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Global existence for 2D nonlinear Schrödinger equations via high–low frequency decomposition method

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Abstract

We study global existence of solutions for the Cauchy problem of the nonlinear Schrödinger equation $iu_t + \Delta u = |u|^{2m}u$ in the 2 dimension case, where m is a positive integer, $m \geqslant 2$. Using the high–low frequency decomposition method, we prove that if $\frac{10m-6}{10m-5} < s < 1$ then for any initial value $\varphi \in H^s(R^2)$, the Cauchy problem has a global solution in $C(R, H^s(R^2))$, and it can be split into $u(t) = e^{it\Delta}\varphi + y(t)$, with $y \in C(R, H^1(R^2))$ satisfying $\|y(t)\|_{H^1} \leqslant (1+|t|)^{\frac{2(1-s)}{(10m-5)s-(10m-6)}+\epsilon}$, where ϵ is an arbitrary sufficiently small positive number.

Keywords: Nonlinear Schrödinger equation; Initial value problem; Global existence; High–low frequency decomposition method

1. Introduction

In this paper we study global existence of solutions for the Cauchy problem of the nonlinear Schrödinger equation:

$$iu_t + \Delta u = |u|^{2m}u, \quad x \in R^n,\ t \in R, \tag{1.1}$$

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$$u(x,0)=\varphi(x), \quad x\in R^n, \tag{1.2}$$

where m is a positive integer and $\varphi\in H^s(R^n)$ for some $s\in R$. We shall consider 2 dimension case, i.e., the case $n=2$.

Problem (1.1)–(1.2) has been intensively studied for many years. The best local well-posedness result was obtained by Tsutsumi [18] and Cazenave and Weissler [5]. They proved that problem (1.1) is locally well-posed in the Sobolev space $H^s(R^n)$ if $s>s_0\equiv\frac{n}{2}-\frac{1}{m}$ in the case $s_0\geqslant 0$ and $s\geqslant 0$ in the case $s_0<0$, and locally solvable in $H^{s_0}(R^n)$ in the case $s_0\geqslant 0$. These results are optimal in the case $s_0\geqslant 0$, i.e., problem (1.1)–(1.2) is not locally well-posed in $H^s(R^n)$ if $s<s_0$, cf. [4]. (However, in the case $s_0<0$ the condition $s\geqslant 0$ can be weakened, and the problem of finding the lowest s_0' such that local well-posedness holds in $H^s(R^n)$ for all $s>s_0'$ still remains open; see [15].) Since Eq. (1.1) has the following conservation laws:

$$\frac{1}{2}\int_{R^n}\left|u(x,t)\right|^2dx=\text{const}, \tag{1.3}$$

$$\frac{1}{2}\int_{R^n}\left|\nabla u(x,t)\right|^2dx+\frac{1}{2(m+1)}\int_{R^n}\left|u(x,t)\right|^{2(m+1)}dx=\text{const}, \tag{1.4}$$

by a standard argument we see that problem (1.1)–(1.2) is globally well-posed in $H^s(R^n)$ at least in the following cases:

$$\begin{cases} s\geqslant 0, & \text{if } n=1 \text{ and } m\leqslant\frac{2}{n} \text{ or } n=2 \text{ and } m<\frac{2}{n},\\ s\geqslant 1, & \text{if either } n=1,2 \text{ with } m \text{ arbitrary or } n\geqslant 3 \text{ and } m\leqslant\frac{2}{n-2}.\end{cases} \tag{1.5}$$

An interesting problem is whether global well-posedness holds for all cases $s>s_0$ (when $s_0\geqslant 0$). Initiated by Bourgain [1], great efforts have been made by a few authors toward this direction [1–3,7,8]. In [1], Bourgain considered the case $n=2$ and $m=1$. By decomposing the initial function into a sum of two functions, one possessing only low frequencies and the other possessing only high frequencies but having small L^2 norm, he succeeded to prove that global well-posedness holds for $s\geqslant\frac{3}{5}$. Bourgain's method is now commonly referred as *high–low frequency decomposition method* in the literature and has been used to deal with other dispersive equations, including the mKdV equation [10] and the semilinear wave equation [16,17]. Recently, Colliander et al. [7,8] improved Bourgain's result by using the so-called *I-method*, and extended to the case $n=3$. They proved that problem (1.1)–(1.2) is globally well-posed in $H^s(R^n)$ if $n=2$, $m=1$, $s\geqslant\frac{4}{7}$ and $n=3$, $m=1$, $s\geqslant\frac{5}{6}$.

In this paper we consider the case $n=2$ and m a general positive integer not less than 2. We shall use the high–low frequency decomposition method to establish the following result:

Theorem 1.1. *Assume that $n=2$, m is positive integer, $m\geqslant 2$, $\varphi\in H^s(R^2)$ and $\frac{10m-6}{10m-5}<s<1$. Then problem* (1.1)–(1.2) *has a global solution $u\in C(R,H^s(R^n))$ which has the expression*:

$$u(t)=e^{it\Delta}\varphi+y(t), \tag{1.6}$$

$$y\in C\big(R;H^1\big(R^2\big)\big), \qquad \big\|y(t)\big\|_{H^1}\leqslant C\big(1+|t|\big)^{\frac{2(1-s)}{(10m-5)s-(10m-6)}+\epsilon}, \tag{1.7}$$

where ϵ is an arbitrary sufficiently small positive number.

Since the argument for $t<0$ is similar to that for $t>0$, we shall give the proof of the above result only for the part $t>0$.

The arrangement of the rest part is as follows. In Section 2 we introduce some notations and preliminary results related to the Schrödinger operator. In Section 3 we perform the high–low frequency decomposition process and consider the correspondingly decomposed problems. The proof of Theorem 1.1 is arranged in Section 4. In the last section we give some concluding remarks.

2. Notations and preliminaries

We denote by $U(t)$ $(t \in R)$ the fundamental solution operator of the Schrödinger equation, i.e.,

$$U(t)\varphi(x) = F^{-1}\big(e^{-it|\xi|^2}\tilde{\varphi}(\xi)\big) \quad \text{for } \varphi \in S'\big(R^n\big),$$

where $\tilde{\varphi}$ denotes the Fourier transformation of φ, and F^{-1} represents the inverse Fourier transformation. It is well known that $U(t)$ $(t \in R)$ coincides with the unitary group $e^{it\Delta}$ $(t \in R)$ generated by the skew-symmetric operator $i\Delta$ when restricted on the Sobolev space $H^s(R^n)$ (for any $s \in R$).

For $s \in R$, we denote by D_x^s and J_x^s, respectively, the Riesz and the Bessel potentials of order $-s$, i.e.,

$$D_x^s\varphi(x) = F^{-1}\big(|\xi|^s\tilde{\varphi}(\xi)\big), \qquad J_x^s\varphi(x) = F^{-1}\big(\big(1+|\xi|^2\big)^{s/2}\tilde{\varphi}(\xi)\big),$$

whenever the right-hand sides make sense. We shall simply write D_x^1 as D_x. We also denote by $(R_1, R_2, \ldots, R_n)$ the Riesz transformation on R^n, i.e.,

$$R_j\varphi(x) = F^{-1}\big(\xi_j|\xi|^{-1}\tilde{\varphi}(\xi)\big),$$

whenever the right-hand side makes sense. The usual partial derivative $\frac{\partial}{\partial x_j}$ will be abbreviated as ∂_j $(j = 1, 2, \ldots, n)$. It is well known that for any $1 < p < \infty$ and any $\varphi \in W^{1,p}(R^n)$ there hold

$$D_x\varphi = -i\sum_{j=1}^n R_j\partial_j\varphi = -i\sum_{j=1}^n \partial_j R_j\varphi, \quad \partial_j\varphi = iR_jD_x\varphi = iD_xR_j\varphi$$

$(j = 1, 2, \ldots, n)$, regarding both D_x and ∂_j $(j = 1, 2, \ldots, n)$ as bounded linear operators mapping $W^{1,p}(R^n)$ into $L^p(R^n)$.

For $\delta > 0$ and $q, r \in [1, \infty]$, we denote by $\|\cdot\|_{L^q_{t,\delta}L^r_x}$ the norm in the Banach space $L^q([0, \delta], L^r(R^n))$, so that for $1 \leqslant q < \infty$,

$$\|f\|_{L^q_{t,\delta}L^r_x} = \left(\int_0^\delta \|f(t,\cdot)\|_r^q\,dt\right)^{1/q},$$

and for $q = \infty$,

$$\|f\|_{L^\infty_{t,\delta}L^r_x} = \operatorname*{ess\,sup}_{0\leqslant t\leqslant\delta}\big\|f(t,\cdot)\big\|_r.$$

The inner product of functions u, v in $L^2([0, \delta], L^2(R^n))$ will be denoted as $\langle u, v\rangle$.

We denote by $X_{s,b}(R \times R^n)$ the completion of the Schwartz space $S(R \times R^n)$ under the norm

$$\|u\|_{X_{s,b}(R\times R^2)} = \left(\iint \big(1+|\xi|^2\big)^s\big(1+\big|\tau+|\xi|^2\big|^2\big)^b\big|\tilde{u}(\tau,\xi)\big|^2\,d\xi\,d\tau\right)^{1/2}.$$

Note that $-(\tau+|\xi|^2)$ is the symbol of the Schrödinger operator $i\partial_t+\Delta$. We often abbreviate $\|u\|_{s,b}$ for $\|u\|_{X_{s,b}(R\times R^2)}$. It is easy to verify that

$$\|u\|_{s,b}=\left\|J_t^b J_x^s U(-t)u(t,x)\right\|_{L^2(R\times R^2)},$$

where J_t^b and J_x^s respectively denote Bessel potentials in the t and x variables, of orders $-b$ and $-s$, respectively.

For any time interval I, we denote by $X_{s,b}(I\times R^2)$ the restriction of $X_{s,b}(R\times R^n)$ on $I\times R^n$, with norm

$$\|u\|_{X_{s,b}(I\times R^n)}=\inf\bigl\{\|f\|_{s,b}\colon\ f\in X_{s,b}\bigl(R\times R^n\bigr),\ f|_{I\times R^n}=u\bigr\}.$$

We also abbreviate $X_{s,b}^\delta$ for $X_{s,b}([0,\delta]\times R^n)$, and $\|u\|_{X_{s,b}^\delta}$ for $\|u\|_{X_{s,b}([0,\delta]\times R^n)}$. In our arguments we shall be frequently using the trivial embedding

$$\|u\|_{X^\delta_{s_1,b_1}}\leqslant\|u\|_{X^\delta_{s_2,b_2}}\qquad\text{whenever } s_1<s_2,\ b_1<b_2,$$

the dual relation

$$\bigl(X_{s,b}^\delta\bigr)^*=X_{-s,-b}^\delta, \tag{2.1}$$

and the interpolation

$$X_{s,b}^\delta=\bigl(X_{s_0,b_0}^\delta,X_{s_1,b_1}^\delta\bigr)_\theta, \tag{2.2}$$

where $0\leqslant\theta\leqslant1$, $b=(1-\theta)b_0+\theta b_1$, $s=(1-\theta)s_0+\theta s_1$.

Following [5], we say that (q,r) is an *admissible pair* if $2\leqslant q\leqslant\infty$,

$$\frac{2}{q}=\frac{n}{2}\left(1-\frac{2}{r}\right),$$

and

$$\begin{cases}2\leqslant r\leqslant\frac{2n}{n-2}, & \text{if } n>2,\\ 2\leqslant r<\infty, & \text{if } n=2,\\ 2\leqslant r\leqslant\infty, & \text{if } n<2.\end{cases}$$

This concept is connected to the following fundamental result: If (q,r) is a admissible pair then there holds the *Strichartz estimate* (see [4, Theorem 2.3.3]):

$$\bigl\|U(t)\varphi(x)\bigr\|_{L_t^qL_x^r}\leqslant C\|\varphi\|_{L^2(R^n)}. \tag{2.3}$$

Following [1] we shall use the notation $a+$ and $a-$ to, respectively, abbreviate expressions of forms $a+\varepsilon$ and $a-\varepsilon$ with a sufficiently small quantity $\varepsilon>0$. However, to avoid possible mistake and misunderstanding we shall limit our use of these abbreviations to the case that ε can be an *arbitrary* sufficiently small positive number; in the case that ε depends on other small quantities we shall explicitly write out their precise expressions.

Lemma 2.1. *For any admissible pair (q,r) and any $0<\delta<1$ there hold*:

$$\|u\|_{L_{t,\delta}^qL_x^r}\leqslant C\|u\|_{X^\delta_{0,\frac12+}}, \tag{2.4}$$

$$\|u\|_{L_{t,\delta}^{q_\theta}L_x^{r_\theta}}\leqslant C\|u\|_{X^\delta_{0,\frac{\theta}{2}+}}, \tag{2.5}$$

where $\theta\in[0,1]$, $\frac{1}{q_\theta}=\frac{\theta}{q}+\frac{1-\theta}{2}$, $\frac{1}{r_\theta}=\frac{\theta}{r}+\frac{1-\theta}{2}$, and C is independent of δ.

Proof. To establish (2.4) we only need to prove that for any $b > 1/2$ there holds

$$\|f\|_{L_t^q L_x^r} \leqslant C\|f\|_{X_{0,b}}. \tag{2.6}$$

Indeed, for any $u \in X_{0,b}^\delta$ we can find a series of $f_k \in X_{0,b}$ ($k = 1, 2, \ldots$), such that $f_k|_{[0,\delta]\times R^n} = u$, and

$$\|f_k\|_{X_{0,b}} \leqslant \|u\|_{X_{0,b}^\delta} + \frac{1}{k}, \quad k = 1, 2, \ldots.$$

If (2.6) is proved then we have

$$\|u\|_{L_{t,\delta}^q L_x^r} \leqslant \|f_k\|_{L_t^q L_x^r} \leqslant C\|f_k\|_{X_{0,b}} \leqslant C\|u\|_{X_{0,b}^\delta} + \frac{1}{k}, \quad k = 1, 2, \ldots.$$

Letting $k \to \infty$, we obtain (2.4). In the sequel we give the proof of (2.6).

Let (q, r) be an admissible pair and let $b > 1/2$. We denote

$$\hat{\varphi}_\tau(\xi) = \left(1 + |\tau|^2\right)^{b/2} \hat{f}\left(\xi, \tau - |\xi|^2\right),$$
$$g(x, t, \tau) = U(t)\varphi_\tau(x) = \int_R e^{i(x\cdot\xi - t|\xi|^2)} \hat{\varphi}_\tau(\xi)\, d\xi.$$

Then $\|f\|_{X_{0,b}} = (\int_{-\infty}^{+\infty} \int_{R^n} |\hat{\varphi}_\tau(\xi)|^2\, d\xi\, d\tau)^{1/2}$, and

$$f(x, t) = \int_{-\infty}^{+\infty} e^{it\tau} g(x, t, \tau)\left(1 + |\tau|^2\right)^{-b/2} d\tau.$$

The last equality implies that

$$\|f\|_{L_t^q L_x^r} \leqslant \int_{-\infty}^{+\infty} \left\|g(\cdot, \cdot, \tau)\right\|_{L_t^q L_x^r} \left(1 + |\tau|^2\right)^{-b/2} d\tau. \tag{2.7}$$

By the Strichartz estimate (2.3) we have

$$\left\|g(\cdot, \cdot, \tau)\right\|_{L_t^q L_x^r} = \left\|U(t)\varphi_\tau(x)\right\|_{L_t^q L_x^r} \leqslant C\|\varphi_\tau\|_{L^2(R^n)} = C\|\hat{\varphi}_\tau\|_{L^2(R^n)}.$$

Substituting this estimate into (2.7), we get

$$\begin{aligned}\|f\|_{L_t^q L_x^r} &\leqslant C \int_{-\infty}^{+\infty} \|\hat{\varphi}_\tau\|_{L^2(R^n)} \left(1 + |\tau|^2\right)^{-b/2} d\tau \leqslant C\left(\int_{-\infty}^{+\infty} \|\hat{\varphi}_\tau\|_{L^2(R^n)}^2\, d\tau\right)^{1/2} \\ &= C\|f\|_{X_{0,b}},\end{aligned}$$

so that (2.6) holds.

Having proved (2.4), (2.5) easily follows from interpolation between (2.4) and the relation

$$\|u\|_{L_{t,\delta}^2 L_x^2} = \|u\|_{X_{0,0}^\delta}.$$

This completes the proof. $\square$

Corollary 2.2. *Let $n = 2$. Then for any $0 < \delta < 1$ and any sufficiently small $\varepsilon > 0$ there hold*:

$$\|u\|_{L^4_{t,\delta} L^4_x} \leqslant C\|u\|_{X^\delta_{0,\frac{1}{2}+}}, \tag{2.8}$$

$$\|u\|_{L^{4-\varepsilon}_{t,\delta} L^{4-\varepsilon}_x} \leqslant C\|u\|_{X^\delta_{0,\frac{1}{2}-\frac{\varepsilon}{8}}}, \tag{2.9}$$

$$\|u\|_{X^\delta_{0,-\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{L^{q_\varepsilon}_{t,\delta} L^{q_\varepsilon}_x}, \quad q_\varepsilon = \frac{4-8\varepsilon}{3-8\varepsilon}. \tag{2.10}$$

Proof. (2.8) is an immediate consequence of (2.4), because $(4, 4)$ is an admissible pair in the case $n = 2$. (2.9) follows from (2.5) by letting $(q, r) = (4, 4)$, $\theta = \frac{4-2\varepsilon}{4-\varepsilon}$ and noticing that

$$\frac{\theta}{2} = \frac{2-\varepsilon}{4-\varepsilon} < \frac{1}{2} - \frac{\varepsilon}{8}.$$

To prove (2.10) we use the dual relation (2.1) and the Hölder inequality to deduce

$$\|u\|_{X^\delta_{0,-\frac{1}{2}+\varepsilon}} \leqslant \sup_{\|\psi\|_{X^\delta_{0,\frac{1}{2}-\varepsilon}} \leqslant 1} \left|\langle\psi, u\rangle\right| \leqslant \sup_{\|\psi\|_{X^\delta_{0,\frac{1}{2}-\varepsilon}} \leqslant 1} \|\psi\|_{L^{4-8\varepsilon}_{t,\delta} L^{4-8\varepsilon}_x} \|u\|_{L^{q_\varepsilon}_{t,\delta} L^{q_\varepsilon}_x}.$$

Since $\|\psi\|_{L^{4-8\varepsilon}_{t,\delta} L^{4-8\varepsilon}_x} \leqslant C\|\psi\|_{X^\delta_{0,\frac{1}{2}-\varepsilon}}$ (by (2.9)), we see that (2.10) holds. □

Lemma 2.3. *For any real s and $0 < \delta < 1$ there hold*:

$$\left\|U(t)\varphi\right\|_{X^\delta_{s,b}} \leqslant C\|\varphi\|_{H^s(R^2)}, \quad -\infty < b < \infty, \tag{2.11}$$

$$\|u\|_{X^\delta_{s,-b_1}} \leqslant C\delta^{b_1-b_2-}\|u\|_{X^\delta_{s,-b_2}}, \quad 0 \leqslant b_2 < b_1 < \frac{1}{2}, \tag{2.12}$$

$$\left\|\int_0^t U(t-\tau)u(\tau)\,d\tau\right\|_{X^\delta_{s,b}} \leqslant C\delta^{\frac{1}{2}-b}\|u\|_{X^\delta_{s,b-1}}, \quad \frac{1}{2} < b \leqslant 1, \tag{2.13}$$

where C represents a constant independent of δ.

Proof. Take a function $\psi \in C_0^\infty(R)$ such that $\psi(t) = 1$ for $0 \leqslant t \leqslant 1$, and $\psi(t) = 0$ for $t \leqslant -1$ and $t \geqslant 2$. Since $0 < \delta < 1$, we have

$$\begin{aligned}\left\|U(t)\varphi\right\|_{X^\delta_{s,b}} &\leqslant \left\|U(t)\varphi\right\|_{X^1_{s,b}} \leqslant \left\|\psi(t)U(t)\varphi\right\|_{X_{s,b}} = \left\|J^b_t J^s_x U(-t)\big(\psi(t)U(t)\varphi\big)\right\|_{L^2(R\times R^n)} \\ &= \left\|J^b\psi\right\|_{L^2(R)} \cdot \left\|J^s\varphi\right\|_{L^2(R^n)} \leqslant C\|\varphi\|_{H^s(R^2)}.\end{aligned}$$

This proves (2.11).

To establish (2.12) we only need to prove that

$$\left\|\psi\left(\frac{t}{\delta}\right)f\right\|_{X_{s,-b_1}} \leqslant C\delta^{b_1-b_2-}\|f\|_{X_{s,-b_2}}; \tag{2.14}$$

the argument of deducing (2.12) from (2.14) is similar to that in the proof of Lemma 2.1. By dual, (2.14) follows if we establish that

$$\left\|\psi\left(\frac{t}{\delta}\right)f\right\|_{X_{s,b_2}} \leqslant C\delta^{b_1-b_2-}\|f\|_{X_{s,b_1}}. \tag{2.15}$$

To prove (2.15) we denote $g(x,t) = J_t^{b_1} J_x^s U(-t) f(x,t)$. Then a simple computation shows that

$$\left\| \psi\left(\frac{t}{\delta}\right) f \right\|_{X_{s,b_2}} = \left\| J_t^{b_2}\left(\psi\left(\frac{t}{\delta}\right) J_t^{-b_1} g\right)\right\|_{L^2(R\times R^n)}.$$

Since $\|f\|_{X_{s,b_1}} = \|g\|_{L^2(R\times R^n)}$, (2.15) follows if we prove that

$$\left\| J_t^{b_2}\left(\psi\left(\frac{t}{\delta}\right) J_t^{-b_1} g\right)\right\|_{L^2(R\times R^n)} \leqslant C\delta^{b_1-b_2-} \|g\|_{L^2(R\times R^n)}.$$

The last inequality is clearly implied by the following apparently simpler inequality:

$$\left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{H^{b_2}(R)} \leqslant C\delta^{b_1-b_2-} \|\varphi\|_{H^{b_1}(R)}. \tag{2.16}$$

In the sequel we give the proof of (2.16).

First, by Kenig et al. [13, (3.6)], we have

$$\left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{H^{\tilde b}(R)} \leqslant C\delta^{1-2\tilde b} \|\varphi\|_{H^{\tilde b}(R)}, \quad \frac{1}{2} < \tilde b \leqslant 1. \tag{2.17}$$

Next, since

$$\left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{L^2(R)} \leqslant C\left(\int_{-\delta}^{2\delta} |\varphi(t)|^2\, dt\right)^{1/2} \leqslant C\delta^{\frac{1}{2}-\frac{1}{q}} \|\varphi\|_{L^q(R)}$$

and $\|\varphi\|_{L^q(R)} \leqslant C\|\varphi\|_{H^{\bar b}(R)}$ for $2 \leqslant q < \infty$ and $\bar b \geqslant \frac{1}{2} - \frac{1}{q}$, by taking $\bar b = \frac{1}{2} - \frac{1}{q}$ we get

$$\left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{L^2(R)} \leqslant C\delta^{\bar b} \|\varphi\|_{H^{\bar b}(R)}, \quad 0 \leqslant \bar b < \frac{1}{2}. \tag{2.18}$$

The desired result now follows by interpolation between (2.17) and (2.18): For sufficiently small $\varepsilon > 0$ we put $\tilde b = \frac{1}{2} + \varepsilon$, $\bar b = \frac{(b_1-b_2)(1+2\varepsilon)}{1-2b_2+2\varepsilon}$ and $\theta = \frac{2b_2}{1+2\varepsilon}$. Then we have

$$\left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{H^{b_2}(R)} = \left\| \psi\left(\frac{t}{\delta}\right) \varphi \right\|_{H^{\theta\tilde b+(1-\theta)\bar b}(R)} \leqslant C\delta^{\theta(1-2\tilde b)+(1-\theta)\bar b} \|\varphi\|_{H^{\theta\tilde b+(1-\theta)\bar b}(R)}.$$

Since $\theta(1-2\tilde b) + (1-\theta)\bar b = b_1 - b_2 - \frac{4b_2\varepsilon}{1+2\varepsilon}$ and $\theta\tilde b + (1-\theta)\bar b = b_1$, we see that (2.16) follows.

Similarly, (2.13) follows from the following inequality:

$$\left\| \psi\left(\frac{t}{\delta}\right) \int_0^t U(t-\tau) f(\tau)\, d\tau \right\|_{X_{s,b}} \leqslant C\delta^{\frac{1}{2}-b} \|f\|_{X_{s,b-1}}, \quad \frac{1}{2} < b \leqslant 1. \tag{2.19}$$

This inequality has been established by Kenig et al. in the 1 dimension case (see [15, Lemma 2.6, (2.72)]; note that the condition $s \leqslant 0$ imposed in [15, Lemma 2.6] is merely for the application there and is unnecessary). The proof for general n dimension case is rather similar. Actually, this inequality holds for a much wider class of dispersive operators, cf. [7,9,11,14] for instance. We thus omit its proof here. □

Corollary 2.4. *For any $s \in R$, any $0 < \delta < 1$ and sufficiently small $\varepsilon > 0$ there holds*:

$$\left\| \int_0^t U(t-\tau)u(\tau)\,d\tau \right\|_{X^\delta_{s,\frac{1}{2}+\varepsilon}} \leqslant C\|u\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon}}, \tag{2.20}$$

where C is independent of δ.

Proof. Taking $b = \frac{1}{2}+\varepsilon$, $b_1 = \frac{1}{2}-\varepsilon = -(b-1)$ and $b_2 = \frac{1}{2}-3\varepsilon$ in (2.13) and (2.12), we obtain

$$\begin{aligned}\left\| \int_0^t U(t-\tau)u(\tau)\,d\tau \right\|_{X^\delta_{s,\frac{1}{2}+\varepsilon}} &\leqslant C\delta^{-\varepsilon}\|u\|_{X^\delta_{s,-b_1}} \leqslant C\delta^{-\varepsilon}\cdot\delta^{b_1-b_2-\varepsilon}\|u\|_{X^\delta_{s,-b_2}} \\ &= C\|u\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon}}.\end{aligned}$$

Hence the desired assertion holds. □

Lemma 2.5 *(Bourgain). Assume that $0 < s < \frac{1}{2}$. Then for any $\delta > 0$ there holds*:

$$\left\| D_x^s(uv)\right\|_{L^2_{t,\delta}L^2_x} \leqslant C\|u\|_{X^\delta_{s+,\frac{1}{2}+}}\|v\|_{X^\delta_{0,\frac{1}{2}-}}. \tag{2.21}$$

Proof. By Bourgain [1, Corollary 113, (118)] we see that the corresponding inequality without δ is valid. Having seen this fact, the above inequality easily follows from a restriction argument as in the proof of Lemma 2.1. □

Remark. Strictly speaking, in (2.21) $\frac{1}{2}-$ depends on s, i.e., if s is sufficiently near $\frac{1}{2}$ then $\frac{1}{2}-$ is correspondingly sufficiently near $\frac{1}{2}$. More precisely, if $s = \frac{1}{2}-\sigma$ for some $0 < \sigma < \frac{1}{2}$, then there exists $0 < \nu(\sigma) < \frac{1}{2}$ such that (2.21) holds when $s+$, $\frac{1}{2}+$ and $\frac{1}{2}-$ are, respectively, replaced with $s+\varepsilon$, $\frac{1}{2}+\tilde{\varepsilon}$ and $\frac{1}{2}-\bar{\varepsilon}$ for arbitrary $\varepsilon > 0$, $\tilde{\varepsilon} > 0$ and $0 < \bar{\varepsilon} < \nu(\sigma)$; if $\sigma \to 0$ then also $\nu(\sigma) \to 0$ but $C \to \infty$. If $s = \frac{1}{2}$ then $\frac{1}{2}-$ must be replaced with $\frac{1}{2}+$; see [1, Corollary 113, (115)].

Lemma 2.6. *Let $0 < \sigma < \frac{1}{2}$ and assume that $u \in L^2([0,\delta], H^{\frac{1}{2}+\sigma}(R^2))$, $v \in X^\delta_{1,\frac{1}{2}+\varepsilon}$, where $\delta, \varepsilon > 0$. Then there exists $\nu(\sigma) > 0$ (independent of δ and ε) such that for any $\bar{\varepsilon} \in (0, \nu(\sigma))$ we have $uv \in X^\delta_{1,-\frac{1}{2}+\bar{\varepsilon}}$, and*

$$\left\| D_x(uv)\right\|_{X^\delta_{0,-\frac{1}{2}+\bar{\varepsilon}}} \leqslant C\Big(\|u\|_{L^{h_{\bar{\varepsilon}}}_{t,\delta}L^{h_{\bar{\varepsilon}}}_x}\|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon}} + \left\|D_x^{\frac{1}{2}+\sigma}\right\|_{L^2_{t,\delta}L^2_x}\|v\|_{X^\delta_{\frac{1}{2}-\sigma+\tilde{\varepsilon},\frac{1}{2}+\varepsilon}}\Big), \tag{2.22}$$

where $h_{\bar{\varepsilon}} = \frac{2(1-2\bar{\varepsilon})}{1-3\bar{\varepsilon}}$, and $\tilde{\varepsilon}$ is an arbitrary positive number.

Proof. We only need to prove that (2.22) holds for any $u, v \in C^\infty([0,\delta], C_0^\infty(R^2))$; for general u, v satisfying the conditions of the lemma the assertion then follows from a standard density argument. By the dual relation (2.1) and the density of $C^\infty([0,\delta], C_0^\infty(R^2))$ in $X^\delta_{0,\frac{1}{2}-\bar{\varepsilon}}$ we have

$$\left\| D_x(uv)\right\|_{X^\delta_{0,-\frac{1}{2}+\bar{\varepsilon}}} = \sup\Big\{\left|\langle\psi, D_x(uv)\rangle\right|:\ \psi \in C^\infty\big([0,\delta], C_0^\infty\big(R^2\big)\big),\ \|\psi\|_{X^\delta_{0,\frac{1}{2}-\bar{\varepsilon}}} \leqslant 1\Big\}. \tag{2.23}$$

We compute

$$\begin{aligned}\big\langle\psi, D_x(uv)\big\rangle &= i\sum_{j=1}^{2}\big\langle\psi, R_j\partial_j(uv)\big\rangle = i\sum_{j=1}^{2}\langle R_j\psi, v\partial_j u + u\partial_j v\rangle\\ &= \sum_{j=1}^{2}\langle\bar v R_j\psi, D_x R_j u\rangle + \sum_{j=1}^{2}\langle R_j\psi, uD_x R_j v\rangle\\ &= \sum_{j=1}^{2}\big\langle D_x^{\frac12-\sigma}(\bar v R_j\psi), D_x^{\frac12+\sigma}R_j u\big\rangle + \sum_{j=1}^{2}\langle R_j\psi, uR_jD_x v\rangle.\end{aligned}$$

Thus by the Hölder inequality we get

$$\begin{aligned}\big|\big\langle\psi, D_x(uv)\big\rangle\big| &\leqslant C\sum_{j=1}^{2}\big\|D_x^{\frac12-\sigma}(\bar v R_j\psi)\big\|_{L^2_{t,\delta}L^2_x}\big\|D_x^{\frac12+\sigma}R_j u\big\|_{L^2_{t,\delta}L^2_x}\\ &\quad + C\sum_{j=1}^{2}\|R_j\psi\|_{L^{4-8\bar\varepsilon}_{t,\delta}L^{4-8\bar\varepsilon}_x}\|u\|_{L^{h_{\bar\varepsilon}}_{t,\delta}L^{h_{\bar\varepsilon}}_x}\|R_jD_x v\|_{L^4_{t,\delta}L^4_x}.\end{aligned}$$

Using inequalities (2.8), (2.9) and (2.21), we infer that there exists $\nu(\sigma) > 0$ such that if $\bar\varepsilon \in (0, \nu(\sigma))$ then for any $\tilde\varepsilon > 0$ we have

$$\begin{aligned}\big|\big\langle\psi, D_x(uv)\big\rangle\big| &\leqslant C\sum_{j=1}^{2}\|v\|_{X^\delta_{\frac12-\sigma+\tilde\varepsilon,\frac12+\varepsilon}}\|R_j\psi\|_{X^\delta_{0,\frac12-\bar\varepsilon}}\big\|R_jD_x^{\frac12+\sigma}u\big\|_{L^2_{t,\delta}L^2_x}\\ &\quad + C\sum_{j=1}^{2}\|R_j\psi\|_{X^\delta_{0,\frac12-\bar\varepsilon}}\|u\|_{L^{h_{\bar\varepsilon}}_{t,\delta}L^{h_{\bar\varepsilon}}_x}\|R_j v\|_{X^\delta_{1,\frac12+\varepsilon}}\\ &\leqslant C\|\psi\|_{X^\delta_{0,\frac12-\bar\varepsilon}}\Big(\|v\|_{X^\delta_{\frac12-\sigma+\tilde\varepsilon,\frac12+\varepsilon}}\big\|D_x^{\frac12+\sigma}u\big\|_{L^2_{t,\delta}L^2_x} + \|u\|_{L^{h_{\bar\varepsilon}}_{t,\delta}L^{h_{\bar\varepsilon}}_x}\|v\|_{X^\delta_{1,\frac12+\varepsilon}}\Big).\end{aligned}$$

Substituting this estimate into (2.23), we get the desired result. □

3. Decomposition and decomposed problems

For $\varphi \in H^1(R^2)$ we denote

$$E(\varphi) = \frac12\|\varphi\|^2_{L^2(R^2)}, \qquad H(\varphi) = \frac12\|\nabla\varphi\|^2_{L^2(R^2)} + \frac{1}{2(m+1)}\|\varphi\|^{2(m+1)}_{L^{2(m+1)}(R^2)}.$$

Using these notations, the conservation laws (1.3) and (1.4) (when $u \in C(R, H^1(R^2))$) can be rewritten as

$$E\big(u(\cdot,t)\big) = \text{const}, \quad H\big(u(\cdot,t)\big) = \text{const} \quad \text{for } t \in R. \tag{3.1}$$

Lemma 3.1. *Let $1 - \frac1m < s < 1$ and $\varphi \in H^s(R^2)$. Then for any given positive number $N \geqslant 1$, there exists corresponding decomposition of φ:*

$$\varphi = \varphi_1 + \varphi_2, \tag{3.2}$$

such that the following assertions hold:

(i) $\varphi_1 \in H^1(R^2)$, *and*

$$E(\varphi_1) \leqslant C\|\varphi\|^2_{L^2(R^2)}, \qquad H(\varphi_1) \leqslant CN^{2(1-s)}\|\varphi\|^2_{H^s(R^2)}\big(1+\|\varphi\|^{2m}_{H^s(R^2)}\big). \tag{3.3}$$

(ii) $\varphi_2 \in H^s(R^2)$, *and*

$$\|\varphi_2\|_{L^2(R^2)} \leqslant CN^{-s}\|\varphi\|_{H^s(R^2)}, \qquad \|\varphi_2\|_{H^s(R^2)} \leqslant C\|\varphi\|_{H^s(R^2)}. \tag{3.4}$$

In (3.3) *and* (3.4) C *represents a constant independent of* N *and* φ.

Proof. We define

$$\varphi_1(x) = (2\pi)^{-2}\int\limits_{|\xi|\leqslant N}\tilde{\varphi}(\xi)e^{ix\xi}\,d\xi, \qquad \varphi_2(x) = (2\pi)^{-2}\int\limits_{|\xi|>N}\tilde{\varphi}(\xi)e^{ix\xi}\,d\xi. \tag{3.5}$$

It is immediate to see that $\varphi_1 \in H^1(R^2)$, $\varphi_2 \in H^s(R^2)$, and (3.2), (3.4) and the first inequality in (3.3) hold. We now prove the second inequality in (3.3). Clearly,

$$\|\nabla\varphi_1\|^2_{L^2(R^2)} \leqslant N^{2(1-s)}\|\varphi\|^2_{H^s(R^2)}.$$

Thus the desired result follows if we prove that

$$\|\varphi_1\|_{L^{2(m+1)}(R^2)} \leqslant C\|\varphi\|_{H^s(R^2)}N^{\frac{1-s}{m+1}}. \tag{3.6}$$

By the Hausdorff–Young inequality we have

$$\begin{aligned}
&\|\varphi_1\|_{L^{2(m+1)}(R^2)} \\
&\quad\leqslant C\|\tilde{\varphi}_1\|_{L^{\frac{2(m+1)}{2m+1}}(R^2)} = C\Bigg(\int\limits_{|\xi|\leqslant N}\big|\tilde{\varphi}(\xi)\big|^{\frac{2(m+1)}{2m+1}}\,d\xi\Bigg)^{\frac{2m+1}{2(m+1)}} \\
&\quad= C\Bigg(\int\limits_{|\xi|\leqslant N}\big(1+|\xi|^2\big)^{-\frac{(m+1)s}{2m+1}}\cdot\big(1+|\xi|^2\big)^{\frac{(m+1)s}{2m+1}}\big|\tilde{\varphi}(\xi)\big|^{\frac{2(m+1)}{2m+1}}\,d\xi\Bigg)^{\frac{2m+1}{2(m+1)}} \\
&\quad\leqslant C\Bigg(\int\limits_{|\xi|\leqslant N}\big(1+|\xi|^2\big)^{-\frac{(m+1)s}{m}}\,d\xi\Bigg)^{\frac{m}{2(m+1)}}\cdot\Bigg(\int\limits_{|\xi|\leqslant N}\big(1+|\xi|^2\big)^s\big|\tilde{\varphi}(\xi)\big|^2\,d\xi\Bigg)^{1/2} \\
&\quad\leqslant C\Bigg(\int\limits_0^N(1+\rho)^{-\frac{2(m+1)s}{m}}\rho\,d\rho\Bigg)^{\frac{m}{2(m+1)}}\|\varphi\|_{H^s(R^2)}.
\end{aligned}$$

Since

$$\begin{aligned}
\Bigg(\int\limits_0^N(1+\rho)^{-\frac{2(m+1)s}{m}}\rho\,d\rho\Bigg)^{\frac{m}{2(m+1)}} &\leqslant \begin{cases} C, & \text{if } s>\frac{m}{m+1}, \\ (\log(1+N))^{\frac{m}{2(m+1)}}, & \text{if } s=\frac{m}{m+1}, \\ N^{-s+\frac{m}{m+1}}, & \text{if } 1-\frac{1}{m}<s<\frac{m}{m+1}, \end{cases} \\
&\leqslant CN^{\frac{1-s}{m+1}},
\end{aligned}$$

where the last inequality follows from the fact $s > 1-\frac{1}{m}$, we see that (3.6) holds. □

Remark. One can easily verify that the condition $s > \frac{10m-6}{10m-5}$ of Theorem 1.1 implies the condition $s > \frac{m-1}{m}$ of Lemma 3.1.

Let u be the solution of problem (1.1)–(1.2) (recall that $n = 2$). If we use the above decomposition of the initial function φ, we can then accordingly decompose the solution u into

$$u = v + w,$$

where v stands for the solution of the initial value problem

$$\begin{cases} iv_t + \Delta v = |v|^{2m} v, & x \in R^2,\ t \in R, \\ v(x,0) = \varphi_1(x), & x \in R^2, \end{cases} \tag{3.7}$$

and w is the solution of the initial value problem

$$\begin{cases} iw_t + \Delta w = |v+w|^{2m}(v+w) - |v|^{2m} v, & x \in R^2,\ t \in R, \\ w(x,0) = \varphi_2(x), & x \in R^2. \end{cases} \tag{3.8}$$

Conversely, if for decomposition (3.2) of the initial function φ, we have obtained solutions v and w of problems (3.7) and (3.8) in a common region $I \times R^2$, where I is a time interval, then their sum $u = v + w$ is clearly a solution of problem (1.1)–(1.2) in the same region $I \times R^2$. In the sequel we shall use this idea to solve problem (1.1)–(1.2).

We first consider problem (3.7). By (3.3) we assume that the initial function φ_1 belongs to $H^1(R^2)$ and it satisfies

$$E(\varphi_1) \leqslant M_0, \qquad H(\varphi_1) \leqslant M_1 N^{2(1-s)} \tag{3.9}$$

for some constants M_0 and M_1. Since $\varphi_1 \in H^1(R^2)$, by the global result mentioned before, problem (3.7) has a unique global solution $v \in C(R; H^1(R^2))$. In the following we make estimates on v. We shall use the equivalent integral form of (3.7):

$$v(t) = U(t)\varphi_1 - i \int_0^t U(t-\tau)\big(v|v|^{2m}\big)(\tau)\,d\tau. \tag{3.10}$$

Lemma 3.2. *Suppose φ_1 satisfies condition* (3.9), *and let $\delta = cN^{-(4m-2+\varepsilon)(1-s)}$, where c is an arbitrary positive number and ε is a sufficiently small positive number. Then for the solution of* (3.7) *we have*

$$\|v\|_{L_{t,\delta}^{4m+\varepsilon} L_x^{4m+\varepsilon}} \leqslant C c^{\frac{1}{4m+\varepsilon}}, \tag{3.11}$$

where C is a constant depending only on M_0 and M_1; C is independent of φ_1, N and c.

Proof. By the Gagliardo–Nirenberg inequality and condition (3.9) we have

$$\big\|v(\cdot,t)\big\|_{L^{4m+\varepsilon}(R^2)} \leqslant C\big\|v(\cdot,t)\big\|_{L^2(R^2)}^{\frac{2}{4m+\varepsilon}} \big\|v(\cdot,t)\big\|_{H^1(R^2)}^{1-\frac{2}{4m+\varepsilon}} \leqslant CN^{(1-\frac{2}{4m+\varepsilon})(1-s)}. \tag{3.12}$$

Thus

$$\|v\|_{L_{t,\delta}^{4m+\varepsilon} L_x^{4m+\varepsilon}} \leqslant C\delta^{\frac{1}{4m+\varepsilon}} N^{(1-\frac{2}{4m+\varepsilon})(1-s)} \leqslant Cc^{\frac{1}{4m+\varepsilon}}.$$

This proves (3.11). $\square$

Lemma 3.3. *Suppose φ_1 satisfies condition* (3.9). *Let v be the solution of* (3.7) *and let $\delta = cN^{-(4m-2+\varepsilon)(1-s)}$, where c and ε are as before. Then there exists $c_0 > 0$ such that for any $0 < c \leqslant c_0$ there hold*

$$\|v\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} \leqslant C, \qquad \|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}} \leqslant CN^{1-s}, \tag{3.13}$$

where $\varepsilon' = \frac{\varepsilon}{3(4m+3\varepsilon)}$, and C is a constant depending only on M_0 and M_1; C is independent of φ_1, N and c.

Proof. By (3.10), (2.11) and (2.20), we have

$$\begin{aligned}\|v\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} &\leqslant \|U(t)\varphi_1(x)\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} + \left\| \int_0^t U(t-\tau)\big(v|v|^{2m}\big)(\tau)\,d\tau \right\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} \\ &\leqslant C\|\varphi_1\|_{L^2(R^2)} + C\big\|v|v|^{2m}\big\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}}.\end{aligned} \tag{3.14}$$

By (2.10), (3.11) and (2.8), we have

$$\big\|v|v|^{2m}\big\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}} \leqslant C\big\|v|v|^{2m}\big\|_{L^{r_\varepsilon}_{t,\delta}L^{r_\varepsilon}_x} \leqslant C\|v\|_{L^4_{t,\delta}L^4_x}\|v\|^{2m}_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} \leqslant Cc^{\frac{2m}{4m+\varepsilon}}\|v\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}},$$

where (see (2.10))

$$r_\varepsilon = \frac{4-8\times 3\varepsilon'}{3-8\times 3\varepsilon'} = \frac{4(1-6\varepsilon')}{3(1-8\varepsilon')} = \frac{4(4m+\varepsilon)}{12m+\varepsilon}. \tag{3.15}$$

Substituting this estimate into (3.14) and using condition (3.9), we get

$$\|v\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} \leqslant C + Cc^{\frac{2m}{4m+\varepsilon}}\|v\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}.$$

Hence, the first inequality in (3.13) holds for sufficiently small c.

Next, again by (3.10), (2.11) and (2.20), we have

$$\begin{aligned}\|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}} &\leqslant \|U(t)\varphi_1(x)\|_{X^\delta_{1,\frac{1}{2}+\varepsilon}} + \left\| \int_0^t U(t-\tau)\big(v|v|^{2m}\big)(\tau)\,d\tau \right\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}} \\ &\leqslant C\|\varphi_1\|_{H^1(R^2)} + C\big\|v|v|^{2m}\big\|_{X^\delta_{1,-\frac{1}{2}+3\varepsilon'}}.\end{aligned} \tag{3.16}$$

Similarly as before we have

$$\begin{aligned}\big\|v|v|^{2m}\big\|_{X^\delta_{1,-\frac{1}{2}+3\varepsilon'}} &\leqslant C\big\|J^1_x\big(v|v|^{2m}\big)\big\|_{L^{r_\varepsilon}_{t,\delta}L^{r_\varepsilon}_x} \leqslant C\big\|J^1_x v\big\|_{L^4_{t,\delta}L^4_x}\|v\|^{2m}_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} \\ &\leqslant Cc^{\frac{2m}{4m+\varepsilon}}\|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}}.\end{aligned}$$

Substituting this estimates into (3.16) and using condition (3.9) we get

$$\|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}} \leqslant CN^{1-s} + Cc^{\frac{2m}{4m+\varepsilon}}\|v\|_{X^\delta_{1,\frac{1}{2}+\varepsilon'}}.$$

Hence, the second inequality in (3.13) also holds for sufficiently small c. This completes the proof. □

By interpolation (2.2), we see that under the conditions of Lemma 3.3, for any $0 \leqslant \mu \leqslant 1$ there holds:

$$\|v\|_{X^{\delta}_{\mu,\frac{1}{2}+\varepsilon'}} \leqslant CN^{\mu(1-s)}. \tag{3.17}$$

Next we consider the initial value problem (3.8). We assume that v in (3.8) is the solution of problem (3.7) with φ_1 satisfying condition (3.9). We also assume that φ_2 in (3.8) belongs to $H^s(R^2)$ and satisfies the condition (see (3.4)):

$$\|\varphi_2\|_{L^2(R^2)} \leqslant M_2 N^{-s}, \qquad \|\varphi_2\|_{H^s(R^2)} \leqslant M_2. \tag{3.18}$$

We denote

$$F(v,w) = (w+v)|w+v|^{2m} - v|v|^{2m}.$$

Then problem (3.8) can be rewritten as

$$\begin{cases} iw_t + \Delta w = F(v,w), & x \in R^2,\ t \in R, \\ w(x,0) = \varphi_2(x), & x \in R^2. \end{cases} \tag{3.19}$$

In the following we prove the existence of a solution w and make estimates on some norms of it.

Lemma 3.4. *Suppose φ_2 satisfies condition* (3.18) *and v is the solution of problem* (3.7) *with φ_1 satisfying condition* (3.9). *Suppose further that $s > \sqrt{(m-1)/m}$. Let δ, ε and ε' be as in Lemma* 3.3, *i.e., $\delta = cN^{-(4m-2+\varepsilon)(1-s)}$, ε is a sufficiently small positive number, and $\varepsilon' = \frac{\varepsilon}{3(4m+3\varepsilon)}$. Then there exists $N_0 \geqslant 1$ and $c_0 > 0$ such that for $N \geqslant N_0$ and $0 < c \leqslant c_0$, problem* (3.19) *has a unique solution w on* $[0,\delta] \times R^2$, *satisfying $w \in X^{\delta}_{s,\frac{1}{2}+\varepsilon'}$ and*

$$\|w\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} \leqslant CN^{-s}, \qquad \|w\|_{X^{\delta}_{s,\frac{1}{2}+\varepsilon'}} \leqslant C, \tag{3.20}$$

where C is a constant depending only on M_0, M_1 and M_2; C is independent of N, φ_1 and φ_2.

Proof. We shall prove this lemma by using the Banach fixed point theorem. For this purpose we rewrite (3.8) in integral form, namely,

$$w(t) = U(t)\varphi_2 - i\int_0^t U(t-\tau)F\big(v(\tau),w(\tau)\big)\,d\tau. \tag{3.21}$$

Let M be a positive number to be specified later. We denote

$$B^{\delta}_{M,N} = \Big\{w \in X^{\delta}_{s,\frac{1}{2}+\varepsilon'}\colon [w] \equiv N^s\|w\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} + \|w\|_{X^{\delta}_{s,\frac{1}{2}+\varepsilon'}} \leqslant M\Big\},$$

and define a mapping S as follows: For $w \in X^{\delta}_{s,\frac{1}{2}+\varepsilon'}$,

$$Sw(t) = U(t)\varphi_2 - i\int_0^t U(t-\tau)F\big(v(\tau),w(\tau)\big)\,d\tau.$$

In the sequel we prove that S is well-defined and it maps $B^{\delta}_{M,N}$ into itself.

By (2.11), (2.20) and condition (3.18), we have

$$\begin{aligned}\|Sw\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} &= \big\|U(t)\varphi_2\big\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} + \left\|\int_0^t U(t-\tau)F\big(v(\tau),w(\tau)\big)\,d\tau\right\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} \\ &\leqslant C\|\varphi_2\|_{L^2(R^2)} + C\big\|F(v,w)w\big\|_{X^{\delta}_{0,-\frac{1}{2}+3\varepsilon'}} \\ &\leqslant CN^{-s} + C\big\|F(v,w)\big\|_{X^{\delta}_{0,-\frac{1}{2}+3\varepsilon'}}. \end{aligned} \tag{3.22}$$

Clearly,

$$F(v,w) = g_1(v,w)w + g_2(v,w)\bar{w} \equiv F_1(v,w) + F_2(v,w),$$

where

$$g_1(v,w) = (m+1)\int_0^1 \big|\theta(v+w)+(1-\theta)v\big|^{2m}\,d\theta,$$

$$g_2(v,w) = m\int_0^1 \big|\theta(v+w)+(1-\theta)v\big|^{2m-2}\big(\theta(v+w)+(1-\theta)v\big)^2\,d\theta.$$

By (2.10) we have

$$\big\|F_1(v,w)\big\|_{X^{\delta}_{0,-\frac{1}{2}+3\varepsilon'}} \leqslant C\big\|F_1(v,w)\big\|_{L^{r_\varepsilon}_{t,\delta}L^{r_\varepsilon}_x} \leqslant c\|w\|_{L^4_{t,\delta}L^4_x}\big\|g_1(v,w)\big\|_{L^{\frac{4m+\varepsilon}{2m}}_{t,\delta}L_x^{\frac{4m+\varepsilon}{2m}}}$$

(see (3.15) for r_ε). Hence, by (2.8) and (3.11), we see that

$$\begin{aligned}\big\|F_1(v,w)\big\|_{X^{\delta}_{0,-\frac{1}{2}+3\varepsilon'}} &\leqslant C\|w\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}}\Big(\|w\|^{2m}_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} + \|v\|^{2m}_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x}\Big) \\ &\leqslant C\|w\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}}\Big(\|w\|^{2m}_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} + c^{\frac{2m}{4m+\varepsilon}}\Big).\end{aligned}$$

By the Sobolev embedding, embedding (2.4) and interpolation (2.2), we have

$$\begin{aligned}\|w\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} &\leqslant C\big\|J_x^{s_\varepsilon}w\big\|_{L^{4m+\varepsilon}_{t,\delta}L^{p_\varepsilon}_x} \leqslant C\big\|J_x^{s_\varepsilon}w\big\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}} \\ &\leqslant C\|w\|^{1-\frac{s_\varepsilon}{s}}_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}}\|w\|^{\frac{s_\varepsilon}{s}}_{X^{\delta}_{s,\frac{1}{2}+\varepsilon'}},\end{aligned} \tag{3.23}$$

where

$$s_\varepsilon = \frac{4(m-1)+\varepsilon}{4m+\varepsilon}, \qquad p_\varepsilon = \frac{2(4m+\varepsilon)}{2(2m-1)+\varepsilon}, \tag{3.24}$$

and the last inequality holds because we have $0 < s_\varepsilon < s$ for sufficiently small ε. Hence

$$\begin{aligned}\big\|F_1(v,w)\big\|_{X^{\delta}_{0,-\frac{1}{2}+3\varepsilon'}} &\leqslant C\|w\|_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}}\Big(\|w\|^{2m(1-\frac{s_\varepsilon}{s})}_{X^{\delta}_{0,\frac{1}{2}+\varepsilon'}}\|w\|^{\frac{2ms_\varepsilon}{s}}_{X^{\delta}_{s,\frac{1}{2}+\varepsilon'}} + c^{\frac{2m}{4m+\varepsilon}}\Big) \\ &\leqslant CN^{-s-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}N^{-s}[w].\end{aligned} \tag{3.25}$$

By a similar argument we see that a similar estimate also holds for $F_2(v,w)$. Hence

$$\|F(v,w)\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}} \leqslant CN^{-s-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}N^{-s}[w]. \tag{3.26}$$

Substituting this estimate into (3.22) we obtain

$$\|Sw\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} \leqslant CN^{-s} + CN^{-s-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}N^{-s}[w]. \tag{3.27}$$

Next, again by using (2.11), (2.20) and condition (3.18), we obtain

$$\begin{aligned}
\|Sw\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} &= \|U(t)\varphi_2\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} + \left\|\int_0^t U(t-\tau)F\big(v(\tau),w(\tau)\big)\,d\tau\right\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \\
&\leqslant C\|\varphi_2\|_{H^s(R^2)} + C\|F(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}} \\
&\leqslant C + C\|F_1(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}} + C\|F_2(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}}.
\end{aligned} \tag{3.28}$$

To estimate $\|F_1(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}}$, we use (2.8), (2.10), (3.11), (3.17), (3.23), the Leibnitz rule and the chain rule for fractional derivatives (see [6, §3] and [12, Appendix]) to deduce

$$\begin{aligned}
\|D_x^sF_1(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}} &\leqslant \|D_x^sF_1(v,w)\|_{L^{r_\varepsilon}_{t,\delta}L^{r_\varepsilon}_x} \quad \big(\text{see (3.15) for } r_\varepsilon\big) \\
&\leqslant C\big(\|w\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} + \|v\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x}\big)^{2m}\|D_x^sw\|_{L^4_{t,\delta}L^4_x} \\
&\quad + C\big(\|w\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} + \|v\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x}\big)^{2m-1} \\
&\quad \times \big(\|D_x^sw\|_{L^4_{t,\delta}L^4_x} + \|D_x^sv\|_{L^4_{t,\delta}L^4_x}\big)\|w\|_{L^{4m+\varepsilon}_{t,\delta}L^{4m+\varepsilon}_x} \\
&\leqslant C\Big(\|w\|^{1-\frac{s_\varepsilon}{s}}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\|w\|^{\frac{s_\varepsilon}{s}}_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} + c^{\frac{1}{4m+\varepsilon}}\Big)^{2m}\|w\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \\
&\quad + C\Big(\|w\|^{1-\frac{s_\varepsilon}{s}}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\|w\|^{\frac{s_\varepsilon}{s}}_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} + c^{\frac{1}{4m+\varepsilon}}\Big)^{2m-1} \\
&\quad \times \big(\|w\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} + \|v\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}}\big)\|w\|^{1-\frac{s_\varepsilon}{s}}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\|w\|^{\frac{s_\varepsilon}{s}}_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \\
&\leqslant CN^{-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}[w] \\
&\quad + CN^{s(1-s)-2m(s-s_\varepsilon)}[w]^{2m} + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{-(s-s_\varepsilon)}[w]^2 \\
&\quad + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{s(1-s)-(s-s_\varepsilon)}[w].
\end{aligned} \tag{3.29}$$

Note that, by the assumption $s > \sqrt{(m-1)/m}$, in expressions on the right-hand side of the last inequality all exponents of N are negative for sufficiently small ε. From (3.25) and (3.29), we see that

$$\begin{aligned}
&\|F_1(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}} \\
&\quad \leqslant C\|F_1(v,w)\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}} + C\|D_x^sF_1(v,w)\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}}
\end{aligned}$$

$$\leqslant CN^{-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}[w] + CN^{s(1-s)-2m(s-s_\varepsilon)}[w]^{2m} + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{-(s-s_\varepsilon)}[w]^2 + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{s(1-s)-(s-s_\varepsilon)}[w].$$

A similar argument shows that a similar estimate also holds for $\|F_2(v,w)\|_{X^\delta_{s,-\frac{1}{2}+3\varepsilon'}}$. Substituting these estimates into (3.28), we get

$$\|Sw\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \leqslant C + CN^{-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}[w] + CN^{s(1-s)-2m(s-s_\varepsilon)}[w]^{2m} + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{-(s-s_\varepsilon)}[w]^2 + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{s(1-s)-(s-s_\varepsilon)}[w]. \tag{3.30}$$

From (3.27) and (3.30), we get

$$\begin{aligned}[Sw] &= N^s\|Sw\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} + \|Sw\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \\ &\leqslant C + CN^{-2m(s-s_\varepsilon)}[w]^{2m+1} + Cc^{\frac{2m}{4m+\varepsilon}}[w] + CN^{s(1-s)-2m(s-s_\varepsilon)}[w]^{2m} \\ &\quad + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{-(s-s_\varepsilon)}[w]^2 + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{s(1-s)-(s-s_\varepsilon)}[w].\end{aligned} \tag{3.31}$$

Hence, the mapping S is well-defined. Moreover, since all exponents of N on the right-hand side of the last inequality are negative, by taking M sufficiently large (say, larger than the first C on the right-hand side of the last inequality), we can then correspondingly determine a $N_0 > 0$ sufficiently large and a $c_0 > 0$ sufficiently small such that for any $N \geqslant N_0$ and any $0 < c \leqslant c_0$, $[w] \leqslant M$ implies $[Sw] \leqslant M$, namely, S maps $B^\delta_{M,N}$ into itself.

Similarly, we can also prove that

$$\begin{aligned}[Sw_1 - Sw_2] &= N^s\|Sw_1 - Sw_2\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}} + \|Sw_1 - Sw_2\|_{X^\delta_{s,\frac{1}{2}+\varepsilon'}} \\ &\leqslant \big(CN^{-2m(s-s_\varepsilon)}\big([w_1]+[w_2]\big)^{2m} + Cc^{\frac{2m}{4m+\varepsilon}} \\ &\quad + CN^{s(1-s)-2m(s-s_\varepsilon)}\big([w_1]+[w_2]\big)^{2m-1} \\ &\quad + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{-(s-s_\varepsilon)}\big([w_1]+[w_2]\big) \\ &\quad + Cc^{\frac{2m-1}{4m+\varepsilon}}N^{s(1-s)-(s-s_\varepsilon)}\big)[w_1 - w_2].\end{aligned} \tag{3.32}$$

Hence if we take N_0 further large and c_0 further small then S is a contraction mapping of $B^\delta_{M,N}$ into itself. The desired result now follows immediately from Banach's fixed point theorem. □

Remark. One can easily verify that the condition $s > \frac{10m-6}{10m-5}$ of Theorem 1.1 implies the condition $s > \sqrt{(m-1)/m}$ of Lemma 3.4.

By interpolation (2.2) we see that under the conditions of Lemma 3.4, for any $0 \leqslant \mu \leqslant s$ there holds

$$\|w\|_{X^\delta_{\mu,\frac{1}{2}+\varepsilon'}} \leqslant CN^{\mu-s}. \tag{3.33}$$

Lemma 3.5. *Suppose the assumptions of Lemmas* 3.3 *and* 3.4 *are satisfied. Let v, w be solutions of problems* (3.7), (3.8) *and define*

$$z(t)=-i\int_0^t U(t-\tau)F\big(v(\tau),w(\tau)\big)\,d\tau. \tag{3.34}$$

Then the following estimates hold:

$$\|z\|_{L^\infty_{t,\delta}L^2_x}\leqslant CN^{-s}, \tag{3.35}$$

$$\|D_x z\|_{L^\infty_{t,\delta}L^2_x}\leqslant CN^{m-(m+\frac12)s+\sigma s+O(\varepsilon)}, \tag{3.36}$$

where σ is an arbitrary small positive number and C is a constant depending only on M_0, M_1, M_2 and σ; C is independent of N, φ_1 and φ_2.

Proof. The proof of (3.35) is easy. Indeed, by (2.4), (2.20), (3.20) and (3.26), we have

$$\|z\|_{L^\infty_{t,\delta}L^2_x}\leqslant C\|z\|_{X^\delta_{0,\frac12+\varepsilon'}}\leqslant C\big\|F(v,w)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}}\leqslant CN^{-s}.$$

In the sequel we give the proof of (3.36).

We first note that $F(v,w)$ can be split into a finite sum:

$$F(v,w)=G_0(v,w)+G_1(v,w)+\cdots+G_{2m}(v,w),$$

where $G_0(v,w)=G_0(w)=w|w|^{2m}$, $G_k(v,w)=P_j(w,\bar w)Q_k(v,\bar v)$, $j+k=2m+1$, $k=1,\dots,2m$, and P_j, Q_k are homogeneous polynomials of degrees j,k, respectively. Thus

$$\begin{aligned}\|D_x z\|_{L^\infty_{t,\delta}L^2_x}&\leqslant C\|D_x z\|_{X^\delta_{0,\frac12+\varepsilon'}}\leqslant C\big\|D_xF(v,w)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}}\\&\leqslant C\sum_{k=0}^{2m}\big\|D_xG_k(v,w)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}}.\end{aligned}\tag{3.37}$$

We now estimate $\|D_xG_k(v,w)\|_{X^\delta_{0,-\frac12+3\varepsilon'}}$ $(k=0,1,2,\dots,2m)$.

We first consider $1\leqslant k\leqslant 2m$. Since $G_k(v,w)=P_j(w,\bar w)Q_k(v,\bar v)$ $(j+k=2m+1)$ can be written as a finite sum of terms in the form $w^\alpha\bar w^\beta v^{\alpha'}\bar v^{\beta'}$, where α, β, α', β' are nonnegative integers and $\alpha+\beta=j$, $\alpha'+\beta'=k$, we only need to estimate $\|D_x(w^\alpha\bar w^\beta v^{\alpha'}\bar v^{\beta'})\|_{X^\delta_{0,-\frac12+3\varepsilon'}}$. Since $k\geqslant1$, we have either $\alpha'\geqslant1$ or $\beta'\geqslant1$. Suppose first $\alpha'\geqslant1$. By Lemma 2.6, for any $\sigma\in(0,\frac12)$ we have

$$\begin{aligned}\big\|D_x\big(w^\alpha\bar w^\beta v^{\alpha'}\bar v^{\beta'}\big)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}}&\leqslant C\big\|w^\alpha\bar w^\beta v^{\alpha'-1}\bar v^{\beta'}\big\|_{L^{h_{\varepsilon'}}_{t,\delta}L^{h_{\varepsilon'}}_x}\|v\|_{X^\delta_{1,\frac12+\varepsilon'}}\\&\quad+\big\|D_x^{\frac12+\sigma}\big(w^\alpha\bar w^\beta v^{\alpha'-1}\bar v^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x}\|v\|_{X^\delta_{\frac12-\sigma+\varepsilon,\frac12+\varepsilon'}}\\&\equiv I+II,\end{aligned}\tag{3.38}$$

where

$$h_{\varepsilon'}=\frac{2(1-2\times3\varepsilon')}{1-3\times3\varepsilon'}=\frac{2(1-6\varepsilon')}{1-9\varepsilon'}=\frac{4m+\varepsilon}{2m}.$$

In (3.38) we need $3\varepsilon'\in(0,\nu(\sigma))$ (see Lemma 2.6 for $\nu(\sigma)$). We now separately estimate I and II.

Recalling $\alpha+\beta=j$, $\alpha'+\beta'=k$ and $j+k=2m+1$, by the Hölder inequality we have, for $2\leqslant k\leqslant 2m-1$ (or $2\leqslant j\leqslant 2m-1$),

$$
\begin{aligned}
&\big\| w^{\alpha}\bar{w}^{\beta}v^{\alpha'-1}\bar{v}^{\beta'}\big\|_{L_{t,\delta}^{h_{\varepsilon'}}L_x^{h_{\varepsilon'}}} \\
&\quad = \big\| |w|^{j}|v|^{k-1}w\big\|_{L_{t,\delta}^{\frac{4m+\varepsilon}{2m}}L_x^{\frac{4m+\varepsilon}{2m}}} \leqslant C\|w\|^{j}_{L_{t,\delta}^{\frac{j(4m+\varepsilon)}{2m}}L_x^{\frac{j(4m+\varepsilon)}{2m-\varepsilon}}}\|v\|^{k-1}_{L_{t,\delta}^{\infty}L_x^{\frac{(k-1)(4m+\varepsilon)}{\varepsilon}}} \\
&\quad \leqslant C\big\| D_x^{\mu_\varepsilon}w\big\|^{j}_{L_{t,\delta}^{\frac{j(4m+\varepsilon)}{2m}}L_x^{\rho_\varepsilon}}\big\| D_x^{\nu_\varepsilon}v\big\|^{k-1}_{L_{t,\delta}^{\infty}L_x^{2}} \\
&\qquad \left(\text{by Sobolev embedding; } \mu_\varepsilon = \frac{j(4m+\varepsilon)-8m+2\varepsilon}{j(4m+\varepsilon)},\ \rho_\varepsilon = \frac{2j(4m+\varepsilon)}{j(4m+\varepsilon)-4m},\right. \\
&\qquad \left.\nu_\varepsilon = 1-\frac{2\varepsilon}{(k-1)(4m+\varepsilon)}\right) \\
&\quad \leqslant C\big\| D_x^{\mu_\varepsilon}w\big\|^{j}_{X^{\delta}_{0,\frac12+\varepsilon'}}\big\| D_x^{\nu_\varepsilon}v\big\|^{k-1}_{X^{\delta}_{0,\frac12+\varepsilon'}} \quad \big(\text{by (2.8)}\big) \\
&\quad \leqslant CN^{j(\mu_\varepsilon-s)}N^{\nu_\varepsilon(1-s)(k-1)} \quad \big(\text{by (3.17) and (3.33)}\big) \\
&\quad = CN^{2(m-1)-2ms+O(\varepsilon)},
\end{aligned}
$$

for $k=1$ (or $j=2m$),

$$
\begin{aligned}
\big\| w^{\alpha}\bar{w}^{\beta}v^{\alpha'-1}\bar{v}^{\beta'}\big\|_{L_{t,\delta}^{h_{\varepsilon'}}L_x^{h_{\varepsilon'}}} &= \big\| |w|^{2m}\big\|_{L_{t,\delta}^{\frac{4m+\varepsilon}{2m}}L_x^{\frac{4m+\varepsilon}{2m}}} = C\|w\|^{2m}_{L_{t,\delta}^{4m+\varepsilon}L_x^{4m+\varepsilon}} \\
&\leqslant C\|w\|^{2m(1-\frac{s_\varepsilon}{s})}_{X^{\delta}_{0,\frac12+\varepsilon'}}\|w\|^{\frac{2ms_\varepsilon}{s}}_{X^{\delta}_{s,\frac12+\varepsilon'}} \quad \big(\text{by (3.23)}\big) \\
&\leqslant CN^{-2m(s-s_\varepsilon)} \quad \big(\text{by (3.20)}\big) \\
&= CN^{2(m-1)-2ms+O(\varepsilon)},
\end{aligned}
$$

and for $k=2m$ (or $j=1$),

$$
\begin{aligned}
&\big\| w^{\alpha}\bar{w}^{\beta}v^{\alpha'-1}\bar{v}^{\beta'}\big\|_{L_{t,\delta}^{h_{\varepsilon'}}L_x^{h_{\varepsilon'}}} \\
&\quad = \big\| |w||v|^{2m-1}\big\|_{L_{t,\delta}^{\frac{4m+\varepsilon}{2m}}L_x^{\frac{4m+\varepsilon}{2m}}} \leqslant C\|w\|_{L_{t,\delta}^{\frac{4m+\varepsilon}{2m}}L_x^{\frac{4m+\varepsilon}{2m-\varepsilon}}}\|v\|^{2m-1}_{L_{t,\delta}^{\infty}L_x^{\frac{(2m-1)(4m+\varepsilon)}{\varepsilon}}} \\
&\quad \leqslant C\delta^{\frac{2m}{4m+\varepsilon}-\frac{3\varepsilon}{2(4m+\varepsilon)}}\|w\|_{L_{t,\delta}^{\frac{2(4m+\varepsilon)}{3\varepsilon}}L_x^{\frac{4m+\varepsilon}{2m-\varepsilon}}}\big\| D_x^{\nu_\varepsilon}\big\|^{2m-1}_{L_{t,\delta}^{\infty}L_x^{2}} \quad \left(\nu_\varepsilon = 1-\frac{2\varepsilon}{(2m-1)(4m+\varepsilon)}\right) \\
&\quad \leqslant C\delta^{\frac{4m-3\varepsilon}{2(4m+\varepsilon)}}\|w\|_{X^{\delta}_{0,\frac12+\varepsilon'}}\big\| D_x^{\nu_\varepsilon}v\big\|^{2m-1}_{X^{\delta}_{0,\frac12+\varepsilon'}} \\
&\quad \leqslant CN^{-(2m-1)(1-s)+O(\varepsilon)}\cdot N^{-s}\cdot N^{(2m-1)(1-s)+O(\varepsilon)} \\
&\quad = CN^{-s+O(\varepsilon)}.
\end{aligned}
$$

Thus, since $s > \sqrt{\frac{m-1}{m}} > \frac{2m-2}{2m-1}$ so that $\max\{2(m-1)-2ms, -s\} = -s$, we have

$$
I \leqslant C\big\| w^{\alpha}\bar{w}^{\beta}v^{\alpha'-1}\bar{v}^{\beta'}\big\|_{L_{t,\delta}^{h_{\varepsilon'}}L_x^{h_{\varepsilon'}}}\|v\|_{X^{\delta}_{1,\frac12+\varepsilon'}} \leqslant CN^{1-2s+O(\varepsilon)}. \tag{3.39}
$$

To estimate II we first use the Leibnitz rule for fractional derivatives (see [6, §3] and [12, Appendix]) to deduce, for $3\leqslant k\leqslant 2m-1$ (or $2\leqslant j\leqslant 2m-2$),

$$
\begin{aligned}
&\big\|D_x^{\frac{1}{2}+\sigma}\big(w^\alpha\bar{w}^\beta v^{\alpha'-1}\bar{v}^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x}\\
&\quad\leqslant C\Big(\|w\|^j_{L^{4j}_{t,\delta}L_x^{\frac{4j}{1-2\varepsilon}}}\|v\|^{k-2}_{L^\infty_{t,\delta}L_x^{\frac{2(k-2)}{\varepsilon}}}\big\|D_x^{\frac{1}{2}+\sigma}\big\|_{L^4_{t,\delta}L^4_x}\\
&\qquad+\|w\|^{j-1}_{L^{4(j-1)}_{t,\delta}L_x^{\frac{4(j-1)}{1-2\varepsilon}}}\|v\|^{k-1}_{L^\infty_{t,\delta}L_x^{\frac{2(k-1)}{\varepsilon}}}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{L^4_{t,\delta}L^4_x}\Big)\\
&\quad\leqslant C\big\|D_x^{\frac{j-1+\varepsilon}{j}}w\big\|^j_{L^{4j}_{t,\delta}L_x^{\frac{4j}{2j-1}}}\big\|D_x^{\frac{k-2-\varepsilon}{k-2}}v\big\|^{k-2}_{L^\infty_{t,\delta}L^2_x}\big\|D_x^{\frac{1}{2}+\sigma}v\big\|_{L^4_{t,\delta}L^4_x}\\
&\qquad+C\big\|D_x^{\frac{j-2+\varepsilon}{j-1}}w\big\|^{j-1}_{L^{4(j-1)}_{t,\delta}L_x^{\frac{4(j-1)}{2j-3}}}\big\|D_x^{\frac{k-1-\varepsilon}{k-1}}v\big\|^{k-1}_{L^\infty_{t,\delta}L^2_x}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{L^4_{t,\delta}L^4_x}\\
&\quad\leqslant C\big\|D_x^{\frac{j-1+\varepsilon}{j}}w\big\|^j_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{k-2-\varepsilon}{k-2}}v\big\|^{k-2}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{1}{2}+\sigma}v\big\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\\
&\qquad+C\big\|D_x^{\frac{j-2+\varepsilon}{j-1}}w\big\|^{j-1}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{k-1-\varepsilon}{k-1}}v\big\|^{k-1}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\\
&\quad\leqslant CN^{(\frac{j-1}{j}-s)j+O(\varepsilon)}\cdot N^{(1-s)(k-2)+O(\varepsilon)}\cdot N^{(\frac{1}{2}+\sigma)(1-s)}\\
&\qquad+CN^{(\frac{j-2}{j-1}-s)(j-1)+O(\varepsilon)}\cdot N^{(1-s)(k-1)+O(\varepsilon)}\cdot N^{\frac{1}{2}+\sigma-s}\\
&\quad\leqslant CN^{2m-\frac{3}{2}-(2m-\frac{1}{2})s+\sigma(1-s)+O(\varepsilon)}+CN^{2m-\frac{3}{2}-2ms+\sigma+O(\varepsilon)}\\
&\quad\leqslant CN^{2m-\frac{3}{2}-(2m-\frac{1}{2})s+\sigma+O(\varepsilon)},
\end{aligned}
$$

for $k=2$ (or $j=2m-1$),

$$
\begin{aligned}
&\big\|D_x^{\frac{1}{2}+\sigma}\big(w^\alpha\bar{w}^\beta v^{\alpha'-1}\bar{v}^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x}\\
&\quad\leqslant C\big(\|w\|^{2m-1}_{L^{8m-4}_{t,\delta}L^{8m-4}_x}\big\|D_x^{\frac{1}{2}+\sigma}v\big\|_{L^4_{t,\delta}L^4_x}+\|w\|^{2m-2}_{L^{8(m-1)}_{t,\delta}L_x^{\frac{8(m-1)}{1-2\varepsilon}}}\|v\|_{L^\infty_{t,\delta}L_x^{\frac{2}{\varepsilon}}}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{L^4_{t,\delta}L^4_x}\big)\\
&\quad\leqslant C\big\|D_x^{\frac{2m-2}{2m-1}}w\big\|^{2m-1}_{L^{8m-4}_{t,\delta}L_x^{\frac{8m-4}{4m-3}}}\big\|D_x^{\frac{1}{2}+\sigma}v\big\|_{L^4_{t,\delta}L^4_x}\\
&\qquad+C\big\|D_x^{\frac{2m-3+\varepsilon}{2(m-1)}}w\big\|^{2(m-1)}_{L^{8(m-1)}_{t,\delta}L_x^{\frac{8(m-1)}{4m-5}}}\big\|D_x^{1-\varepsilon}v\big\|_{L^\infty_{t,\delta}L^2_x}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{L^4_{t,\delta}L^4_x}\\
&\quad\leqslant C\big\|D_x^{\frac{2m-2}{2m-1}}w\big\|^{2m-1}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{1}{2}+\sigma}v\big\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\\
&\qquad+C\big\|D_x^{\frac{2m-3+\varepsilon}{2(m-1)}}w\big\|^{2(m-1)}_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{1-\varepsilon}v\big\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\big\|D_x^{\frac{1}{2}+\sigma}w\big\|_{X^\delta_{0,\frac{1}{2}+\varepsilon'}}\\
&\quad\leqslant CN^{2m-\frac{3}{2}-(2m-\frac{1}{2})s+\sigma(1-s)}+CN^{2m-\frac{3}{2}-2ms+\sigma+O(\varepsilon)}\\
&\quad\leqslant CN^{2m-\frac{3}{2}-(2m-\frac{1}{2})s+\sigma+O(\varepsilon)},
\end{aligned}
$$

for $k=1$ (or $j=2m$),

$$\begin{aligned}\big\| D_x^{\frac{1}{2}+\sigma}\big(w^\alpha \bar{w}^\beta v^{\alpha'-1}\bar{v}^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x} &= \big\| D_x^{\frac{1}{2}+\sigma}\big(w^\alpha \bar{w}^\beta\big)\big\|_{L^2_{t,\delta}L^2_x}\\ &\leqslant C\|w\|^{2m-1}_{L^{8m-4}_{t,\delta}L^{8m-4}_x}\big\|D_x^{\frac{1}{2}+\sigma} w\big\|_{L^4_{t,\delta}L^4_x}\\ &\leqslant C\big\|D_x^{\frac{2m-2}{2m-1}} w\big\|^{2m-1}_{L^{8m-4}_{t,\delta}L_x^{\frac{8m-4}{4m-3}}}\big\|D_x^{\frac{1}{2}+\sigma} w\big\|_{L^4_{t,\delta}L^4_x}\\ &\leqslant CN^{2m-2-(2m-1)s}\cdot N^{\frac{1}{2}+\sigma-s} = CN^{2m-\frac{3}{2}-2ms+\sigma}\end{aligned}$$

and for $k=2m$ (or $j=1$),

$$\begin{aligned}&\big\| D_x^{\frac{1}{2}+\sigma}\big(w^\alpha \bar{w}^\beta v^{\alpha'-1}\bar{v}^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x}\\ &\quad\leqslant C\|w\|_{L^4_{t,\delta}L_x^{\frac{4}{1-\varepsilon}}}\|v\|^{2m-2}_{L^\infty_{t,\delta}L_x^{\frac{8(m-1)}{\varepsilon}}}\big\|D_x^{\frac{1}{2}+\sigma} v\big\|_{L^4_{t,\delta}L^4_x} + C\|v\|^{2m-1}_{L^\infty_{t,\delta}L^{8m-4}_x}Q\big\|D_x^{\frac{1}{2}+\sigma} w\big\|_{L^2_{t,\delta}L^4_x}\\ &\quad\leqslant C\big\|D_x^{\frac{\varepsilon}{2}} w\big\|_{L^4_{t,\delta}L^4_x}\big\|D_x^{1-\frac{\varepsilon}{4(m-1)}} v\big\|^{2m-2}_{L^\infty_{t,\delta}L^2_x}\big\|D_x^{\frac{1}{2}+\sigma} v\big\|_{L^4_{t,\delta}L^4_x}\\ &\qquad + C\big\|D_x^{\frac{4m-3}{4m-2}} v\big\|^{2m-1}_{L^\infty_{t,\delta}L^2_x}\cdot\delta^{\frac{1}{4}}\big\|D_x^{\frac{1}{2}+\sigma} w\big\|_{L^4_{t,\delta}L^4_x}\\ &\quad\leqslant CN^{\frac{\varepsilon}{2}-s}\cdot N^{(2m-2)(1-s)+O(\varepsilon)}\cdot N^{(\frac{1}{2}+\sigma)(1-s)}\\ &\qquad + CN^{(2m-1)\cdot\frac{4m-3}{4m-2}(1-s)}\cdot N^{-(m-\frac{1}{2})(1-s)+O(\varepsilon)}\cdot N^{\frac{1}{2}+\sigma-s}\\ &\quad\leqslant CN^{2m-\frac{3}{2}-(2m-\frac{1}{2})s+\sigma(1-s)+O(\varepsilon)} + CN^{m-\frac{1}{2}-ms+\sigma+O(\varepsilon)}\\ &\quad\leqslant CN^{m-\frac{1}{2}-ms+\sigma+O(\varepsilon)}.\end{aligned}$$

In getting the last inequality we used the condition $s>\sqrt{\frac{m-1}{m}}>\frac{2m-2}{2m-1}$. Thus

$$II\leqslant\big\| D_x^{\frac{1}{2}+\sigma}\big(w^\alpha \bar{w}^\beta v^{\alpha'-1}\bar{v}^{\beta'}\big)\big\|_{L^2_{t,\delta}L^2_x}\|v\|_{X^\delta_{\frac{1}{2}-\sigma+\varepsilon,\frac{1}{2}+\varepsilon'}}\leqslant CN^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)}. \tag{3.40}$$

To treat the case $\beta'\geqslant 1$ we only need to interchange the role of v and $\bar{v}$, and the result is the same. Since the bound for I does not exceed the bound for II, we thus have proved that for $1\leqslant k\leqslant 2m$ there holds

$$\big\|D_xG_k(v,w)\big\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}}\leqslant CN^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)}. \tag{3.41}$$

Next we consider $\|D_xG_0(w)\|_{X^\delta_{0,-\frac{1}{2}+3\varepsilon'}}$. We need the following identity:

$$\partial_j\big(|u|^{2m}u\big)=\frac{m+1}{2m}u\partial_j\big(|u|^{2m}\big)+\frac{1}{2}\bar{u}\partial_j\big(u^2|u|^{2(m-1)}\big) \tag{3.42}$$

(for smooth u), which is proved as follows: Since

$$\partial_j\big(|u|^{2m}u\big)=\partial_j\big(|u|^{2m}\big)u+|u|^{2m}\partial_ju$$

and

$$\begin{aligned}|u|^{2m}\partial_ju&=u|u|^{2(m-1)}\bar{u}\partial_ju=u|u|^{2(m-1)}\partial_j\big(|u|^2\big)-u^2|u|^{2(m-1)}\partial_j\bar{u}\\ &=\frac{1}{m}u\partial_j\big(|u|^{2m}\big)-\partial_j\big(|u|^{2m}u\big)+\partial_j\big(u^2|u|^{2(m-1)}\big)\bar{u},\end{aligned}$$

we see that (3.42) follows. By (3.42) we see that for $\psi, w \in C^\infty([0,\delta], C_0^\infty(R^2))$ there hold

$$\begin{aligned}
&\langle \psi, D_x G_0(w)\rangle \\
&\quad = i\sum_{j=1}^{2}\langle \psi, R_j\partial_j\big(|w|^{2m}w\big)\rangle \\
&\quad = \frac{m+1}{2m} i\sum_{j=1}^{2}\langle R_j\psi, w\partial_j\big(|w|^{2m}\big)\rangle + \frac{1}{2} i\sum_{j=1}^{2}\langle R_j\psi, \bar w\partial_j\big(w^2|w|^{2(m-1)}\big)\rangle \\
&\quad = \frac{m+1}{2m}\sum_{j=1}^{2}\langle \bar w R_j\psi, D_x R_j\big(|w|^{2m}\big)\rangle + \frac{1}{2}\sum_{j=1}^{2}\langle w R_j\psi, D_x R_j\big(w^2|w|^{2(m-1)}\big)\rangle \\
&\quad = \frac{m+1}{2m}\sum_{j=1}^{2}\langle D_x^{\frac12-\sigma}(\bar w R_j\psi), D_x^{\frac12+\sigma}R_j\big(|w|^{2m}\big)\rangle \\
&\qquad + \frac{1}{2}\sum_{j=1}^{2}\langle D_x^{\frac12-\sigma}(w R_j\psi), D_x^{\frac12+\sigma}R_j\big(w^2|w|^{2(m-1)}\big)\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
\big|\langle \psi, D_x G_0(w)\rangle\big| \leqslant{}& C\sum_{j=1}^{2}\big\|D_x^{\frac12-\sigma}(\bar w R_j\psi)\big\|_{L^2_{t,\delta}L^2_x}\big\|D_x^{\frac12+\sigma}R_j\big(|w|^{2m}\big)\big\|_{L^2_{t,\delta}L^2_x} \\
&+ C\sum_{j=1}^{2}\big\|D_x^{\frac12-\sigma}(w R_j\psi)\big\|_{L^2_{t,\delta}L^2_x}\big\|D_x^{\frac12+\sigma}R_j\big(w^2|w|^{2(m-1)}\big)\big\|_{L^2_{t,\delta}L^2_x}.
\end{aligned}$$

By Lemma 2.5 we have

$$\begin{aligned}
\big\|D_x^{\frac12-\sigma}(\bar w R_j\psi)\big\|_{L^2_{t,\delta}L^2_x} &\leqslant C\|\bar w\|_{X^\delta_{\frac12-\sigma+\varepsilon,\frac12+\varepsilon'}}\|R_j\psi\|_{X^\delta_{0,\frac12-3\varepsilon'}} \\
&\leqslant C\|w\|_{X^\delta_{\frac12-\sigma+\varepsilon,\frac12+\varepsilon'}}\|\psi\|_{X^\delta_{0,\frac12-3\varepsilon'}},
\end{aligned}$$

and similarly for $\|D_x^{\frac12-\sigma}(\bar w R_j\psi)\|_{L^2_{t,\delta}L^2_x}$. Thus by the dual relation (2.1) we obtain

$$\begin{aligned}
\big\|D_x G_0(w)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}} \leqslant{}& C\sum_{j=1}^{2}\|w\|_{X^\delta_{\frac12-\sigma+\varepsilon,\frac12+\varepsilon'}}\big\|D_x^{\frac12+\sigma}\big(|w|^{2m}\big)\big\|_{L^2_{t,\delta}L^2_x} \\
&+ C\sum_{j=1}^{2}\|w\|_{X^\delta_{\frac12-\sigma+\varepsilon,\frac12+\varepsilon'}}\big\|D_x^{\frac12+\sigma}\big(w^2|w|^{2(m-1)}\big)\big\|_{L^2_{t,\delta}L^2_x}.
\end{aligned}$$

Using this inequality, we can deduce similarly as before to get an estimate for $\|D_x G_0(w)\|_{X^\delta_{0,-\frac12+3\varepsilon'}}$. We write out the result:

$$\big\|D_x G_0(w)\big\|_{X^\delta_{0,-\frac12+3\varepsilon'}} \leqslant CN^{2m-1-(2m+\frac12)s+O(\varepsilon)}. \tag{3.43}$$

Substituting (3.41) and (3.43) into (3.37) we obtain

$$\|D_x z\|_{L^\infty_{t,\delta} L^2_x} \leqslant C N^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)}.$$

Thus the desired result is verified. This completes the proof. □

4. The proof of Theorem 1.1

Assume that $\frac{10m-6}{10m-5} < s < 1$. Given $\varphi \in H^s(R^2)$, let

$$M_0 = 2C\|\varphi\|_{L^2(R^2)}, \qquad M_1 = 2C\|\varphi\|^2_{H^s(R^2)}\big(1+\|\varphi\|^{2m}_{H^s(R^2)}\big), \qquad M_2 = C\|\varphi\|_{H^s(R^2)},$$

where C is the constant in (3.3) and (3.4). Let N be a sufficiently large positive number to be specified later, and let $\varphi = \varphi_1 + \varphi_2$ be the decomposition of φ ensured by Lemma 3.1. Then by (3.3) and (3.4) we have

$$E(\varphi_1) \leqslant \frac{1}{2} M_0, \qquad H(\varphi_1) \leqslant \frac{1}{2} M_1 N^{1-s}, \tag{4.1}$$

$$\|\varphi_2\|_{L^2(R^2)} \leqslant M_2 N^{-s}, \qquad \|\varphi_2\|_{H^s(R^2)} \leqslant M_2. \tag{4.2}$$

Thus conditions (3.9) and (3.18) are satisfied. It follows from Lemmas 3.2–3.5 that there exist $c_0 > 0$ and $N_0 > 0$ such that for any $0 < c \leqslant c_0$, $N \geqslant N_0$ and sufficiently small $\varepsilon > 0$, if we take $\delta = cN^{-(4m-2+\varepsilon)(1-s)}$, then problem (3.8) has a unique solution w in the region $[0,\delta] \times R^2$. It follows that problem (1.1)–(1.2) has a unique solution u in the same region, and

$$u(t) = v(t) + w(t). \tag{4.3}$$

From the proof of Lemma 3.4 we know that w can be split into

$$w(t) = U(t)\varphi_2 + z(t),$$

where, by Lemma 3.5, z satisfies

$$\|z\|_{L^\infty_{t,\delta} L^2_x} \leqslant C N^{-s}, \qquad \|D_x z\|_{L^\infty_{t,\delta} L^2_x} \leqslant C N^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)}, \tag{4.4}$$

where σ is an arbitrary positive number in $(0, \frac{1}{2})$ and C is a constant depending only on M_0, M_1, M_2 and σ; C is independent of either N or specific decomposition of φ. Later we shall specify σ. Now shrift the initial time from $t = 0$ to $t = \delta$, and put

$$\varphi_1(\delta) = v(\delta) + z(\delta), \qquad \varphi_2(\delta) = U(\delta)\varphi_2.$$

By the conservation laws in (3.1) we have

$$E\big(v(\delta)\big) = E(\varphi_1) \leqslant \frac{1}{2} M_0, \tag{4.5}$$

and

$$H\big(v(\delta)\big) = H(\varphi_1) \leqslant \frac{1}{2} M_1 N^{2(1-s)}. \tag{4.6}$$

By (4.1), (4.4) and (4.5), it is easy to see that

$$\big|E\big(\varphi_1(\delta)\big) - E\big(v(\delta)\big)\big| \leqslant \big\|v(\delta)\big\|_{L^2(R^2)} \big\|z(\delta)\big\|_{L^2(R^2)} + \frac{1}{2}\big\|z(\delta)\big\|^2_{L^2(R^2)} \leqslant C N^{-s}$$

(for large N), so that

$$E\big(\varphi_1(\delta)\big) \leqslant E\big(v(\delta)\big) + CN^{-s} \leqslant \frac{1}{2}M_0\big(1 + CN^{-s}\big). \tag{4.7}$$

Similarly, we have also

$$\begin{aligned}
\big|H\big(\varphi_1(\delta)\big) - H(\varphi_1)\big| &\leqslant \big|H\big(\varphi(\delta) + z(\delta)\big) - H\big(\varphi(\delta)\big)\big| \\
&\leqslant C\Big(\big\|\big(\big|\nabla v(\delta)\big| + \big|\nabla z(\delta)\big|\big)\big|\nabla z(\delta)\big|\big\|_{L^1} \\
&\quad + \big\|\big(\big|v(\delta)\big| + \big|z(\delta)\big|\big)^{2m+1}\big|z(\delta)\big|\big\|_{L^1}\Big) \\
&\leqslant C\Big(\big\|v(\delta)\big\|_{H^1}\big\|z(\delta)\big\|_{H^1} + \big\|z(\delta)\big\|_{H^1}^2 \\
&\quad + \big\|v(\delta)\big\|_{L^{2(2m+1)}}^{2m+1}\big\|z(\delta)\big\|_{L^2} + \big\|z(\delta)\big\|_{L^{2m+2}}^{2m+2}\Big) \\
&\leqslant C\Big(\big\|v(\delta)\big\|_{H^1}\big\|z(\delta)\big\|_{H^1} + \big\|z(\delta)\big\|_{H^1}^2 + \big\|v(\delta)\big\|_{L^2}\big\|v(\delta)\big\|_{H^1}^{2m}\big\|z(\delta)\big\|_{L^2} \\
&\quad + \big\|z(\delta)\big\|_{L^2}^2\big\|z(\delta)W\big\|_{H^1}^{2m}\Big) \\
&\leqslant CN^{1-s}N^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)} + CN^{2(m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon))} \\
&\quad + CN^{2m(1-s)}N^{-s} + CN^{-2s}N^{2m(m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon))} \\
&\leqslant CN^{m+1-(m+\frac{3}{2})s+\sigma+O(\varepsilon)}
\end{aligned}$$

(for large N), which implies that

$$\begin{aligned}
H\big(\varphi_1(\delta)\big) &\leqslant H(\varphi_1) + CN^{m+1-(m+\frac{3}{2})s+\sigma+O(\varepsilon)} \\
&\leqslant \frac{1}{2}M_1N^{2(1-s)}\big(1 + CN^{m-1-(m-\frac{1}{2})s+\sigma+O(\varepsilon)}\big).
\end{aligned} \tag{4.8}$$

Since $s > \frac{10m-6}{10m-5} > \frac{2m-2}{2m-1}$, we have $m-1-(m-\frac{1}{2})s < 0$ and also $m-1-(m-\frac{1}{2})s+\sigma < 0$ for sufficiently small σ. We now choose a $\sigma > 0$ sufficiently small (depending only on s) such that the last condition is satisfied and fix it, and then take $\varepsilon > 0$ sufficiently small (depending only on s and σ) such that both the conditions $3\varepsilon' \in (0, \nu(\sigma))$ (required by Lemma 2.6) and $m-1-(m-\frac{1}{2})s+\sigma+O(\varepsilon) < 0$ hold, where $\varepsilon' = \frac{\varepsilon}{3(4m+3\varepsilon)}$. With σ and ε specified in this way (but see the argument following (4.11) below), we now take $N \geqslant N_0$ large enough such that

$$CN^{m-1-(m-\frac{1}{2})s+\sigma+O(\varepsilon)} \leqslant 1,$$

where C is the constant on the right-hand side of the second inequality in (4.8), which is now specific because σ and ε have been specified (but see the argument following (4.11) below). Then, by (4.8), $\varphi_1(\delta)$ still satisfies condition (3.9). Besides, since $U(t)$ ($t \in R$) is a unitary group on each $H^s(R^2)$, we have

$$\big\|\varphi_2(\delta)\big\|_{L^2(R^2)} = \|\varphi_2\|_{L^2(R^2)}, \qquad \big\|\varphi_2(\delta)\big\|_{H^s(R^2)} = \|\varphi_2\|_{H^s(R^2)},$$

so that, by (4.2),

$$\big\|\varphi_2(\delta)\big\|_{L^2(R^2)} \leqslant M_2N^{-s}, \qquad \big\|\varphi_2(\delta)\big\|_{H^s(R^2)} \leqslant M_2. \tag{4.9}$$

Hence $\varphi_2(\delta)$ still satisfies condition (3.18). It follows that when φ_1 and φ_2 in (3.7) and (3.8) are respectively replaced with $\varphi_1(\delta)$ and $\varphi_2(\delta)$, all arguments in the proofs of Lemmas 3.2–3.5 still hold, and we thus get a new pair of solutions v, w of those problems corresponding to new initial values. Moreover, since the estimates ensured by Lemmas 3.2–3.5 depend only on conditions (3.9) and (3.18), this new pair of solutions also exist on the common region $[0,\delta]\times R^2$ with the same δ as for old pair, and they also enjoy all the same estimates as for the old pair. Since $u(\delta)=\varphi_1(\delta)+\varphi_2(\delta)$, by putting $u(t)$ to be the sum of $v(t-\delta)$ and $w(t-\delta)$ for $\delta\leqslant t\leqslant 2\delta$, where v and w now refer to the new pair of solutions of (3.7) and (3.8), we see that the solution u of problem (1.1)–(1.2) is extended to the region $[0,2\delta]\times R^2$.

The above argument can be repeated for k steps, where k is the maximal integer satisfying

$$k\cdot CN^{-s}\leqslant 1\quad\text{and}\quad k\cdot CN^{m-1-(m-\frac{1}{2})s+\sigma+O(\varepsilon)}\leqslant 1, \tag{4.10}$$

where C is the constant appearing in (4.7) and (4.8). From (4.10) we see that

$$k=C\min\bigl\{N^s,N^{(m-\frac{1}{2})s-(m-1)-\sigma-O(\varepsilon)}\bigr\}=CN^{(m-\frac{1}{2})s-(m-1)-\sigma-O(\varepsilon)}.$$

Accordingly, the maximal extended time interval is equal to $[0,k\delta]$, and

$$\begin{aligned}k\delta&=CN^{(m-\frac{1}{2})s-(m-1)-\sigma-O(\varepsilon)}\cdot cN^{-(4m-2-\varepsilon)(1-s)}\\&=CN^{(5m-\frac{5}{2})s-(5m-3)-\sigma-O(\varepsilon)}.\end{aligned}\tag{4.11}$$

Since $s>\frac{10m-6}{10m-5}$, we see that the exponent of N in the last expression is positive if we replace σ and ε with smaller numbers (depending only on s) when necessary, so that $k\delta\to\infty$ as $N\to\infty$. Hence, for any given $T>0$ we can find a corresponding $N\geqslant N_0$ sufficiently large such that $k\delta\geqslant T$ and, as a result, the solution of problem (1.1)–(1.2) can be extended to the region $[0,T]\times R^2$.

To complete the proof of Theorem 1.1 we still need to verify (1.6) and (1.7). To this end for every integer $1\leqslant n\leqslant k$ we respectively denote by $v_n(t)$, $w_n(t)$, $z_n(t)$, φ_{n1} and φ_{n2} the corresponding functions $v(t)$, $w(t)$, $z(t)$, φ_1 and φ_2 obtained at the nth step (so that $v_n(t)$, $w_n(t)$, $z_n(t)$ are defined on the interval $[0,\delta]$, and $\varphi_{n1}=v_{n-1}(\delta)+z_{n-1}(\delta)$, $\varphi_{n2}=U((n-1)\delta)\varphi_2=e^{i(n-1)\delta\Delta}\varphi_2$ for $2\leqslant n\leqslant k$). We claim that

$$\begin{aligned}&u(t)=e^{it\Delta}\varphi+v_n\bigl(t-(n-1)\delta\bigr)+z_n\bigl(t-(n-1)\delta\bigr)-e^{it\Delta}\varphi_1,\\&\quad\text{for } t\in\bigl[(n-1)\delta,n\delta\bigr]\end{aligned}\tag{4.12}$$

$(n=1,2,\dots,k)$. Indeed, since $w_n(t)=U(t)\varphi_{n2}+z_n(t)$ and $U(t)=e^{it\Delta}$ $(t\in R)$ is a group on $H^s(R^2)$, we have

$$\begin{aligned}u(t)&=v_n\bigl(t-(n-1)\delta\bigr)+w_n\bigl(t-(n-1)\delta\bigr)\\&=v_n\bigl(t-(n-1)\delta\bigr)+\bigl(U\bigl(t-(n-1)\delta\bigr)\varphi_{n2}+z_n\bigl(t-(n-1)\delta\bigr)\bigr)\\&=v_n\bigl(t-(n-1)\delta\bigr)+z_n\bigl(t-(n-1)\delta\bigr)+e^{i(t-(n-1)\delta)\Delta}e^{i(n-1)\delta\Delta}\varphi_2\\&=v_n\bigl(t-(n-1)\delta\bigr)+z_n\bigl(t-(n-1)\delta\bigr)+e^{it\Delta}\varphi_2,\end{aligned}$$

and (4.12) immediately follows by replacing φ_2 with $\varphi-\varphi_1$. By (4.12) we see that, in particular,

$$u(t)=e^{it\Delta}\varphi+\bigl(v_k\bigl(t-(k-1)\delta\bigr)+z_k\bigl(t-(k-1)\delta\bigr)-e^{it\Delta}\varphi_1\bigr)\equiv e^{it\Delta}\varphi+y(t)$$

for $(k-1)\delta\leqslant t\leqslant k\delta$. Hence, for any $t\geqslant 1$ by taking $T=t$ and then choosing N so large that $k\delta\geqslant T$ but $(k-1)\delta<T$, or $N\sim CT^{\frac{2}{(10m-5)s-(10m-6)}+O(\sigma)+O(\varepsilon)}$ (by (4.11)), we get, by Lemmas 3.1, 3.3 and 3.4, that

$$\begin{aligned}\|y(t)\|_{H^1(R^n)} &\leqslant \|v_k(t-(k-1)\delta)\|_{H^1(R^2)} + \|z_k(t-(k-1)\delta)\|_{H^1(R^2)} + \|e^{it\Delta}\varphi_1\|_{H^1(R^2)}\\ &\leqslant CN^{1-s} + CN^{m-(m+\frac{1}{2})s+\sigma s+O(\varepsilon)} + CN^{1-s}\\ &\leqslant CN^{1-s} \leqslant CT^{\frac{2(1-s)}{(10m-5)s-(10m-6)}+\epsilon},\end{aligned}$$

where $\epsilon = O(\sigma) + O(\varepsilon)$. Since σ and ε can be taken arbitrarily small, ϵ can also be taken arbitrarily small. This ends the proof of Theorem 1.1.

5. Conclusions

We have shown that, in the case $n = 2$, problem (1.1)–(1.2) has a global solution in $C(R, H^s(R^2))$ if the initial data $\varphi \in H^s(R^2)$ and $\frac{10m-6}{10m-5} < s < 1$. Moreover, the solution has the property that $u(t) - e^{it\Delta}\varphi \in C(R, H^1(R^2))$, and $\|u(t) - e^{it\Delta}\varphi\|_{H^1}$ does not grow faster than $(1+|t|)^{\kappa_m}$ as $|t| \to \infty$, where κ_m is a positive constant. The proof uses Bourgain's high–low frequency decomposition (HLFD) method. In this proof a new Leibnitz rule owing to Bourgain (see Lemma 2.5) plays a fundamental role. Besides, the proof is also based on the fact that Eq. (1.1) conserves both L^2 and H^1 norms of the solution. From the rigorous analysis performed in this paper we see that this property of the equation is essential for the application of the HLFD approach. Thus, if an equation does have a similar property, then the HLFD approach does not work to it.

The condition $\frac{10m-6}{10m-5} < s < 1$ is not optimal, even by the approach employed in this paper. It is possible to weaken this condition by making more careful estimates than those given by Lemmas 3.2–3.4, or by choosing more carefully the exponent of N in the expression of δ. However, this will not lead us to much progress. It seems that high–low frequency decomposition method cannot lead us to the final solution of the conjecture that (1.1)–(1.2) is globally well-posed in $H^s(R^n)$ for any $s > s_0$, where s_0 is the lowest index for local well-posedness. To get the final solution, new approaches must be developed.

Finally, we would like to point out that the analysis made in this paper can surely be extended to the other dimension case. In order to do so, a similar Leibnitz rule as that given in Lemma 2.5 should be first established.

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