A}} \gamma_i(a,s)\,da, \quad i = 1,2. \end{cases}$$

Then it is not difficult to show that $(\hat{\beta}_1, \hat{\beta}_2, \hat{p}_1 + \gamma_1, \hat{p}_2 + \gamma_2)$ solves system (22). Now the tangent directions cone K_2 consists of the kernel of $G'(\beta^*, p^*)$.

Define the linear subspaces of X by

$$K_{11} = \{(\beta, p) \in X:\ v_i(a,t) \equiv 0,\ i = 1,2\},$$
$$K_{12} = \{(\beta, p) \in X:\ p_i(a,T) \equiv 0,\ i = 1,2\},$$

where v_1 and v_2 are given by (20) and (21). Then $K_2 = K_{11} \cap K_{12}$, $K_2^* = K_{11}^* + K_{12}^*$.

For any $f_2 \in K_2^*$, $f_2 = f_{11} + f_{12}$, $f_{1i} \in K_{1i}^*$, $i = 1,2$, there exists $\alpha_i(a) \in L^2(0,A)$, $i = 1,2$, such that

$$f_{12}(\beta, p) = \sum_{i=1}^{2} \int_0^A \alpha_i(a)p_i(a,T)\,da. \tag{24}$$

According to Dubovitskii–Milyutin's theorem [10, Theorem 6.1], there exist functionals $f_0 \in K_0^*$, $f_1 \in K_1^*$, $f_{1i} \in K_{1i}^*$, $i = 1, 2$, not all zero, such that

$$f_0 + f_1 + f_{11} + f_{12} = 0. \tag{25}$$

For any $\beta \in L^\infty(0, T)$, select p such that the first two equations in (22) holds. Then $(\beta, p) \in K_{11}$ and $f_{11}(\beta, p) = 0$ [10, Theorem 10.1], from which

$$\begin{aligned} f_1(\beta, p) &= -f_0(\beta, p) - f_{12}(\beta, p) \\ &= \sum_{i=1}^{2} \left\{ \lambda_0 \int_0^T \int_0^A \left[\beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + p_i(a, t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \right] da\, dt \right. \\ &\qquad \left. - \int_0^A \alpha_i(a) p_i(a, T)\, da \right\}. \end{aligned} \tag{26}$$

Define the adjoint system

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1 \beta_1^* q_1(0, t) + \lambda_1 q_1 P_2^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) + \int_0^A (\lambda_2 p_2^* q_2)(a, t)\, da, \\ \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} = \mu_2 q_2 - m_2 \beta_2^* q_2(0, t) + \lambda_2 q_2 P_1^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_2}(\beta^*, p^*, a, t) + \int_0^A (\lambda_1 p_1^* q_1)(a, t)\, da, \\ q_i(a, T) = \alpha_i(a), \\ q_i(A, t) = 0, \quad (a, t) \in Q_T. \end{cases} \tag{27}$$

Then we can prove that

$$\begin{aligned} &\sum_{i=1}^{2} \left[\lambda_0 \int_0^T \int_0^A p_i(a, t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\, da\, dt - \int_0^A \alpha_i(a) p_i(a, T)\, da \right] \\ &\quad = -\sum_{i=1}^{2} \int_0^T q_i(0, t) \int_0^A m_i(a, t) p_i^*(a, t)\, da \cdot \beta_i(t)\, dt. \end{aligned} \tag{28}$$

From (26) and (28),

$$f_1(\beta, p) = \sum_{i=1}^{2} \int_0^T \int_0^A \left[\lambda_0 \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) - q_i(0, t) m_i(a, t) p_i^*(a, t) \right] da \cdot \beta_i(t)\, dt. \tag{29}$$

Consequently (18) leads us to

$$\begin{cases} \sum_{i=1}^{2} \int_0^A \left[\lambda_0 \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) - q_i(0, t) m_i(a, t) p_i^*(a, t) \right] da \\ \quad \times [\beta_i - \beta_i^*(t)] \geqslant 0, \\ \quad \forall \beta_i \in [\beta_0, \beta^0],\ \text{a.e. } t \in [0, T],\ i = 1, 2. \end{cases} \tag{30}$$

We claim that there is no possibility for both λ_0 and $\alpha(a) = (\alpha_1(a), \alpha_2(a))$ being zero. Otherwise $f_0 = 0$, $f_{12} = 0$, $q_i(a,t) = 0$, $f_1 = 0$ (Ref. [10]). Then from (25), $f_{11} = 0$. This contradicts the fact that f_0, f_1, f_{11}, f_{12} are not all identically zero.

On the other hand, if $K_0 = \emptyset$, then for any $(\beta, p) \in X$,

$$\sum_{i=1}^{2} \int_0^T \int_0^A \left[\beta_i(t) \frac{\partial L}{\partial \beta_i}(\beta^*, p^*, a, t) + p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t) \right] da\, dt = 0. \tag{31}$$

Choosing $\lambda_0 = 1$ and $\alpha(a) = 0$ in (28) yields

$$\begin{cases} \sum_{i=1}^2 \int_0^T \int_0^A p_i(a,t) \frac{\partial L}{\partial p_i}(\beta^*, p^*, a, t)\, da\, dt \\ \quad = -\sum_{i=1}^2 \int_0^T q_i(0,t) \int_0^A m_i(a,t) p_i^*(a,t)\, da \cdot \beta_i(t)\, dt. \end{cases} \tag{32}$$

Combining (31) with (32), we still get inequality (30).

Finally, if the adjoint system (27) has a nonzero solution such that

$$\int_0^A q_i(0,t) m_i(a,t) p_i^*(a,t)\, da = 0, \quad \text{a.e. } t \in [0,T],\ i = 1,2, \tag{33}$$

then let $\lambda_0 = 0$, inequality (30) is also satisfied. If for any nonzero solution of (27), we always have

$$\left(\int_0^A q_1(0,t) m_1(a,t) p_1^*(a,t)\, da, \int_0^A q_2(0,t) m_2(a,t) p_2^*(a,t)\, da \right) \neq 0, \tag{34}$$

then the system (23) must be controllable; otherwise there exists $\alpha(a) \in L^2(0, A; R^2)$ such that

$$\sum_{i=1}^2 \int_0^A \alpha_i(a) p_i(a,t)\, da = 0, \quad \alpha(a) \neq 0.$$

Choosing $\lambda_0 = 0$ in (28), we obtain that

$$\sum_{i=1}^2 \int_0^T q_i(0,t) \int_0^A m_i(a,t) p_i^*(a,t)\, da \cdot \beta_i(t)\, dt = 0$$

holds for arbitrary $\beta_i(t) \in [\beta_0, \beta^0]$, which yields (33). This contradicts (34). Therefore the system (23) is controllable.

In all cases, inequality (30) remains valid. We have proved

Theorem 3. *If (β^*, p^*) is a solution to the problem* (13)–(14)*, then there exists $\lambda_{0T} \geqslant 0$, $\alpha_T(a) \in L^2(0, A; R^2)$, not all zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\{\beta \cdot H(\beta^*, p^*) \colon \beta \in [\beta_0, \beta^0]\}, \quad \textit{a.e. } t \in [0,T],$$

where $\cdot$ denotes the scalar product in R^2,

$$H(\beta^*, p^*) = \left(\int_0^A \left[q_1(0,t)m_1(a,t)p_1^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_1}(\beta^*, p^*, a, t) \right] da, \right.$$
$$\left. \int_0^A \left[q_2(0,t)m_2(a,t)p_2^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_2}(\beta^*, p^*, a, t) \right] da \right),$$

q_i, $i = 1, 2$, *is the solution of the adjoint system* (27) *corresponding to* $\lambda_0 = \lambda_{0T}$, $\alpha_i = \alpha_{iT}$.

Now return to the infinite time problem (12). We suppose

$$\lambda_{0T} + \left\| q_T(a,\cdot) \right\|_{L^2(0,T;R^2)} \leqslant M, \quad \text{a.e. } a \in [0, A], \tag{35}$$

where $M > 0$ is a constant. Choose $T_N \to \infty$ such that $\lambda_{0T_N} \to \lambda_\infty$. For any fixed $t > 0$ and T_N large enough, by means of characteristic line, we derive out that

$$\begin{cases} q_{1T_N}(0,t) = \int_t^{T_N} \exp\{-\int_t^s [\mu_1(\rho - t, \rho) + \lambda_1(\rho - t, \rho)P_2^*(\rho)]\, d\rho\} \\ \qquad \times \big[m_1(s-t, s)\beta_1^*(s) q_{1T_N}(0,s) \\ \qquad - \int_0^A (\lambda_2 p_2^* q_{2T_N})(a,s)\, da - \lambda_{0T_N} \frac{\partial L}{\partial p_1}(\beta^*, p^*, s-t, s) \big]\, ds, \\ q_{2T_N}(0,t) = \int_t^{T_N} \exp\{-\int_t^s [\mu_2(\rho - t, \rho) + \lambda_2(\rho - t, \rho)P_1^*(\rho)]\, d\rho\} \\ \qquad \times \big[m_2(s-t, s)\beta_2^*(s) q_{2T_N}(0,s) \\ \qquad - \int_0^A (\lambda_1 p_1^* q_{1T_N})(a,s)\, da - \lambda_{0T_N} \frac{\partial L}{\partial p_2}(\beta^*, p^*, s-t, s) \big]\, ds. \end{cases} \tag{36}$$

Note that $\int_t^s \mu_1(\rho - t, \rho)\, d\rho = +\infty$ when $s \geqslant t + A$. So the integration interval $[t, T_N]$ in (36) can be replaced by $[t, t+A]$. From (35) it follows that $\|q_{T_N}(a,\cdot)\|_{L^2(t,t+A;R^2)} \leqslant M$. Thus there is a subsequence of time (also denoted by $\{T_N\}$) such that

$$q_{T_N}(a,\cdot) \to q_\infty(a,\cdot) \quad \text{weakly in } L^2(t, t+A; R^2). \tag{37}$$

From (36) and (37), it is not difficult to prove that

$$\begin{cases} q_{1\infty}(0,t) = \int_t^{t+A} \exp\{-\int_t^s [\mu_1(\rho - t, \rho) + \lambda_1(\rho - t, \rho)P_2^*(\rho)]\, d\rho\} \\ \qquad \times \big[m_1(s-t, s)\beta_1^*(s) q_{1\infty}(0,s) \\ \qquad - \int_0^A (\lambda_2 p_2^* q_{2\infty})(a,s)\, da - \lambda_\infty \frac{\partial L}{\partial p_1}(\beta^*, p^*, s-t, s) \big]\, ds, \\ q_{2\infty}(0,t) = \int_t^{t+A} \exp\{-\int_t^s [\mu_2(\rho - t, \rho) + \lambda_2(\rho - t, \rho)P_1^*(\rho)]\, d\rho\} \\ \qquad \times \big[m_2(s-t, s)\beta_2^*(s) q_{2\infty}(0,s) \\ \qquad - \int_0^A (\lambda_1 p_1^* q_{1\infty})(a,s)\, da - \lambda_\infty \frac{\partial L}{\partial p_2}(\beta^*, p^*, s-t, s) \big]\, ds, \end{cases}$$

which enables us to state

Theorem 4. *Let* (β^*, p^*) *be a solution for the problem* (12), *then there exist* $\lambda_\infty \geqslant 0$ *and a function* $q : [0,\infty) \to R^2$, *not simultaneously zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\big\{ \beta \cdot H(\beta^*, p^*) \colon \beta \in [\beta_0, \beta^0] \big\}, \quad \textit{a.e. } t \in [0, \infty],$$

where

$$H(\beta^*,p^*)=\left(\int_0^A\left[q_1(0,t)m_1(a,t)p_1^*(a,t)-\lambda_\infty\frac{\partial L}{\partial\beta_1}(\beta^*,p^*,a,t)\right]da,\right.$$
$$\left.\int_0^A\left[q_2(0,t)m_2(a,t)p_2^*(a,t)-\lambda_\infty\frac{\partial L}{\partial\beta_2}(\beta^*,p^*,a,t)\right]da\right),$$

$q(a,t)$ *is given by the following adjoint system*:

$$\begin{cases}\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial a}=\mu_1q_1-m_1\beta_1^*q_1(0,t)+\lambda_1q_1P_2^*(t)\\ \qquad\qquad+\lambda_\infty\frac{\partial L}{\partial p_1}(\beta^*,p^*,a,t)+\int_0^A(\lambda_2p_2^*q_2)(a,t)\,da,\\ \frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial a}=\mu_2q_2-m_2\beta_2^*q_2(0,t)+\lambda_2q_2P_1^*(t)\\ \qquad\qquad+\lambda_\infty\frac{\partial L}{\partial p_2}(\beta^*,p^*,a,t)+\int_0^A(\lambda_1p_1^*q_1)(a,t)\,da,\\ q_i(a,\infty)=0,\\ q_i(A,t)=0,\quad (a,t)\in Q_T,\ i=1,2.\end{cases}$$

5. Constrained end point problem

Problem (13)–(14) leads us to the following problem:

$$\text{minimize } J(\beta,p)=\int_0^T\int_0^A L\big(\beta_1(t),\beta_2(t),p_1(a,t),p_2(a,t),a,t\big)\,da\,dt, \tag{38}$$

where $T>0$ is fixed, $\beta\in U$ and (β,p) is subject to the system (1) and

$$p_i(\cdot,T)\in V_i,\quad V_i=\left\{p\in L^2(0,A):\ \|p-p_i^0\|\leqslant\varepsilon\right\},\ i=1,2, \tag{39}$$

in which p_i^0 and ε are prescribed. The assumptions on L and the definition of X and Ω_1 are as before.

Let

$$\Omega_2=\left\{(\beta,p)\in X:\ p_i(\cdot,T)\in V_i,\ i=1,2\right\},$$
$$\Omega_3=\left\{(\beta,p)\in X:\ (\beta,p)\text{ satisfies (1)}\right\}.$$

Suppose that (β^*,p^*) solves the problem (38)–(39). Clearly the cone of directions of decrease and its dual cone are as in Section 4; so are the feasible directions cone for Ω_1 and its dual. Since Ω_2 is a closed convex set and $\mathrm{int}(\Omega_2)\neq\emptyset$, any functional f_2 in the dual of the feasible directions cone for Ω_2 is supporting; that is,

$$f_2(\beta,p)\geqslant f_2(\beta^*,p^*),\quad \forall p(a,T)\in V_1\times V_2.$$

Obviously there exists $\alpha\in L^2(0,A;R^2)$ such that

$$f_2(\beta,p)=\int_0^A\alpha(a)\cdot p(a,T)\,da.$$

Therefore [11, p. 300]

$$\alpha(a) = \tilde{\lambda}_0\big[p^0(a) - p^*(a, T)\big], \quad \tilde{\lambda}_0 \geqslant 0.$$

Then by a reasoning similar to that in Section 4, we arrive at

Theorem 5. *If* (β^*, p^*) *is a solution to the problem* (38)–(39)*, then there exist* $\lambda_0 \geqslant 0$ *and* $\tilde{\lambda}_0 \geqslant 0$*, not both zero, such that*

$$\beta^*(t) \cdot H(\beta^*, p^*) = \max\big\{\beta \cdot H(\beta^*, p^*):\ \beta \in [\beta_0, \beta^0]\big\}, \quad a.e.\ t \in [0, T],$$

$$H(\beta^*, p^*) = \left(\int_0^A \left[q_1(0,t)m_1(a,t)p_1^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_1}(\beta^*, p^*, a, t) \right] da, \right.$$
$$\left. \int_0^A \left[q_2(0,t)m_2(a,t)p_2^*(a,t) - \lambda_0 \frac{\partial L}{\partial \beta_2}(\beta^*, p^*, a, t) \right] da \right),$$

and q *is the solution of the following adjoint system*:

$$\begin{cases} \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial a} = \mu_1 q_1 - m_1\beta_1^* q_1(0,t) + \lambda_1 q_1 P_2^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_1}(\beta^*, p^*, a, t) + \int_0^A (\lambda_2 p_2^* q_2)(a,t)\, da, \\ \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial a} = \mu_2 q_2 - m_2\beta_2^* q_2(0,t) + \lambda_2 q_2 P_1^*(t) \\ \qquad\qquad + \lambda_0 \frac{\partial L}{\partial p_2}(\beta^*, p^*, a, t) + \int_0^A (\lambda_1 p_1^* q_1)(a,t)\, da, \\ q_i(a,T) = \tilde{\lambda}_0[p_i^0(a) - p_i^*(a,T)], \\ q_i(A,t) = 0, \quad (a,t) \in Q_T,\ i = 1, 2. \end{cases}$$

6. Concluding remarks

Note that just for the sake of simplicity, the average fertility of female individuals, $\beta_i(t)$, in the system (1) is chosen to be independent of age a. Replacing $\beta_i(t)$ with $\beta_i(a,t)$ forms no essential obstacles to the previous treatment.

On the other hand, the situations seem rather difficult to deal with for the symbiotic system, we will investigate the problem in another work.

Finally one can easily check that the all results but the time-optimal problem in Ref. [11] are contained by this work.

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