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On a structural acoustic model with interface a Reissner–Mindlin plate or a Timoshenko beam

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Abstract

This paper is concerned with a model which describes the interaction of sound and elastic waves in a structural acoustic chamber in which one "wall" is flexible and flat. The model is new in the sense that the composite dynamics of the three-dimensional structure is described by the linearized equations for a gas defined on the interior of the chamber and the Reissner–Mindlin plate equations on the two-dimensional flat wall of the chamber, while, if a two-dimensional acoustic chamber is considered, the Timoshenko beam equations describe the deflections of the one-dimensional "wall." With a view to achieving uniform stabilization of the structure linear feedback boundary damping is incorporated in the model, viz. in the wave equation for the gas and in the system of equations for the vibrations of the elastic medium. We present the uniform stability result for the case of a two-dimensional chamber and outline the method for the three-dimensional model which shows strong resemblance with the system of dynamic plane elasticity.

Keywords: Structural acoustic model; Reissner–Mindlin plate; Timoshenko beam; Uniform stabilization

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1. Introduction and statement of the problem

Well-posedness results, with a strong focus on stabilization, for the equations governing the coupled dynamic behaviour of three-dimensional acoustic chambers in which one "flat" wall is a flexible plate, have attracted considerable attention in recent years. Pioneering contributions considering linear as well as nonlinear models in which the elastic wall is an isothermal Bernoulli, Kirchoff or a Von Kármán plate, are due to Avalos and Lasiecka [2–4,19,21], while Lasiecka and Lebiedzik [24] considered nonlinear models with thermal effects on the interface. Whereas in the earlier models nonlinear boundary dissipation, acting at the free edge of the isothermal plate, is introduced while additionally viscous or boundary damping is incorporated in the equations modelling the linearized gas flow, the introduction of thermal effects into the later models requires a minimal amount of damping: it turns out that potential damping (friction) in the boundary conditions for the wave equation on part of the hard wall and boundary damping at the edge of the interface between the acoustic medium and the elastic medium yields stability with no mechanical damping imposed on the interface itself or in the wave equation in the three-dimensional region.

Related work is the investigations of the interactive problem occurring in aeroelasticity by Boutet de Monvel and Chueshov [6]. In particular, in [8] the dynamics of a Von Kármán plate subject to aerodynamical pressure of a subsonic flow of gas in $\mathbb{R}^3_+$ is considered and stabilization of the entire structure achieved by incorporating internal damping in the clamped plate.

It appears that in none of the structural acoustic models in the literature, the Reissner–Mindlin plate equations or Timoshenko beam equations are used to describe the deflections of the elastic medium. As is known, the Reissner–Mindlin equations arise when the Kirchoff hypothesis [12, p. 16] is discarded, i.e., the assumption that filaments (assumed straight and without strain deformation) of the plate remain perpendicular to the deformed surface. Thus two additional degrees of freedom, ψ and ϕ, which represent the angles of rotation of filaments, come into play and one has a model in (w, ψ, ϕ) with w the displacement variable. The reader is referred to [28] for the classical Reissner–Mindlin equations, to [31] for the Timoshenko beam equations, i.e., the one-dimensional analogue of the Reissner–Mindlin plate equations, and to [15] where plate-beam models in which the energy of the plate is that associated with linear Reissner–Mindlin plate theory, are considered. More recently a hybrid structure consisting of a rectangular Reissner–Mindlin plate in interaction with a Timoshenko beam was studied by Grobbelaar-Van Dalsen [9].

With regard to the use of the Reissner–Mindlin plate equations in a three-dimensional structural acoustics model, it is important to note that, apart from the fact that the Reissner–Mindlin system provides improved accuracy over the whole frequency by including both rotary inertia as well as transverse shear deformation effects, the Euler–Bernoulli model ceases to be valid at high frequencies, when the wave length of flexural motions becomes comparable to the thickness of the plate [10, p. 21].[1] The implication of this is that while

[1] The cited reference uses the Mindlin model [27] in the displacement variable w—see our later remark on the equivalence, by decoupling, of this equation to the Reissner–Mindlin system.

the literature provides for large deflections of the plate in a structural acoustics model, i.e., in [20] in which the deflections of the plate are described by the full Von Kármán system, it does not provide for the occurrence of high frequencies when the wave length of flexural motions becomes comparable to the thickness of the plate. In view of the above considerations it appears that research on structural acoustic models which use the Reissner–Mindlin equations or, in the case of a two-dimensional acoustic chamber, the Timoshenko equations, to describe the transversal deflections of the elastic medium, could be a meaningful addition to the existing literature on the subject. Moreover, private communication with I. Lasiecka inspired the investigation of the following question:

If in a structural acoustics model a Reissner–Mindlin plate or a Timoshenko beam is used at the interface between the acoustic and the structural medium, is the entire structure stabilizable with the aid of feedback boundary damping?

This paper is a first attempt to address the question of stabilization of the entire structure by the implementation of a simple linear feedback boundary damping scheme. The techniques used to obtain estimates of the energies associated with the more general problem are to a large extent adopted from the work of I. Lasiecka (see, e.g., [20,21]). The plan of the paper is as follows. In Section 2 we present a model for the three-dimensional case, i.e., in which the acoustic chamber is three-dimensional and the elastic medium at the flat wall a two-dimensional Reissner–Mindlin plate which is free on part of its edge and clamped on the remaining part—the two-dimensional model in which the acoustic chamber is two-dimensional and the elastic medium a Timoshenko beam which is clamped at one end and free at the other, is then derived immediately. In Section 3 we present existence and uniqueness results and show energy decrease for both the three- and the two-dimensional models. In Section 4 uniform stability of the energy associated with a weak solution of the two-dimensional model is shown with the aid of the method of multipliers. The approach of "lowering" the dimension of the problem and the resulting absence of boundary derivatives in the tangential direction, rule out the need for highly technical microlocal estimates which are required in the three-dimensional case [21, p. 208], [20, p. 1383], while the power of the method of multipliers is illustrated equally effectively. On the other hand it should be noted that the results can be extended to the three-dimensional case. Guidelines in this respect are provided in Remark 4.10.

The significance of the paper is firstly the novelty of the model, which is not only a more faithful model over the whole frequency range, at least as far as the flexural vibrations of the elastic medium are concerned, but also valid at high frequencies, and secondly the fact that, with more comprehensive energy functionals, uniform stabilization of the structure is still attainable by applying only feedback boundary damping (with the result in this study presented only for a two-dimensional acoustic chamber). More precisely a linear modification of the feedback boundary stabilization scheme applied in the structural acoustic model [21] in which the deflections of the plate are governed by the Euler–Bernoulli equation, yields uniform stabilization of the energy associated with the problem. Once more our results can be extended by implementing a nonlinear stabilization scheme along with a possible nonlinearity in the model and proceeding analogously as in [21].

2. Constitutive equations

Let $\Omega \subset \mathbb{R}^3$ denote an open bounded domain with boundary Γ sufficiently smooth. We assume that $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_0 and Γ_1 disjoint and each Γ_i, $i = 0, 1$, simply connected, and Γ_0 an open bounded region in $\mathbb{R}^2$ with boundary $\partial\Gamma_0$ consisting of two disjoint portions $\partial_0\Gamma_0$ and $\partial_1\Gamma_0$. The variable z will describe the dynamics in the acoustic medium Ω and ηz_t, with $\eta \ll 1$ the density of the gas, denotes the acoustic pressure. The pair (w, ψ, ϕ) will describe the deflections of the interface Γ_0 between the acoustic and the structural medium. We assume that the plate is clamped along $\partial_0\Gamma_0$ and free along $\partial_1\Gamma_0$. By incorporating feedback boundary damping into the equation for the gas and into the system of equations for the Reissner–Mindlin plate on $\partial_1\Gamma_0$, the constitutive equations comprise the following interactive system in z, w, ψ and ϕ:

$$
\begin{aligned}
&z_{tt} - c^2 \Delta z = 0 && \text{in } \Omega \times (0, \infty),\\
&\frac{\partial z}{\partial n} + \ell_0 z + d z_t = 0 && \text{on } \Gamma_1 \times (0, \infty),\\
&\frac{\partial z}{\partial n} + d z_t - w_t = 0 && \text{on } \Gamma_0 \times (0, \infty);
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\rho h^3}{12}(\psi_{tt}, \phi_{tt}) - D \operatorname{div} M + KF = 0,\\
&\rho h w_{tt} - K \operatorname{div} F + \eta z_t = 0 \quad \text{in } \Gamma_0 \times (0, \infty);
\end{aligned}
$$

$$
\psi = \phi = w = 0 \quad \text{on } \partial_0\Gamma_0 \times (0, \infty);
$$

$$
\begin{aligned}
&DM \cdot n = (-k_0 \psi_t, -k_1 \phi_t),\\
&KF \cdot n = -k_2 w_t \quad \text{on } \partial_1\Gamma_0 \times (0, \infty),
\end{aligned}
$$

where

$$
M = (m_{ij}) = \begin{pmatrix} \psi_x + \mu\phi_y & \left(\frac{1-\mu}{2}\right)(\psi_y + \phi_x) \\ \left(\frac{1-\mu}{2}\right)(\psi_y + \phi_x) & \phi_y + \mu\psi_x \end{pmatrix}
$$

and

$$
F = (\psi + w_x, \phi + w_y), \qquad n = (n_1, n_2)
$$

denote respectively the moment matrix, the shear force vector and the unit outward normal vector to $\partial_1\Gamma_0$. The constant $c = \sqrt{p/\eta}$, p the pressure, denotes the speed of sound while ρ and h denote respectively the density and thickness of the plate. The coefficients K and $D = \frac{Eh^3}{12(1-\mu^2)}$, with $0 < \mu < 1$ Poisson's ratio, denote respectively the shear modulus and the modulus of flexural rigidity of the plate. We assume that the constants ℓ_0, d are strictly positive while k_0, k_1, k_2 are nonnegative with $k_0 + k_1 + k_2 > 0$. In this study we shall retain the physical parameters c^2, ρ, h, K, D, etc., instead of taking them as unity. This seems more true to the physics of the problem and may also be useful in a numerical approach to the problem.

2.1. Initial-boundary-value problem for the three-dimensional case

In the three-dimensional case the above system of constitutive equations, appended by initial conditions, yields the following initial-boundary-value problem in (z, w, ψ, ϕ), to be referred to in what follows as $Pr(P)$:

$$
\begin{aligned}
& z_{tt} - c^2 \Delta z = 0 && \text{in } \Omega \times (0,\infty), \\
& \frac{\partial z}{\partial n} + \ell_0 z + d z_t = 0 && \text{on } \Gamma_1 \times (0,\infty), \\
& \frac{\partial z}{\partial n} + d z_t - w_t = 0 && \text{on } \Gamma_0 \times (0,\infty);
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\rho h^3}{12}\psi_{tt} - D\psi_{xx} - D\left(\frac{1-\mu}{2}\right)\psi_{yy} - D\left(\frac{1+\mu}{2}\right)\phi_{xy} + K(\psi + w_x) = 0, \\
& \frac{\rho h^3}{12}\phi_{tt} - D\phi_{yy} - D\left(\frac{1-\mu}{2}\right)\phi_{xx} - D\left(\frac{1+\mu}{2}\right)\psi_{xy} + K(\phi + w_y) = 0, \\
& \rho h w_{tt} - K(\psi + w_x)_x - K(\phi + w_y)_y + \eta z_t = 0 \quad \text{on } \Gamma_0 \times (0,\infty);
\end{aligned}
$$

$$
\psi = \phi = w = 0 \quad \text{on } \partial_0 \Gamma_0;
$$

$$
\begin{aligned}
& D\left[n_1\psi_x + \mu n_1 \phi_y + \left(\frac{1-\mu}{2}\right)(\psi_y + \phi_x) n_2\right] = -k_0 \psi_t, \\
& D\left[n_2\phi_y + \mu n_2 \psi_x + \left(\frac{1-\mu}{2}\right)(\psi_y + \phi_x) n_1\right] = -k_1 \phi_t, \\
& K\left[(\psi + w_x)n_1 + (\phi + w_y)n_2\right] = -k_2 w_t \quad \text{on } \partial_1 \Gamma_0 \times (0,\infty);
\end{aligned}
$$

$$
\begin{aligned}
& z(x,0) = z_0, \quad z_t(x,0) = z_1, \quad w(x,0) = w_0, \quad w_t(x,0) = w_1, \\
& \psi(x,0) = \psi_0, \quad \psi_t(x,0) = \psi_1, \quad \phi(x,0) = \phi_0, \quad \phi_t(x,0) = \phi_1.
\end{aligned}
$$

Remarks on the model

(i) Only linear feedback boundary damping is introduced into the model, viz. in the boundary conditions on $\Gamma_0 \cup \Gamma_1$ of the z-equations and in the boundary conditions on $\partial_1 \Gamma_0$ of the Reissner–Mindlin system. In the latter case the feedback boundary controls act via the higher order free mechanical boundary conditions for the Reissner–Mindlin plate (see [21, p. 208] for the difficulties arising from this situation). The clamped conditions imposed on $\partial_0 \Gamma_0$ ensure the positive definiteness property of a specific elliptic operator, as does the presence of the constant ℓ_0 in the boundary condition on Γ_1 for the z-equations.

(ii) Two boundary conditions at the interface Γ_0 relate the displacement and shear angles of the plate and the velocity of the gas: firstly in the coupled stationary boundary condition in the z problem the inclusion of a damping term causes an adjustment in the velocity of the gas to comply with the "no slip" boundary condition. Secondly the interaction between the gas and the plate is reflected by the presence of the term $\eta z_t|_{\Gamma_0}$, the back pressure against the moving wall Γ_0, in the equation for the displacement w of the wall. The coupling between the acoustic and the structural medium which takes place on the interface between the two media, plays a crucial part in what follows (see Proposition 3.3).

(iii) It is to be expected that, to achieve uniform stabilization, a geometric condition will be needed on the clamped portion of the boundary which is not subject to dissipation, viz.

$$\big((x,y)-(x_0,y_0)\big)\cdot n \leqslant 0 \quad \text{on } \partial_0\Gamma_0,\ (x_0,y_0)\in\mathbb{R}^2$$

(cf., e.g., [20, p. 1379]).

2.2. Model for a two-dimensional structural acoustic chamber

Assume that $\Omega \subset \mathbb{R}^2$ is an open bounded domain with boundary Γ sufficiently smooth (or rectangular). Assume $\Gamma = \Gamma_1 \cup \Gamma_0$, with Γ_0 and Γ_1 separate parts, Γ_1 simply connected and Γ_0 the flat portion acting as the flexible wall with boundary $\partial\Gamma_0 = P_1 \cup P_2$, where $P_1 = (a,0)$ and $P_2 = (a,\ell)$ are two points in the plane. By using as the elastic medium along Γ_0 a Timoshenko beam of length ℓ which is clamped at P_1 and free at P_2, we arrive at the following initial-boundary-value problem in (z,w,ϕ), to be denoted in what follows as $Pr(P_{2-D})$:

$$\begin{aligned}
&z_{tt} - c^2\Delta z = 0 && \text{in } \Omega\times(0,\infty),\\
&\frac{\partial z}{\partial n} + \ell_0 z + d z_t = 0 && \text{on } \Gamma_1\times(0,\infty),\\
&\frac{\partial z}{\partial n} + d z_t - w_t = 0 && \text{on } \Gamma_0\times(0,\infty);
\end{aligned}$$

$$\begin{aligned}
&\frac{\rho h^3}{12}\phi_{tt} - EI\phi_{yy} + K(\phi + w_y) = 0,\\
&\rho h w_{tt} - K(\phi + w_y)_y + \eta z_t = 0 \quad \text{on } \Gamma_0\times(0,\infty);
\end{aligned}$$

$$w = \phi = 0 \quad \text{at } P_1\times(0,\infty);$$

$$\begin{aligned}
&EI\phi_y = -k_0\phi_t,\\
&K(\phi + w_y) = -k_1 w_t \quad \text{at } P_2\times(0,\infty);
\end{aligned}$$

$$\begin{aligned}
&z(x,0) = z_0, \quad z_t(x,0) = z_1, \quad w(x,0) = w_0, \quad w_t(x,0) = w_1,\\
&\phi(x,0) = \phi_0, \quad \phi_t(x,0) = \phi_1
\end{aligned}$$

in which EI denotes the flexural rigidity of the beam, $K = kGA$ with G the shear modulus, A the cross-sectional area, k a correction factor, and $k_0 \geqslant 0$, $k_1 \geqslant 0$, $k_0 + k_1 > 0$.

Remarks. (i) It is readily shown that the system in (w,ϕ) may be uncoupled formally to yield a single partial differential equation, of order four in t and x, viz.

$$\begin{aligned}
&\rho h w_{tt} - \left(\frac{\rho h^3}{12} + \frac{\rho h}{K}EI\right) w_{ttyy} + EI w_{yyyy} + \frac{\rho h}{K}\left(\frac{\rho h^3}{12}\right) w_{tttt}\\
&\quad = \left[1 - \frac{EI}{K}\left(\frac{\partial^2}{\partial y^2} - \frac{\rho h^3}{12EI}\frac{\partial^2}{\partial t^2}\right)\right](-\eta z_t)
\end{aligned}$$

or

$$\rho h w_{tt} + EI\left(\frac{\partial^2}{\partial y^2} - \frac{\rho h^3}{12EI}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial y^2} - \frac{\rho h}{K}\frac{\partial^2}{\partial t^2}\right)w$$
$$= \left[1 - \frac{EI}{K}\left(\frac{\partial^2}{\partial y^2} - \frac{\rho h^3}{12EI}\frac{\partial^2}{\partial t^2}\right)\right](-\eta z_t).$$

This single equation in w clearly displays the inclusion of the effects of rotary inertia and shear deformation, over and above the bending effect in the classical Euler–Bernoulli equation. The reader is referred to [10, p. 21] to see that this is precisely the one-dimensional analogue of the Mindlin equation used in the cited reference as mentioned in the footnote earlier on.

(ii) Models that arise in the problem of noise control in a two-dimensional chamber with piezo-ceramic patches attached to the vibrating wall, have been considered in [23, Section 9.10]. The reader is referred to, e.g., [23, p. 897] for the analysis of a damped model in which the elastic feedback control acts via the hinged boundary conditions for the Kirchoff equation on the wall.

3. Energy decrease for $Pr(P)$ and $Pr(P_{2-D})$

Norms and inner products in general Sobolev spaces $H^s(D)$, $s > 0$, will be denoted by $\|.\|_{s,D}$ and $(,)_{s,D}$. $H^{-s}(D)$ equipped with the norm $\|.\|_{-s,D}$ will denote the dual $(H^s(D))'$ of $H^s(D)$ with respect to the $L^2(D)$ inner product. We shall also use the spaces $H^s(0,T;X)$ of measurable functions u, defined everywhere in $(0,T)$ with values in the Banach space X, for which $\|u\|_X \in H^s(0,T)$.

We now proceed to showing that $Pr(P)$ exhibits energy decrease—due to the interaction between the gas and the Reissner–Mindlin plate, it is obvious that the energy functional $\mathcal{E}(t)$ will comprise an $\mathcal{E}_z$ and an $\mathcal{E}_{(w,\psi,\phi)}$ component. By using standard methods it is easily seen that the energy functional associated with $Pr(P)$ is

$$\mathcal{E}(t) = \eta\mathcal{E}_z(t) + c^2\mathcal{E}_{(w,\psi,\phi)}(t)$$

with

$$2\mathcal{E}_z(t) = \|z_t\|_{0,\Omega}^2 + c^2\ell_0\|z\|_{0,\Gamma_1}^2 + c^2\|\nabla z\|_{0,\Omega}^2,$$
$$\begin{aligned}2\mathcal{E}_{(w,\psi,\phi)}(t) = {} & \rho h\|w_t\|_{0,\Gamma_0}^2 + \frac{\rho h^3}{12}\left(\|\psi_t\|_{0,\Gamma_0}^2 + \|\phi_t\|_{0,\Gamma_0}^2\right) \\ & + K\left(\|\psi + w_x\|_{0,\Gamma_0}^2 + \|\phi + w_y\|_{0,\Gamma_0}^2\right) \\ & + D\int_{\Gamma_0}\left(\psi_x^2 + \phi_y^2 + 2\mu\psi_x\phi_y + \left(\frac{1-\mu}{2}\right)(\psi_y + \phi_x)^2\right)d\Gamma_0.\end{aligned}$$

This naturally leads to the following choice of a space of finite energy for a weak solution (z, w, ψ, ϕ) of $Pr(P)$:

$$\mathcal{Y} = H^1(\Omega) \times \left(H^1(\Gamma_0)\right)^3.$$

For $Pr(P_{2-D})$ we have the energy functional

$$\mathcal{E}(t) = \eta\mathcal{E}_z(t) + c^2\mathcal{E}_{(w,\phi)}(t)$$

with

$$2\mathcal{E}_z(t) = \|z_t\|_{0,\Omega}^2 + c^2\ell_0\|z\|_{0,\Gamma_1}^2 + c^2\|\nabla z\|_{0,\Omega}^2,$$

$$2\mathcal{E}_{(w,\phi)}(t) = \rho h\|w_t\|_{0,\Gamma_0}^2 + K\|\phi + w_y\|_{0,\Gamma_0}^2 + \frac{\rho h^3}{12}\|\phi_t\|_{0,\Gamma_0}^2 + EI\|\phi_y\|_{0,\Gamma_0}^2$$

and the energy space

$$\mathcal{Z} = H^1(\Omega) \times \big(H^1(\Gamma_0)\big)^2$$

in which $H^1(\Omega)$ and $(H^1(\Gamma_0))^2$ are endowed with the norms derived from

$$a_{Ac}(z) := c^2\|\nabla z\|_{0,\Omega}^2 + c^2\ell_0\|z\|_{0,\Gamma_1}^2,$$
$$a_{Int}(w,\phi) := K\|\phi + w_y\|_{0,\Gamma_0}^2 + EI\|\phi_y\|_{0,\Gamma_0}^2,$$

respectively, which, as is well known, are equivalent to the usual first order Sobolev space norms in $H^1(\Omega)$ and $(H^1(\Gamma_0))^2$.

Our first observation is on the existence of unique weak solutions of $Pr(P)$ in $\mathcal{Y}$ and $Pr(P_{2-D})$ in $\mathcal{Z}$.

Proposition 3.1. *Let $(z_0, z_1, w_0, w_1, \psi_0, \psi_1, \phi_0, \phi_1)$ be an element of $H^1(\Omega) \times L^2(\Omega) \times (H^1(\Gamma_0) \times L^2(\Gamma_0))^3$. Then there exists a unique weak solution $(z, w, \psi, \phi) \in \mathcal{Y}$ of $Pr(P)$ such that*

$$(z, z_t, w, w_t, \psi, \psi_t, \phi, \phi_t) \in C\big((0,\infty); H^1(\Omega) \times L^2(\Omega) \times \big(H^1(\Gamma_0) \times L^2(\Gamma_0)\big)^3\big).$$

Proposition 3.2. *Let $(z_0, z_1, w_0, w_1, \phi_0, \phi_1)$ be an element of $H^1(\Omega) \times L^2(\Omega) \times (H^1(\Gamma_0) \times L^2(\Gamma_0))^2$. Then there exists a unique weak solution $(z, w, \phi) \in \mathcal{Z}$ of $Pr(P_{2-D})$ such that*

$$(z, z_t, w, w_t, \phi, \phi_t) \in C\big((0,\infty); H^1(\Omega) \times L^2(\Omega) \times \big(H^1(\Gamma_0) \times L^2(\Gamma_0)\big)^2\big).$$

The proofs are achieved by constructing C_0 contraction semigroups on $\mathcal{Y} \times L^2(\Omega) \times (L^2(\Gamma_0))^3$ and $\mathcal{Z} \times L^2(\Omega) \times (L^2(\Gamma_0))^2$, respectively. As the procedure is standard we omit the proof—the reader is referred to [9, Theorem 3.1] where a similar result is shown for a hybrid structure consisting of a Reissner–Mindlin plate with a Timoshenko beam attached to its free edge. It should be noted that, for instance in the case of $Pr(P_{2-D})$, the underlying elliptic operators A_{Ac} and A_{Int} which are instrumental in constructing the semigroup on $\mathcal{Z} \times L^2(\Omega) \times (L^2(\Gamma_0))^2$ are the canonical isomorphisms of $H^1(\Omega)$ and $(H^1(\Gamma_0))^2$ endowed with the norms $\sqrt{a_{Ac}}$ and $\sqrt{a_{Int}(w,\phi)}$ respectively, onto $H^{-1}(\Omega)$ and $(H^{-1}(\Gamma_0))^2$. For the complete coupled system in (z, w, ϕ) we now define the operator $A = \langle A_{Ac}, A_{Int}\rangle : \mathfrak{D}(A) \subset \mathcal{Z} \to L^2(\Omega) \times (L^2(\Gamma_0))^2$. In accordance with standard procedure the semigroup on $\mathcal{Z} \times L^2(\Omega) \times (L^2(\Gamma_0))^2$ is now defined by formulating a first order evolution equation in $\mathcal{Z} \times L^2(\Omega) \times (L^2(\Gamma_0))^2$.

We now state energy identities for $Pr(P)$ and $Pr(P_{2-D})$ that are of cardinal importance in what follows. The proofs are achieved with the aid of the usual "formal" energy method, i.e., for instance in the case of $Pr(P)$, by multiplication of the equations in z, w, ψ, ϕ by z_t, w_t, ψ_t, ϕ_t, integration on (s,t) and application of Green's formula. To justify the use of the formal calculations, some clarification is appropriate here: the delicate point is that, with the focus in this study on stabilization estimates, which are inverse estimates, the regularity of solutions becomes an "inverse problem." Thus the route to pursue here is an approximation-regularization argument. This technique was developed by Lasiecka and Tataru [18] for the wave equation and then in [25] presented in a more general form (see the section "Regularization" in [25]), which applies to our models $Pr(P)$ and $Pr(P_{2-D})$. Accordingly one applies the standard formal procedure to approximating equations with regular solutions and obtains the result for the original weak solution by passage to the limit in a specific sense, i.e., on the desired energy identity.

Proposition 3.3. *Let (z, w, ψ, ϕ) be a weak solution of $Pr(P)$. For $0 \leqslant s < t$ the following energy relation holds*:

$$\begin{aligned}\mathcal{E}(t) - \mathcal{E}(s) = {} & -2\eta c^2 \int_s^t \left[d \left\| z_t(\tau) \right\|_{0,\Gamma}^2 - \left(w_t(\tau), \vec{z}_t(\tau) \right)_{0,\Gamma_0} \right] d\tau \\ & - 2c^2 \int_s^t \left[k_2 \left\| w_t(\tau) \right\|_{0,\partial_1\Gamma_0}^2 + k_0 \left\| \psi_t(\tau) \right\|_{0,\partial_1\Gamma_0}^2 + k_1 \left\| \phi_t(\tau) \right\|_{0,\partial_1\Gamma_0}^2 \right] d\tau \\ & - 2\eta c^2 \int_s^t \left(z_t(\tau), \vec{\psi}_t(\tau) \right)_{0,\Gamma_0} d\tau.\end{aligned}$$

It follows that

$$\frac{d\mathcal{E}}{dt} = -2c^2 \left[\eta d \| z_t \|_{0,\Gamma}^2 + k_2 \| w_t \|_{0,\partial_1\Gamma_0}^2 + k_0 \| \psi_t \|_{0,\partial_1\Gamma_0}^2 + k_1 \| \phi_t \|_{0,\partial_1\Gamma_0}^2 \right].$$

Thus decrease of energy of the entire system is established—note that the incorporation of the constants η and c^2 in the definition of $\mathcal{E}(t)$ yields cancellation of the terms arising from the coupling of the acoustic and the elastic medium.

It is clear that for $Pr(P_{2-D})$ we can formulate

Proposition 3.4. *Let (z, w, ϕ) be a weak solution of $Pr(P_{2-D})$. For $0 \leqslant s < t$ we have*

$$\begin{aligned}\mathcal{E}(t) - \mathcal{E}(s) = {} & -2\eta c^2 \int_s^t \left[d \left\| z_t(\tau) \right\|_{0,\Gamma}^2 - \left(w_t(\tau), \vec{z}_t(\tau) \right)_{0,\Gamma_0} \right] d\tau \\ & - 2c^2 \int_s^t \left[k_1 w_t^2(\ell,\tau) + k_0 \phi_t^2(\ell,\tau) \right] d\tau\end{aligned}$$

$$- 2\eta c^2 \int_s^t \big(z_t(\tau), \psi\!\!\!\nearrow_t(\tau)\big)_{0,\Gamma_0}\, d\tau, \tag{3.1}$$

$$\frac{d\mathcal{E}}{dt} = -2c^2\big[\eta d\|z_t\|_{0,\Gamma}^2 + k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big].$$

4. Uniform stabilization for $Pr(P_{2-D})$

In this section we present our main result on the uniform stabilization of the energy associated with $Pr(P_{2-D})$. This will entail establishing appropriate estimates of the energies $\mathcal{E}_z(t)$ and $\mathcal{E}_{(w,\phi)}(t)$. Whenever the constant C is used, it denotes a generic positive constant, dependent on, e.g., k_0, k_1, the physical parameters in $Pr(P_{2-D})$, or constants in applications of Young's inequality, and different in each instance—where necessary, dependence on, say T, will be denoted by using a subscript, i.e., C_T.

Theorem 4.1 *(Uniform stabilization). There exist constants $C > 0$, $\omega > 0$ such that*

$$\mathcal{E}(t) \leqslant C\exp(-\omega t)\mathcal{E}(0), \quad \forall t \geqslant 0.$$

Together with Proposition 3.4 which forms the crux of the proof of Theorem 4.1, we shall need the following lemmata.

Lemma 4.2. *Let (z, w, ϕ) be a solution of $Pr(P_{2-D})$ and let T be an arbitrary constant. Then there exists a constant C such that for any $\epsilon_0 > 0$ we have*

$$\begin{aligned}\int_0^T \mathcal{E}_{(w,\phi)}(t)\, dt \leqslant{}& \epsilon_0\big[\mathcal{E}_{(w,\phi)}(0) + \mathcal{E}_{(w,\phi)}(T)\big] + C_\epsilon \eta\|z_t\|^2_{[H^\delta(0,T;H^{1-\delta}(\Gamma_0))]'} \\ &+ C\int_0^T \big[\phi_t^2(\ell,t) + w_t^2(\ell,t)\big]\, dt \\ &+ C_{\epsilon_0}\big(\|w\|^2_{L^\infty(0,T;L^2(\Gamma_0))} + \|\phi\|^2_{L^\infty(0,T;L^2(\Gamma_0))}\big).\end{aligned}$$

Proof. We use the method of multipliers as in [21, Lemma 2.4]. Once more justification for the ensuing calculations are provided by the results of [25], i.e. the multipliers are applied to appropriate smooth "approximations" of the equations in ϕ and w. Passage to the limit in the stability estimates then reconstructs the estimates for the original weak solutions. Led by the results of Lagnese and Leugering [14] for the one-dimensional analogue of the full Von Kármán system, we apply the multipliers $y\phi_y + (1-2\alpha)\phi$ and $yw_y - \alpha w$, where $\alpha \in \mathbb{R}^1$ is to be specified later on. This yields

$$0 = \frac{\rho h^3}{12}\int_0^T\int_0^\ell \phi_{tt}\big(y\phi_y + (1-2\alpha)\phi\big)\, dy\, dt + \rho h\int_0^T\int_0^\ell w_{tt}(yw_y - \alpha w)\, dy\, dt$$

$$- EI\int_0^T\int_0^\ell \phi_{yy}\big(y\phi_y + (1-2\alpha)\phi\big)\,dy\,dt$$

$$+ K\int_0^T\int_0^\ell (\phi + w_y)\big(y\phi_y + (1-2\alpha)\phi\big)\,dy\,dt$$

$$- K\int_0^T\int_0^\ell (\phi + w_y)_y(yw_y - \alpha w)\,dy\,dt + \eta\int_0^T\int_0^\ell z_t(yw_y - \alpha w)\,dy\,dt. \tag{4.1}$$

Integration by parts and the boundary conditions in $Pr(P_{2-D})$ furnish

$$\frac{\rho h^3}{12}\int_0^T\int_0^\ell \phi_{tt}\big(y\phi_y + (1-2\alpha)\phi\big)\,dy\,dt$$

$$= \frac{\rho h^3}{12}\int_0^\ell \phi_t\big(y\phi_y + (1-2\alpha)\phi\big)\big|_0^T\,dy + \frac{\rho h^3}{12}\left(2\alpha - \frac{1}{2}\right)\int_0^T\int_0^\ell \phi_t^2\,dy\,dt$$

$$- \frac{\rho h^3\ell}{24}\int_0^T \phi_t^2(\ell,t)\,dt. \qquad (4.1)_1$$

Similarly we get

$$\rho h\int_0^T\int_0^\ell w_{tt}(yw_y - \alpha w)\,dy\,dt$$

$$= \rho h\int_0^\ell w_t(yw_y - \alpha w)\big|_0^T\,dy + \rho h\left(\alpha + \frac{1}{2}\right)\int_0^T\int_0^\ell w_t^2\,dy\,dt$$

$$- \frac{\rho h\ell}{2}\int_0^T w_t^2(\ell,t)\,dt \qquad (4.1)_2$$

and

$$-EI\int_0^T\int_0^\ell \phi_{yy}\big(y\phi_y + (1-2\alpha)\phi\big)\,dy\,dt$$

$$= -\frac{k_0^2\ell}{2EI}\int_0^T \phi_t^2(\ell,t)\,dt + EI\left(\frac{3}{2} - 2\alpha\right)\int_0^T\int_0^\ell \phi_y^2\,dy\,dt$$

$$- EI(1-2\alpha)\int_0^T (\phi_y\phi)(\ell,t)\,dt. \tag{4.1$_3$}$$

The fourth and fifth terms of (4.1), on the other hand, furnish

$$\begin{aligned}
&K\int_0^T\int_0^\ell (\phi+w_y)\big(y\phi_y+(1-2\alpha)\phi\big)\,dy\,dt - K\int_0^T \big[(\phi+w_y)(\ell w_y-\alpha w)\big](\ell,t)\,dt\\
&\quad + K\int_0^T\int_0^\ell (\phi+w_y)(yw_y-\alpha w)_y\,dy\,dt\\
&= K\int_0^T\int_0^\ell (\phi+w_y)\big[y(\phi+w_y)_y\big]\,dy\,dt + K\int_0^T\int_0^\ell (\phi+w_y)\big((1-2\alpha)\phi\big)\,dy\,dt\\
&\quad + K\int_0^T\int_0^\ell (\phi+w_y)(w_y)(1-\alpha)\,dy\,dt - K\int_0^T\big[(\phi+w_y)(\ell w_y-\alpha w)\big](\ell,t)\,dt\\
&= \frac{K}{2}\int_0^T\int_0^\ell \frac{\partial}{\partial y}\big\{y(\phi+w_y)^2\big\}\,dy\,dt + K\left(\frac{1}{2}-\alpha\right)\int_0^T\int_0^\ell (\phi+w_y)^2\,dy\,dt\\
&\quad - \alpha K\int_0^T\int_0^\ell \big[(\phi+w_y)\phi\big]\,dy\,dt + k_1\int_0^T \big[w_t(\ell w_y-\alpha w)\big](\ell,t)\,dt
\end{aligned} \tag{4.1$_4$}$$

in which the first term in the second last line equals

$$\frac{K\ell}{2}\int_0^T (\phi+w_y)^2(\ell,t)\,dt$$

and by Young's inequality

$$-\alpha K\int_0^T\int_0^\ell \big[(\phi+w_y)\phi\big]\,dy\,dt \geqslant -\frac{\alpha K}{2}\left[\int_0^T\int_0^\ell \left[(\phi+w_y)^2+\frac{\ell^2}{2}\phi_y^2\right]dy\,dt\right]$$

by invoking the Poincaré inequality

$$\int_0^\ell \phi^2\,dy \leqslant \frac{\ell^2}{2}\int_0^\ell \phi_y^2\,dy$$

(recall the clamped boundary conditions at P_1 in $Pr(P_{2-D})$).

Combining $(4.1)_1$–$(4.1)_4$, we arrive at

$$
\begin{aligned}
&\frac{\rho h^3}{24}(4\alpha-1)\int_0^T \|\phi_t\|^2_{0,\Gamma_0}\,dt+\frac{\rho h}{2}(2\alpha+1)\int_0^T \|w_t\|^2_{0,\Gamma_0}\,dt\\
&\quad+\left[\frac{EI}{2}(3-4\alpha)-\frac{K\ell^2\alpha}{4}\right]\int_0^T \|\phi_y\|^2_{0,\Gamma_0}\,dt+\frac{K}{2}(1-3\alpha)\int_0^T \|\phi+w_y\|^2_{0,\Gamma_0}\,dt\\
&\leqslant -\frac{K\ell}{2}\int_0^T (\phi+w_y)^2(\ell,t)\,dt-k_1\int_0^T \big[w_t(\ell w_y-\alpha w)(\ell,t)\big]\,dt\\
&\quad-\frac{\rho h^3}{12}\int_0^\ell \big[\phi_t\big(y\phi_y+(1-2\alpha)\phi\big)\big]\big|_0^T\,dy+\frac{\rho h^3\ell}{24}\int_0^T \phi_t^2(\ell,t)\,dt\\
&\quad-\rho h\int_0^\ell w_t(yw_y-\alpha w)\big|_0^T\,dy+\frac{\rho h\ell}{2}\int_0^T w_t^2(\ell,t)\,dt\\
&\quad-k_0(1-2\alpha)\int_0^T (\phi_t\phi)(\ell,t)\,dt+\frac{k_0^2\ell}{2EI}\int_0^T \phi_t^2(\ell,t)\,dt\\
&\quad-\eta\int_0^T (z_t,yw_y-\alpha w)_{0,\Gamma_0}\,dt.
\end{aligned}
\tag{4.2}
$$

To consider the terms on the right-hand side of (4.2), we denote the five lines concerned by $\sum_{i=1}^5 L_i$. Repeated applications of Young's inequality yield for L_1, in which we start with the second term therein writing $\ell w_y(\ell,t)=\ell[(\phi+w_y)-\phi](\ell,t)$,

$$
\begin{aligned}
L_1 &\leqslant \frac{k_1^2\alpha^2}{2\delta_1}\int_0^T w_t^2(\ell,t)\,dt+\frac{\delta_1}{2}\int_0^T w^2(\ell,t)\,dt+\frac{k_1^2}{2\delta_2}\int_0^T w_t^2(\ell,t)\,dt\\
&\quad+\int_0^T \frac{\delta_2\ell^2}{2}(\phi+w_y)^2(\ell,t)\,dt+\frac{k_1^2\ell^2}{2\delta_3}\int_0^T w_t^2(\ell,t)\,dt+\frac{\delta_3}{2}\int_0^T \phi^2(\ell,t)\,dt\\
&\quad-\frac{k_1^2\ell}{2K}\int_0^T w_t^2(\ell,t)\,dt
\end{aligned}
$$

(in which the term

$$
\frac{\delta_2\ell^2}{2}\int_0^T (\phi+w_y)^2(\ell,t)\,dt \quad \text{equals} \quad \frac{\delta_2 k_1^2\ell^2}{2K^2}\int_0^T w_t^2(\ell,t)\,dt
$$

). We also obtain

$$L_2 \leqslant \left[\frac{\alpha_1 \rho h^3}{24}\|\phi_t\|^2_{0,\Gamma_0} + \frac{\rho h^3(1-2\alpha)^2}{24\alpha_1}\|\phi\|^2_{0,\Gamma_0} + \frac{\alpha_2 \rho h^3}{24}\|\phi_t\|^2_{0,\Gamma_0} + \frac{\rho h^3 \ell^2}{24\alpha_2}\|\phi_y\|^2_{0,\Gamma_0}\right]\Bigg|_0^T + \frac{\rho h^3 \ell}{24}\int_0^T \phi_t^2(\ell,t)\,dt,$$

$$L_3 \leqslant \left[\frac{\beta_1 \rho h}{2}\|w_t\|^2_{0,\Gamma_0} + \frac{\rho h \alpha^2}{2\beta_1}\|w\|^2_{0,\Gamma_0} + \frac{\beta_2 \rho h}{2}\|w_t\|^2_{0,\Gamma_0} + \frac{\rho h \ell^2}{2\beta_2}\|\phi + w_y\|^2_{0,\Gamma_0} + \frac{\gamma_1 \rho h}{2}\|w_t\|^2_{0,\Gamma_0} + \frac{\rho h \ell^2}{2\gamma_1}\|\phi\|^2_{0,\Gamma_0}\right]\Bigg|_0^T + \frac{\rho h \ell}{2}\int_0^T w_t^2(\ell,t)\,dt,$$

$$L_4 \leqslant \frac{\delta_4}{2}\int_0^T \phi^2(\ell,t)\,dt + \frac{k_0^2(1-2\alpha)^2}{2\delta_4}\int_0^T \phi_t^2(\ell,t)\,dt + \frac{k_0^2 \ell}{2EI}\int_0^T \phi_t^2(\ell,t)\,dt$$

with the real number α in the definition of the multipliers and α_i, β_i, $i = 1, 2$, γ_1 and δ_j, $j = 1, 2, 3, 4$ all positive numbers still to be chosen appropriately and L_5 to be considered below.

Firstly the left-hand side of (4.2) dictates that α satisfies $\frac{1}{4} < \alpha < \frac{1}{3}$ as well as

$$\left[\frac{EI}{2}(3-4\alpha) - \frac{K\ell^2\alpha}{4}\right] > 0, \quad \text{i.e.} \quad \alpha < \frac{3}{4 + K\ell^2/(2EI)}.$$

Note that $\alpha \in \mathbb{R}^1$ exists provided

$$\frac{3}{4 + K\ell^2/(2EI)} \geqslant \frac{1}{4}, \quad \text{i.e.,} \ K\ell^2 \leqslant 16EI.$$

Thus subject to the constraint $K\ell^2 \leqslant 16EI$ we fix

$$\alpha \in \left(\frac{1}{4}, \beta\right), \qquad \beta = \min\left(\frac{1}{3}, \frac{3}{4 + K\ell^2/(2EI)}\right).$$

We devote a moment of reflection to the constraint $K\ell^2 \leqslant 16EI$: in view of $K = kGA$ the constraint may be restated as

$$\frac{3\ell^2}{4h^2}kG \leqslant E$$

(taking $I = Ah^2/12$, $A = bh$). It is clear that for the ratio $3\ell^2/(4h^2)$ to be large the ratio $E/(kG)$ has to be large. In particular whenever $3\ell^2/(4h^2) > 1$ we obtain $kG < E$. It is interesting to note that this restriction is imposed in a study of the nonlinear Timoshenko–Kirchoff equation by A. Arosio [1]—moreover, since $G < E$ holds and $k \leqslant 1$ for beams, $kG < E$ is always satisfied for beams [1, p. 502]. Thus our restriction $K\ell^2 \leqslant 16EI$ is related to what is found in the literature in the field and may be regarded as "mild."

Next in the estimates for the terms in the right-hand side of (4.2) we choose α_2, β_2 such that

$$\frac{\rho h^3 \ell^2}{12} = \alpha_2^2 EI, \qquad \rho h \ell^2 = \beta_2^2 K$$

whence

$$\frac{\rho h^3\ell^2}{24\alpha_2}\|\phi_y\|^2_{0,\Gamma_0}=\frac{\alpha_2}{2}EI\|\phi_y\|^2_{0,\Gamma_0},$$
$$\frac{\rho h\ell^2}{2\beta_2}\|\phi+w_y\|^2_{0,\Gamma_0}=\frac{\beta_2}{2}K\|\phi+w_y\|^2_{0,\Gamma_0}.$$

The constants $\delta_1,\delta_3,\delta_4$, on the other hand, are determined by the requirement that each is sufficiently small to ensure that the $\phi^2(\ell,t)$, $w^2(\ell,t)$ terms are absorbed into the terms comprising the energy $\mathcal{E}_{(w,\phi)}(t)$ on the left-hand side by using trace theory as well as the inequality

$$\|w_y\|^2_{0,\Gamma_0}\leqslant 2\left(\|\phi+w_y\|^2_{0,\Gamma_0}+\frac{\ell^2}{2}\|\phi_y\|^2_{0,\Gamma_0}\right).$$

Next we choose α_1,β_1 and γ_1 arbitrarily small. It follows that the sum of the $\|\phi_t\|^2_{0,\Gamma_0}$, $\|w_t\|^2_{0,\Gamma_0}$, $\|\phi_y\|^2_{0,\Gamma_0}$ and $\|\phi+w_y\|^2_{0,\Gamma_0}$ terms between the parentheses $[\cdot]^0_T$ on the right-hand sides of the estimates for L_1-L_4 is dominated by $C[\mathcal{E}_{(w,\phi)}(0)+\mathcal{E}_{(w,\phi)}(T)]$ with C a generic constant. It is clear that the sum of the $\|w\|^2_{0,\Gamma_0}$ and $\|\phi\|^2_{0,\Gamma_0}$ terms between the parentheses $[\cdot]^0_T$ is dominated by (another) generic constant

$$C_{\frac{1}{\alpha_1},\frac{1}{\beta_1},\frac{1}{\gamma_1}}\left(\|w\|^2_{L^\infty(0,T;L^2(\Gamma_0))}+\|\phi\|^2_{L^\infty(0,T;L^2(\Gamma_0))}\right).$$

Finally we collect the dissipative terms

$$\int_0^T\phi_t^2(\ell,t)\,dt\quad\text{and}\quad\int_0^T w_t^2(\ell,t)\,dt$$

on the right-hand side of the same estimates to obtain

$$\left[\frac{k_1^2}{2}\left(\frac{\alpha^2}{\delta_1}+\frac{1}{\delta_2}+\frac{\ell^2}{\delta_3}-\frac{\ell}{2K}+\frac{\delta_2\ell^2}{K^2}\right)+\frac{\rho h\ell}{2}\right]\int_0^T w_t^2(\ell,t)\,dt$$
$$+\left[\frac{k_0^2}{2}\left(\frac{(1-2\alpha)^2}{\delta_4}+\frac{\ell}{EI}\right)+\frac{\rho h^3\ell}{24}\right]\int_0^T\phi_t^2(\ell,t)\,dt. \tag{4.3}$$

It is clear that in compliance with $k_0\geqslant 0$, $k_1\geqslant 0$, $k_0+k_1>0$, in particular by taking $k_1=0$, there exists a positive constant C such that the total of the dissipative boundary terms is dominated by

$$C\int_0^T\left[\phi_t^2(\ell,t)+w_t^2(\ell,t)\right]dt.$$

The coefficients of the integrals in (4.3) deserve attention. Firstly the positivity of the coefficient of the term

$$\int_0^T \phi_t^2(\ell, t)\,dt$$

for any k_0, ρ, h, ℓ, EI such that $K\ell^2 \leqslant 16EI$, $\alpha \in (\frac{1}{4}, \beta)$ seems to indicate that $k_0 > 0$ is the more obvious choice if only one boundary control is applied, at least according to our full multiplier method away from Lyapunov functionals. Secondly application of a shear force control, i.e., $k_1 \neq 0$ and retention of the negative term $-k_1^2 \frac{\ell}{4K}$ (instead of discarding it), requires that, to validate

$$\left[\frac{k_1^2}{2}\left(\frac{\alpha^2}{\delta_1} + \frac{1}{\delta_2} + \frac{\ell^2}{\delta_3} - \frac{\ell}{2K} + \frac{\delta_2 \ell^2}{K^2}\right) + \frac{\rho h \ell}{2}\right] > 0$$

in (4.3), k_1 should be small or $\delta_1, \delta_2, \delta_3$ chosen sufficiently small to yield

$$\frac{\alpha^2}{\delta_1} + \frac{1}{\delta_2} + \frac{\ell^2}{\delta_3} - \frac{\ell}{2K} > 0.$$

Unless k_1 is small enough, the magnitude of the terms $\frac{1}{\delta_1}, \frac{1}{\delta_2}, \frac{1}{\delta_3}$ may be crucial here.

It remains to consider $L_5 = -\eta \int_0^T (z_t, yw_y - \alpha w)_{0,\Gamma_0}\,dt$: we have

$$\int_0^T (z_t, yw_y - \alpha w)_{0,\Gamma_0}\,dt \leqslant C\|z_t\|_{[H^\delta(0,T;H^{1-\delta}(\Gamma_0))]'} \|w\|_{H^\delta(0,T;H^{1-\delta}(\Gamma_0))}$$

in which $0 \leqslant \delta < 1$, by making use of interpolation [26, Chapter 1, Section 14.2]. By Young's inequality

$$\|z_t\|_{[H^\delta(0,T;H^{1-\delta}(\Gamma_0))]'} \|w\|_{H^\delta(0,T;H^{1-\delta}(\Gamma_0))} \leqslant C\epsilon_1 \|z_t\|^2_{[H^\delta(0,T;H^{1-\delta}(\Gamma_0))]'} + \epsilon_1 \|w\|^2_{H^\delta(0,T;H^{1-\delta}(\Gamma_0))}$$

in which

$$\begin{aligned}
\|w\|^2_{H^\delta(0,T;H^{1-\delta}(\Gamma_0))} &\leqslant C\|w\|^{2(1-\delta)}_{L^2(0,T;H^1(\Gamma_0))} \|w\|^{2\delta}_{H^1(0,T;L^2(\Gamma_0))} \\
&\leqslant C\big[\|w\|^2_{L^2(0,T;H^1(\Gamma_0))} + \|w\|^2_{H^1(0,T;L^2(\Gamma_0))}\big] \\
&\leqslant C \int_0^T \mathcal{E}_{(w,\phi)}(t)\,dt
\end{aligned}$$

by applying an interpolation result [30, p. 13], and once more Young's inequality and

$$\|w_y\|^2_{0,\Gamma_0} \leqslant 2\left(\|\phi + w_y\|^2_{0,\Gamma_0} + \frac{\ell^2}{2}\|\phi_y\|^2_{0,\Gamma_0}\right).$$

Collecting the above estimates and rescaling by ϵ_0, (4.1) yields

$$\int_0^T \mathcal{E}_{(w,\phi)}(t)\,dt \leqslant \epsilon_0\big[\mathcal{E}_{(w,\phi)}(0)+\mathcal{E}_{(w,\phi)}(T)\big] + C_\epsilon \eta \|z_t\|^2_{[H^\delta(0,T;H^{1-\delta}(\Gamma_0))]'} + \epsilon\int_0^T \mathcal{E}_{(w,\phi)}(t)\,dt + C\int_0^T \big[\phi_t^2(\ell,t)+w_t^2(\ell,t)\big]\,dt + C_{\epsilon_0}\big(\|w\|^2_{L^\infty(0,T;L^2(\Gamma_0))} + \|\phi\|^2_{L^\infty(0,T;L^2(\Gamma_0))}\big). \tag{4.4}$$

The validity of Lemma 4.2 now follows by selecting ϵ sufficiently small. The result should be compared with the result of [21, Lemma 2.4]. Note that with no need for microlocal estimates corresponding to the plate equation in our case, we will not apply the estimate (4.4) on a smaller time interval. □

Lemma 4.3. *Let (z,w,ϕ) be a weak solution of $Pr(P_{2-D})$. Then for any $T>0$ there exists a constant $C>0$ such that*

$$\int_0^T \mathcal{E}_z(t)\,dt \leqslant C\big[\mathcal{E}_z(0)+\mathcal{E}_z(T)\big] + C\int_0^T\left[(d+1)\|z_t\|^2_{0,\Gamma} + \|w_t\|^2_{0,\Gamma_0} + \left\|\frac{\partial z}{\partial \tau}\right\|^2_{0,\Gamma}\right]dt,$$

where $\frac{\partial z}{\partial \tau}$ denotes the derivative in the tangential direction to the boundary Γ.

Proof. The lemma is the two-dimensional analogue of [21, Lemma 2.2, p. 210]. We only outline the method for the sake of completeness: by applying the multipliers z and $2m\cdot\nabla z$, $m=\{x,y\}-\{x_0,y_0\}$ to the z-equation of $Pr(P_{2-D})$ and recalling the identities (see, e.g., [16, p. 285])

$$\int_0^T (z_{tt}, m\cdot\nabla z)_{0,\Omega}\,dt = (z_t, m\cdot\nabla z)_{0,\Omega}\big|_0^T - \int_0^T\left[\int_\Gamma \frac{1}{2} m\cdot n(z_t)^2\,d\Gamma - (z_t,z_t)_{0,\Omega}\right]dt$$

and

$$\int_0^T (\Delta z, m\cdot\nabla z)_{0,\Omega}\,dt = \int_0^T\int_\Gamma\left[\frac{\partial z}{\partial n}(m\cdot\nabla z|_\Gamma) - \frac{1}{2}(\nabla z|_\Gamma)^2 m\cdot n\right]d\Gamma\,dt$$

and the relation

$$\nabla z|_\Gamma = \frac{\partial z}{\partial n}n + \frac{\partial z}{\partial \tau}\tau \quad \text{whence} \quad (\nabla z|_\Gamma)^2 = \left(\frac{\partial z}{\partial n}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2$$

[17, p. 218], we get

$$\int_0^T\int_\Gamma\left[2c^2\frac{\partial z}{\partial n}\left(m\cdot\left[\frac{\partial z}{\partial n}n + \frac{\partial z}{\partial \tau}\tau\right]\right) - c^2\left[\left(\frac{\partial z}{\partial n}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2\right]m\cdot n + m\cdot n(z_t)^2\right]d\Gamma\,dt$$

$$- (z_t, z)_{0,\Omega}\big|_0^T - 2(z_t, m \cdot \nabla z)_{0,\Omega}\big|_0^T - c^2 \int_0^T \big[d(z_t, z)_{0,\Gamma} - (w_t, z)_{0,\Gamma_0}\big]\,dt$$

$$= \int_0^T \big[(z_t, z_t)_{0,\Omega} + c^2(\nabla z, \nabla z)_{0,\Omega} + c^2\ell_0(z, z)_{0,\Gamma_1}\big]\,dt.$$

This gives

$$2\int_0^T \mathcal{E}_z(t)\,dt$$

$$= \big[-(z_t, z)_{0,\Omega} - 2(z_t, m \cdot \nabla z)_{0,\Omega}\big]\Big|_0^T - c^2 \int_0^T \big[d(z_t, z)_{0,\Gamma} - (w_t, z)_{0,\Gamma_0}\big]\,dt$$

$$+ \int_0^T \int_\Gamma \left[c^2\left[\left(\frac{\partial z}{\partial n}\right)^2 - \left(\frac{\partial z}{\partial \tau}\right)^2\right] m \cdot n + 2c^2 \frac{\partial z}{\partial n}\frac{\partial z}{\partial \tau} m \cdot \tau + m \cdot n (z_t)^2\right] d\Gamma\,dt$$

from which the result of the lemma follows readily by proceeding similarly as in the proof of Lemma 4.2 (see also [18, Proposition 3.1]). □

The following result, adopted from [21] provides an estimate for the term $\|\frac{\partial z}{\partial \tau}\|_{0,\Gamma}^2$ on the right-hand side of the result of Lemma 4.3, in terms of velocity traces and terms below the energy level.

Lemma 4.4. *Let* (z, w, ϕ) *be a solution of* $Pr(P_{2-D})$. *Let* $T > 0$ *be an arbitrary constant and* α' *an arbitrary small constant such that* $\alpha' < \frac{T}{2}$. *Then we have*:

$$\int_{\alpha'}^{T-\alpha'} \left\|\frac{\partial z}{\partial \tau}\right\|_{0,\Gamma}^2 d\Gamma \leqslant C_{T,\alpha'}\left[\int_0^T \left[\left\|\frac{\partial z}{\partial n}\right\|_{0,\Gamma}^2 + \|z_t\|_{0,\Gamma}\right] dt + lot(z)\right]$$

$$\leqslant C_{T,\alpha'}\left[\int_0^T \big[\|w_t\|_{0,\Gamma_0}^2 + (1+d)\|z_t\|_{0,\Gamma}^2\big]\,dt + lot(z)\right],$$

where

$$lot(z) \leqslant C \int_0^T \big[\|z\|_{1-\epsilon,\Omega}^2 + \|z_t\|_{-\epsilon,\Omega}^2\big]\,dt, \quad \epsilon > 0.$$

For the proof the reader is referred to [21, Lemma 4.1] and [17, Lemma 7.2]. The key tool is a pseudodifferential analysis which enforces the diminished time interval on the left-hand side.

Lemma 4.5. *Let z be a solution of $Pr(P_{2-D})$ and $T>0$ an arbitrary constant. Let $0<\alpha'<\frac{T}{2}$. Then there exists a constant C such that*

$$\int_{\alpha'}^{T-\alpha'} \mathcal{E}_z(t)\,dt \leqslant C\big[\mathcal{E}_z(\alpha')+\mathcal{E}_z(T-\alpha')\big]+C_{T,\alpha'}\int_0^T\big[(d+1)\|z_t\|_{0,\Gamma}^2+\|w_t\|_{0,\Gamma_0}^2\big]\,dt + C_{T,\alpha'}lot(z).$$

The result follows by assembling the results of Lemmas 4.3–4.4. Note that the right-hand side contains the term $\|w_t\|_{0,\Gamma_0}^2$. This term is eliminated in the next result by using

$$\int_0^T \|w_t\|_{0,\Gamma_0}^2\,dt \leqslant \int_0^T \mathcal{E}_{(w,\phi)}(t)\,dt.$$

Using to this end Lemma 4.2 in which we take $\delta=0$, we obtain

Corollary 4.6. *Under the assumptions of Lemma* 4.5 *there exists a constant C such that*

$$\begin{aligned}\int_{\alpha'}^{T-\alpha'} \mathcal{E}_z(t)\,dt \leqslant{}& C\big[\mathcal{E}_z(\alpha')+\mathcal{E}_z(T-\alpha')\big]+C_{T,\alpha'}\epsilon_0\big[\mathcal{E}_{(w,\phi)}(0)+\mathcal{E}_{(w,\phi)}(T)\big]\\ &+C_{T,\alpha'}\int_0^T\big[(d+1)\big\|z_t(t)\big\|_{0,\Gamma}^2\big]\,dt + C_{T,\alpha'}lot(z)\\ &+C_{T,\alpha'}\int_0^T\big[\|z_t\|_{0,\Gamma_0}^2+\phi_t^2(\ell,t)+w_t^2(\ell,t)\big]\,dt\\ &+C_{\epsilon_0,T,\alpha'}\big(\|w\|_{L^\infty(0,T;L^2(\Gamma_0))}^2+\|\phi\|_{L^\infty(0,T;L^2(\Gamma_0))}^2\big)\end{aligned}$$

by using $\eta<1$ in the second last line. By now combining Lemma 4.2 and Corollary 4.6 with $\epsilon_0=C_{T,\alpha'}^{-1}$, we obtain, by proceeding similarly as in [21], invoking the energy dissipation relation (3.1), and using the fact that $d>0$,

$$\begin{aligned}T\mathcal{E}(T) &\leqslant \int_0^T \mathcal{E}(t)\,dt\\ &= \eta\int_0^T \mathcal{E}_z(t)\,dt + c^2\int_0^T \mathcal{E}_{(w,\phi)}(t)\,dt\\ &\leqslant C\left(\left[\int_0^{\alpha'}+\int_{\alpha'}^{T-\alpha'}+\int_{T-\alpha'}^{T}\right]\eta\mathcal{E}_z(t)\,dt+\int_0^T\mathcal{E}_{(w,\phi)}(t)\,dt\right)\end{aligned}$$

$$\leqslant C(\alpha'+1)\mathcal{E}(T) + C_{T,\alpha'}\int_0^T \big[\eta(d+1)\|z_t\|_{0,\Gamma}^2\big]\,dt$$

$$+ C_{T,\alpha'}\big[lot(z) + lot(w,\phi)\big] + C_{T,\alpha'}\int_0^T \big[k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big]\,dt,$$

where

$$lot(w,\phi) = \|\phi\|_{L^\infty(0,T;L^2(\Gamma_0))}^2 + \|w\|_{L^\infty(0,T;L^2(\Gamma_0))}^2.$$

By now choosing $\alpha' < \frac{T}{2}$, but independent of T, and T sufficiently large, viz. $T > 2C(\alpha'+1)+2$, we obtain the estimate.

Lemma 4.7. *Assume $\alpha' < \frac{T}{2}$ and $T > 2C(\alpha'+1)+2$. Then*

$$\int_0^T \mathcal{E}(t)\,dt + \mathcal{E}(T) + \mathcal{E}(0)$$

$$\leqslant C_{T,\alpha'}\int_0^T \eta(d+1)\|z_t\|_{0,\Gamma}^2\,dt + C_{T,\alpha'}\big[lot(z) + lot(w,\phi)\big]$$

$$+ C_{T,\alpha'}\int_0^T \big[k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big]\,dt.$$

To attain the absorption of the lower order terms $lot(z) + lot(w,\phi)$ in this result, we need

Lemma 4.8. *For T sufficiently large there exists a constant $C_{T,\mathcal{E}(0)}$ such that the solution (z,w,ϕ) of $Pr(P_{2-D})$ satisfies*

$$\begin{aligned} lot(z) + lot(w,\phi) &\leqslant C_{T,\mathcal{E}(0)}\left[\int_0^T \big[d\,\big\|z_t(t)\big\|_{0,\Gamma}^2 + k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big]\,dt\right] \\ &:= C_{T,\mathcal{E}(0)} P(z,w,\phi). \end{aligned} \tag{4.5}$$

Proof. The method using "compactness–uniqueness" arguments is well known (see, e.g., [7, Lemma 19]). We proceed by contradiction, assuming that there exists a sequence $\{(z_{0,n}, w_{0,n}, \phi_{0,n})\}_n$ of initial data in $\mathcal{Z}$ and a corresponding sequence $\{(z_n, w_n, \phi_n)\}_n$ of solutions of $Pr(P_{2-D})$ such that

$$\frac{lot(z_n) + lot(w_n,\phi_n)}{P(z_n,w_n,\phi_n)} \to \infty \quad \text{as } n\to\infty.$$

We may assume without loss of generality that $lot(z_n) + lot(w_n,\phi_n) \to 1$ and $P(z_n, w_n, \phi_n) \to 0$ as $n\to\infty$. In view of $\mathcal{E}(0) \leqslant M$ for a solution (z,w,ϕ) we can conclude that the energy $\mathcal{E}_n(t) = \eta\mathcal{E}_{n,z_n}(t) + c^2\mathcal{E}_{n,(w_n,\phi_n)}(t)$ satisfies $\mathcal{E}_n(t) \leqslant M$ for $0 \leqslant t \leqslant T$. Consequently there exists a subsequence again denoted by $\{(z_n, w_n, \phi_n)\}_n$ such that

$$\begin{aligned}
&z_n \to z \quad \text{in } L^\infty\big(0, T; H^1(\Omega)\big) \text{ weakly star,}\\
&z_{n,t} \to z_t \quad \text{in } L^\infty\big(0, T; L^2(\Omega)\big) \text{ weakly star,}\\
&(w_n, \phi_n) \to z \quad \text{in } \big(L^\infty\big(0, T; \big(H^1(\Gamma_0)\big)\big)\big)^2 \text{ weakly star,}\\
&(w_{n,t}, \phi_{n,t}) \to (w_t, \phi_t) \quad \text{in } \big(L^\infty\big(0, T; L^2(\Gamma_0)\big)\big)^2 \text{ weakly star.}
\end{aligned}$$

By using Simon's lemma [29] together with classical compact embeddings, e.g., $H^1(\Omega) \subset H^{1-\delta}(\Omega)$, $0 < \delta < 1$, we obtain that

$$z_n \to z \quad \text{strongly in } C\big(0, T; H^{1-\delta_1}(\Omega)\big),\ 0 < \delta_1 < 1. \tag{4.6}$$

Since $\{z_{n,tt}\}_n$ is a bounded sequence in $L^\infty(0, T; H^{-1}(\Omega))$, we conclude once more by Simon's lemma that

$$z_{n,t} \to z_t \quad \text{strongly in } C\big(0, T; H^{-\frac{1}{2}+\delta_2}(\Omega)\big),\ \delta_2 < \frac{1}{2}. \tag{4.7}$$

Putting $\epsilon = \frac{1}{2} - \delta_2$ in (4.7) settles the terms $\|z_{n,t}\|^2_{-\epsilon,(\Omega)}$ in $lot(z_n)$. By the same argument we can conclude that

$$(w_n, \phi_n) \to (w, \phi) \quad \text{strongly in } \big(C\big(0, T; H^{1-\delta_3}(\Gamma_0)\big)\big)^2,\ 0 < \delta_3 < 1,$$

whence

$$(w_n, \phi_n) \to (w, \phi) \quad \text{strongly in } \big(C\big(0, T; L^2(\Gamma_0)\big)\big)^2. \tag{4.8}$$

On the strength of (4.6)–(4.8) and (4.5) we conclude that

$$lot(z_n) + lot(w_n, \phi_n) \to lot(z) + lot(w, \phi) = 1$$

and

$$\begin{aligned}
&z_{n,t} \to 0 \quad \text{in } L^2\big([0, T]; \Gamma\big),\\
&w_{n,t}(\ell, t) \to 0 \quad \text{in } L^2\big([0, T]; \mathbf{C}\big),\\
&\phi_{n,t}(\ell, t) \to 0 \quad \text{in } L^2\big([0, T]; \mathbf{C}\big).
\end{aligned}$$

This implies that the limit functions z, w, ϕ satisfy the original system with homogeneous boundary conditions on Γ and at P_2 and the overdetermined boundary conditions

$$z_t \equiv 0 \quad \text{on } \Gamma, \qquad w_t \equiv 0, \quad \phi_t \equiv 0 \quad \text{at } P_1 \cup P_2.$$

It follows from standard uniqueness arguments [11] (these are applied to the Timoshenko beam equations in [13, p. 106]) that $z = 0$ and $w = \phi = 0$—this is in contradiction with $lot(z) + lot(w, \phi) = 1$ and the proof is complete. □

By combining Lemma 4.8 with Lemma 4.7 we arrive at our final auxiliary result.

Lemma 4.9. *Assume* $\alpha' < \frac{T}{2}$, $T > 2C(\alpha'+1)+2$. *Then*

$$\int_0^T \mathcal{E}(t)\,dt + \mathcal{E}(T) + \mathcal{E}(0) \leqslant C_{T,\mathcal{E}(0)} \int_0^T \big[\eta d \|z_t(t)\|_{0,\Gamma}^2 + k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big]\,dt.$$

We are now ready to complete the proof of Theorem 4.1. By Lemma 4.9 and (3.1), we have

$$\begin{aligned} T\mathcal{E}(T) &\leqslant C_{T,\mathcal{E}(0)}\left[\int_0^T \big[\eta d \|z_t(t)\|_{0,\Gamma}^2 + k_0\phi_t^2(\ell,t) + k_1 w_t^2(\ell,t)\big]\,dt\right] \\ &= C_{T,\mathcal{E}(0)}\big(\mathcal{E}(0) - \mathcal{E}(T)\big) \end{aligned}$$

whence

$$K'\mathcal{E}(T) \leqslant \mathcal{E}(0) - \mathcal{E}(T), \qquad K' = K'\big(\mathcal{E}(0), T\big) = \frac{T}{C_{T,\mathcal{E}(0)}}.$$

Thus we have shown that there exists a T, i.e., $\alpha' < T/2$, $T > 2C(\alpha'+1)+2$, such that

$$\mathcal{E}(T) \leqslant \frac{1}{K'+1}\mathcal{E}(0).$$

Application of the semigroup property (see, e.g., [5], [22, p. 408]) now yields the uniform stability result of Theorem 4.1.

Finally, as promised, we comment on the uniform stabilizability of the three-dimensional acoustic structure modelled by $Pr(P)$.

Remark 4.10. We observe that if the three-dimensional acoustic structure modelled by $Pr(P)$ is considered, microlocal estimates for the tangential derivatives $\frac{\partial\psi}{\partial\tau}$, $\frac{\partial\phi}{\partial\tau}$ and $\frac{\partial w}{\partial\tau}$ will be needed. Prof. I. Lasiecka has kindly pointed out to the author that these can be attained with the aid of modifications of the techniques used in her paper [20] on the uniform stabilizability of the full Von Kármán system with nonlinear feedback. This becomes more clear by observing that the equations for the in-plane displacement (u, v) in the system of dynamic plane elasticity may be written in the form

$$\rho h(u_{tt}, v_{tt}) - \left(\frac{Eh}{1-\mu^2}\right)\operatorname{div}(H+N) = 0$$

with

$$H = (h_{ij}) = \begin{pmatrix} u_x + \mu v_y & \big(\frac{1-\mu}{2}\big)(u_y + v_x) \\ \big(\frac{1-\mu}{2}\big)(u_y + v_x) & v_y + \mu u_x \end{pmatrix}$$

and $N(w) = (N_{ij})(w)$ a matrix with nonlinear entries. Although the coupling between ψ, ϕ and w in $Pr(P)$ is linear, it is more intricate in view of additional terms due to the introduction of shear force in the model. On the other hand the w-equation in $Pr(P)$ is, like the equations in (u, v) in plane elasticity, of order 2, so that one will encounter boundary terms

$$(\nabla w|_{\Gamma})^2 = \left(\frac{\partial w}{\partial n}\right)^2 + \left(\frac{\partial w}{\partial \tau}\right)^2$$

which resemble those emanating in the study of the (u, v)-equations in [20].

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