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On the existence of periodic solutions for a class of p-Laplacian system

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Abstract

By using generalized Borsuk theorem in coincidence degree theory, some criteria to guarantee the existence of ω-periodic solutions for a class of p-Laplacian system are derived.

Keywords: Periodic solutions; p-Laplacian; Coincidence degree

1. Introduction

Throughout this paper, $1 < p < \infty$ is a fixed real number. The conjugate exponent of p is denoted by q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let $\phi_p : \mathbf{R}^n \to \mathbf{R}^n$ be the mapping defined by

$$\phi_p(u) = \phi_p(u_1, \ldots, u_n) := \left(|u_1|^{p-2}u_1, \ldots, |u_n|^{p-2}u_n\right)^{\mathrm{T}}.$$

Then ϕ_p is a homeomorphism of $\mathbf{R}^n$ with the inverse ϕ_q.

In this paper, we will consider the existence of periodic solutions of the following system:

$$\left(\phi_p\left(u'(t)\right)\right)' + \frac{d}{dt}\operatorname{grad} F\left(u(t)\right) + \operatorname{grad} G\left(u(t)\right) = e(t), \tag{1.1}$$

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where $F \in C^2(\mathbf{R}^n, \mathbf{R})$, $G \in C^1(\mathbf{R}^n, \mathbf{R})$, $e \in C(\mathbf{R}, \mathbf{R}^n)$, $e(t+\omega) \equiv e(t)$ for all $t \in \mathbf{R}$, $\omega > 0$ is fixed.

In recent years, the existence of periodic solutions of (1.1) for $p = 2$ has been extensively studied (see [1–7]). In [5], by using Krasnoselskii's fixed point theorem, Ding proved the following result.

Theorem A. (See [5].) *Suppose that there exist constants $c_0 > 0$, $a_0 > 0$, $a_1 > 0$, $b_0 \geqslant 0$, $b_1 \geqslant 0$, and $\alpha > 1$, such that*

(a) $y^{\mathrm{T}} \frac{\partial^2 F(x)}{\partial x^2} y \geqslant c_0 |y|_2^2$, $\forall x,\ y \in \mathbf{R}^n$,
(b) $G(x) \geqslant 0$ *and* $G(x) \geqslant a_0 |x|_2^{\alpha} - b_0$, $\forall x \in \mathbf{R}^n$,
(c) $x^{\mathrm{T}} \operatorname{grad} G(x) \geqslant a_1 G(x) - b_1$, $\forall x \in \mathbf{R}^n$.

Then (1.1) *has at least one ω-periodic solution for $p = 2$.*

Many results were also given by using topological degree theory; see, for example, [1–4,6,7] and the references therein. Some researchers discussed the existence of periodic solutions to scalar p-Laplacian differential equations in [8–11,13,14]. But the existence of periodic solutions of (1.1) for $p \neq 2$ and $n > 1$, as far as we know, has rarely been studied. For general differential systems of p-Laplacian type, M.R. Zhang [12] has considered the Dirichlet boundary value problems

$$-\big(\phi_p\big(u'(t)\big)\big)' = f\big(t, u(t), u'(t)\big), \quad t \in [0,\omega], \qquad u(0) = u(\omega) = 0. \tag{1.2}$$

R. Manásevich and J. Mawhin [13] have discussed the periodic boundary value problems

$$\big(\phi\big(u'(t)\big)\big)' = f\big(t, u(t), u'(t)\big), \quad t \in [0,\omega], \qquad u(0) = u(\omega), \qquad u'(0) = u'(\omega), \tag{1.3}$$

where the function $\phi : \mathbf{R}^n \to \mathbf{R}^n$ satisfies some monotonicity conditions which ensure that ϕ is an homeomorphism onto $\mathbf{R}^n$. They have also given some applications for $\phi = \phi_p$ in [13]. On basis of application of Schauder's fixed point theorem, Mawhin [14] generalized the Hartman–Knobloch results on the periodic boundary value problem in [15,16] to perturbations of the vector p-Laplacian ordinary operator of the form

$$\big(\psi_p(u')\big)' = f(t,u), \tag{1.4}$$

where $\psi_p : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $\psi_p(u) = |u|^{p-2}u$.

The purpose of this paper is to establish some criteria to guarantee the existence of ω-periodic solutions for (1.1) by using coincidence degree theory. The methods used to estimated a priori bound of periodic solutions are different from the corresponding ones in [1–7]. Furthermore, the significance of this paper is that Theorems 3.2 and 3.3 do not impose any other condition on the function $F(x)$ besides F is twice continuously differentiable. When $p = 2$, the results in this paper are also different from those in [1–7].

In what follows, we will use $\langle \cdot,\cdot \rangle$ to denote the Euclidean inner product in $\mathbf{R}^n$, $|\cdot|_p$ denotes the l^p-norm in $\mathbf{R}^n$, i.e.,

$$|x|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

The norm in $\mathbf{R}^{n\times n}$ is defined by

$$\|A\|_p = \sup_{|x|_p=1,\ x\in\mathbf{R}^n} |Ax|_p.$$

The corresponding L^p-norm in $L^p([0,\omega],\mathbf{R}^n)$ is defined by

$$\|x\|_p = \left(\sum_{i=1}^n \int_0^\omega \left|x_i(t)\right|^p dt\right)^{1/p}.$$

The L^∞-norm in $L^\infty([0,\omega],\mathbf{R}^n)$ is

$$\|x\|_\infty = \max_{1\leqslant i\leqslant n} \|x_i\|_\infty,$$

where $\|x_i\|_\infty = \sup_{t\in[0,\omega]} |x_i(t)|$ $(i=1,\dots,n)$.

2. Preliminaries

Let X and Z be real normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I-Q)$, it follows that $L|_{\operatorname{Dom} L\cap\ker P} : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact.

In the proof of our results on existence of periodic solutions below, we will use the following generalized Borsuk theorem in coincidence degree of Gaines and Mawhin [17, p. 31].

Lemma 2.1. *Let L be a Fredholm mapping of index zero. Ω is an open bounded subset of X and Ω is symmetric with respect to the origin and contains it. Let $\widetilde{N} : \bar{\Omega} \times [0,1] \to Z$ be L-compact and such that*

(a) $\widetilde{N}(-x,0) = -\widetilde{N}(x,0)$, $\forall x \in \bar{\Omega}$,
(b) $Lx \neq \widetilde{N}(x,\lambda)$, $\forall x \in \operatorname{Dom} L \cap \partial\Omega$.

Then for every $\lambda \in [0,1]$, equation

$$Lx = \widetilde{N}(x,\lambda)$$

has at least one solution in Ω.

Let $W = W^{1,p}([0,\omega],\mathbf{R}^n)$ be the Sobolev space.

Lemma 2.2. (See [12].) *Suppose $u \in W$ and $u(0) = u(\omega) = 0$, then*

$$\|u\|_p \leqslant \frac{\omega}{\pi_p}\|u'\|_p,$$

where

$$\pi_p = 2\int_0^{(p-1)^{1/p}} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\frac{\pi}{p})}.$$

In order to use coincidence degree theory to study the existence of ω-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\begin{cases} x'(t) = \phi_q(y(t)), \\ y'(t) = -\frac{d}{dt}\operatorname{grad} F(x(t)) - \operatorname{grad} G(x(t)) + e(t). \end{cases} \tag{2.1}$$

If $z(t) = (x^{\mathrm{T}}(t), y^{\mathrm{T}}(t))^{\mathrm{T}}$ is an ω-periodic solution of (2.1), then $x(t)$ must be an ω-periodic solution of (1.1). Thus, the problem of finding an ω-periodic solution for (1.1) reduces to finding one for (2.1).

Let $C_\omega = \{x \in C(\mathbf{R}, \mathbf{R}^n)\colon\ x(t+\omega) \equiv x(t)\}$ with norm $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} \|x_i\|_\infty$, $X = Z = \{z = (x^{\mathrm{T}}(\cdot), y^{\mathrm{T}}(\cdot))^{\mathrm{T}} \in C(\mathbf{R}, \mathbf{R}^{2n})\colon\ z(t+\omega) \equiv z(t)\}$ with norm $\|z\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. Clearly, X and Z are Banach spaces. Meanwhile, let

$$L : \operatorname{Dom} L \subset X \to Z,\ (Lz)(t) = z'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix},$$

$$N : X \to Z, \quad (Nz)(t) = \begin{pmatrix} \phi_q(y(t)) \\ -\frac{d}{dt}\operatorname{grad} F(x(t)) - \operatorname{grad} G(x(t)) + e(t) \end{pmatrix} := H(z,t).$$

It is easy to see that $\ker L = \mathbf{R}^{2n}$, $\operatorname{Im} L = \{z \in Z\colon\ \int_0^\omega z(s)\,ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to \ker L$ and $Q : Z \to \operatorname{Im} Q$ be defined by

$$Pu = \frac{1}{\omega}\int_0^\omega u(s)\,ds, \quad u \in X; \qquad Qv = \frac{1}{\omega}\int_0^\omega v(s)\,ds, \quad v \in Z,$$

and let K_P denote the inverse of $L|_{\ker P \cap \operatorname{Dom} L}$. Obviously, $\ker L = \operatorname{Im} Q = \mathbf{R}^{2n}$ and

$$(K_P z)(t) = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt. \tag{2.2}$$

From (2.2), one can easily see that N is L-compact on $\bar{\Omega}$, where Ω is an open bounded subset of X.

3. Existence of periodic solutions

Theorem 3.1. *Suppose that there exist constants $a > 0$, $b > 0$, $c \geqslant 0$ and $\alpha > 1$, such that*

(i) $y^{\mathrm{T}} \frac{\partial^2 F(x)}{\partial x^2} y \geqslant a|y|_2^2$ *or* $y^{\mathrm{T}} \frac{\partial^2 F(x)}{\partial x^2} y \leqslant -a|y|_2^2$, $\forall x,\ y \in \mathbf{R}^n$,
(ii) $\langle x, \operatorname{grad} G(x)\rangle \geqslant b|x|_\alpha^\alpha - c$, $\forall x \in \mathbf{R}^n$.

Then (1.1) *has at least one ω-periodic solution for* $1 < p \leqslant 2$.

Proof. For any $\lambda \in [0,1]$, let

$$\widetilde{N}(z,\lambda)(t) = \frac{1+\lambda}{2} H(z,t) - \frac{1-\lambda}{2} H(-z,t).$$

Consider the following parameter equation:

$$(Lz)(t) = \widetilde{N}(z,\lambda)(t). \tag{3.1}$$

Let $z(t) = \binom{x(t)}{y(t)}$ be a possible ω-periodic solution of (3.1) for some $\lambda \in [0, 1]$. One can see $x = x(t)$ is an ω-periodic solution of the following system:

$$\begin{aligned}\big(\phi_p\big(x'(t)\big)\big)' &+ \frac{1+\lambda}{2}\frac{d}{dt}\operatorname{grad} F\big(x(t)\big) - \frac{1-\lambda}{2}\frac{d}{dt}\operatorname{grad} F\big(-x(t)\big)\\ &+ \frac{1+\lambda}{2}\operatorname{grad} G\big(x(t)\big) - \frac{1-\lambda}{2}\operatorname{grad} G\big(-x(t)\big) = \lambda e(t).\end{aligned} \tag{3.2}$$

Noticing that $x(t)$ is an ω-periodic solution, we have

$$-\|x'\|_p^p = \int_0^\omega \big\langle x, \big(\phi_p(x')\big)'\big\rangle\, dt, \tag{3.3}$$

and

$$\begin{aligned}\int_0^\omega \Big\langle x(t), \frac{d}{dt}\operatorname{grad} F\big(x(t)\big)\Big\rangle\, dt &= \big\langle x(t), \operatorname{grad} F\big(x(t)\big)\big\rangle\big|_0^\omega - \int_0^\omega \big\langle \operatorname{grad} F\big(x(t)\big), x'(t)\big\rangle\, dt\\ &= -F\big(x(t)\big)\big|_0^\omega = 0.\end{aligned} \tag{3.4}$$

From (ii), by (3.3) and (3.4), we can use (3.2) to obtain

$$-\|x'\|_p^p + b\|x\|_\alpha^\alpha - c\omega \leqslant \lambda \int_0^\omega \big\langle x(t), e(t)\big\rangle\, dt \leqslant \|e\|_\beta \|x\|_\alpha, \tag{3.5}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

On the other hand,

$$\int_0^\omega \big\langle x'(t), \big(\phi_p\big(x'(t)\big)\big)'\big\rangle\, dt = \int_0^\omega \big\langle \phi_q\big(y(t)\big), y'(t)\big\rangle\, dt = 0.$$

From (3.2), we have

$$\int_0^\omega \Big\langle x'(t), \frac{1+\lambda}{2}\frac{d}{dt}\operatorname{grad} F\big(x(t)\big) - \frac{1-\lambda}{2}\frac{d}{dt}\operatorname{grad} F\big(-x(t)\big)\Big\rangle\, dt = \lambda \int_0^\omega \big\langle x', e(t)\big\rangle\, dt.$$

By (i), one can get

$$a\|x'\|_2^2 \leqslant \|e\|_2 \|x'\|_2.$$

So, we have

$$\|x'\|_2 \leqslant \frac{\|e\|_2}{a} := R_1. \tag{3.6}$$

It is obvious that there exist $c_1 > 0$ and $c_2 > 0$ such that

$$c_1|x|_2 \leqslant |x|_p \leqslant c_2|x|_2, \quad x \in \mathbf{R}^n.$$

Thus,

$$\|x'\|_p^p = \int_0^\omega \left|x'(t)\right|_p^p dt \leqslant c_2^p \int_0^\omega \left|x'(t)\right|_2^p dt \leqslant c_2^p \left(\int_0^\omega \left|x'(t)\right|_2^2 dt \right)^{p/2} \omega^{(2-p)/2}$$
$$\leqslant (c_2 R_1)^p \omega^{(2-p)/2} := R_2, \tag{3.7}$$

where $1 < p \leqslant 2$.

From (3.5), we can see

$$b\|x\|_\alpha^\alpha - \|e\|_\beta \|x\|_\alpha - c\omega \leqslant R_2,$$

from which it follows that there exists a positive number R_3 such that

$$\|x\|_\alpha \leqslant R_3. \tag{3.8}$$

From (3.8), there exists $t_0 \in [0, \omega)$, such that $|x(t_0)|_\alpha \leqslant R_3 \omega^{-1/\alpha}$, so

$$\left|x_i(t)\right| = \left| x_i(t_0) + \int_{t_0}^t x_i'(s)\,ds \right| \leqslant R_3 \omega^{-1/\alpha} + \sqrt{\omega} \left(\int_0^\omega \left(x_i'(s)\right)^2 ds \right)^{1/2}$$
$$\leqslant R_3 \omega^{-1/\alpha} + \sqrt{\omega} R_1 := R_4.$$

Therefore $\|x\|_\infty \leqslant R_4$ and $|x(t)|_p \leqslant n^{1/p} R_4$.

Since $F \in C^2(\mathbf{R}^n, \mathbf{R})$, $G \in C^1(\mathbf{R}^n, \mathbf{R})$, there exist R_5 and R_6 such that $\|\frac{\partial^2 F(x)}{\partial x^2}\|_p \leqslant R_5$, $|\operatorname{grad} G(x)|_p \leqslant R_6$ for $|x|_p \leqslant n^{1/p} R_4$. From (3.2), we have

$$\int_0^\omega \left|\left(\phi_p(x')\right)'\right|_p dt \leqslant R_5 \int_0^\omega |x'|_p\,dt + R_6\omega + \int_0^\omega \left|e(t)\right|_p dt$$
$$\leqslant R_5 \omega^{1/q} \|x'\|_p + R_6 \omega + \int_0^\omega \left|e(t)\right|_p dt$$
$$\leqslant R_5 \omega^{1/q} R_2^{1/p} + R_6 \omega + \int_0^\omega \left|e(t)\right|_p dt := R_7. \tag{3.9}$$

Clearly, for each $i = 1, \ldots, n$, there exists $t_i \in (0, \omega)$, such that $x_i'(t_i) = 0$. Thus, for any $t \in [0, \omega]$, we have

$$\left|y_i(t)\right| = \left|\phi_p\left(x_i'(t)\right)\right| = \left|\phi_p\left(x_i'(t)\right) - \phi_p\left(x_i'(t_i)\right)\right| = \left| \int_{t_i}^t \left(\phi_p\left(x_i'(s)\right)\right)' ds \right| \leqslant R_7.$$

Therefore $\|y\|_\infty \leqslant R_7$.

Choose a number $R_8 > \max(R_4, R_7)$, let $\Omega = \{z \in X\colon \|z\| < R_8\}$, then $Lz \neq \widetilde{N}(z, \lambda)$ for any $z \in \operatorname{Dom} L \cap \partial\Omega$, $\lambda \in [0, 1]$. It is easy to see $\widetilde{N}$ is L-compact on $\bar{\Omega} \times [0, 1]$, $Lz = \widetilde{N}(z, 1)$ is (2.1) and $\widetilde{N}(-z, 0) = -\widetilde{N}(z, 0)$. From Lemma 2.1, (2.1) has at least one ω-periodic solution $\tilde{z}(t) = \binom{\tilde{x}(t)}{\tilde{y}(t)}$, $\tilde{x}(t)$ is an ω-periodic solution of (1.1). $\square$

Remark 3.1. We see that the conditions in Theorem A are also valid to (1.1) for $1 < p < 2$. When $p = 2$, the conditions in Theorem 3.1 are weaker than those in Theorem A.

Theorem 3.2. *Suppose that there exist constants $b \geqslant 0$, $c \geqslant 0$ and $d > 0$, such that*

(I) $\langle x, \operatorname{grad} G(x)\rangle \leqslant b|x|_p^p + c$, $\forall x \in \mathbf{R}^n$,
(II) $\forall i \in \{1, \dots, n\}$, *either* $x_i[\frac{\partial G(x)}{\partial x_i} - \bar{e}_i] > 0$ *or* $x_i[\frac{\partial G(x)}{\partial x_i} - \bar{e}_i] < 0$ *for* $|x_i| > d$, *where* $\bar{e}_i = \frac{1}{\omega}\int_0^\omega e_i(t)\,dt$. *Then* (1.1) *has at least one ω-periodic solution for* $b < (\frac{\pi_p}{\omega})^p$.

Proof. We also consider Eqs. (3.1). Let $z(t) = \binom{x(t)}{y(t)}$ be a possible ω-periodic solution of (3.1), from (I) and (3.2), we have

$$-\|x'\|_p^p + b\|x\|_p^p + c\omega \geqslant \lambda \int_0^\omega \langle x(t), e(t)\rangle\,dt \geqslant -\|e\|_q\|x\|_p,$$

i.e.,

$$\|x'\|_p^p \leqslant b\|x\|_p^p + \|e\|_q\|x\|_p + c\omega. \tag{3.10}$$

Integrating both sides of (3.2) over $[0, \omega]$, we get

$$\frac{1+\lambda}{2}\int_0^\omega \left[\frac{\partial G(x(t))}{\partial x_i} - \bar{e}_i\right]dt - \frac{1-\lambda}{2}\int_0^\omega \left[\frac{\partial G(-x(t))}{\partial x_i} - \bar{e}_i\right]dt = 0, \quad i = 1, \dots, n.$$

So there exist $\tilde{t}_i \in [0, \omega]$ such that

$$\frac{1+\lambda}{2}\left[\frac{\partial G(x(\tilde{t}_i))}{\partial x_i} - \bar{e}_i\right] - \frac{1-\lambda}{2}\left[\frac{\partial G(-x(\tilde{t}_i))}{\partial x_i} - \bar{e}_i\right]dt = 0, \quad i = 1, \dots, n.$$

From (II), one can see $|x_i(\tilde{t}_i)| \leqslant d$. Let $\chi_i(t) = x_i(t + \tilde{t}_i) - x_i(\tilde{t}_i)$, $\chi(t) = (\chi_1(t), \dots, \chi_n(t))^{\mathrm{T}}$, then $\chi(0) = \chi(\omega) = 0$, by Lemma 2.2, one can obtain

$$\|\chi\|_p \leqslant \frac{\omega}{\pi_p}\|\chi'\|_p. \tag{3.11}$$

Noticing the periodicity of $x(t)$, we have

$$\|x_i\|_p^p = \int_0^\omega |x_i(t)|^p\,dt = \int_0^\omega |x_i(t + \tilde{t}_i)|^p\,dt \leqslant \int_0^\omega (|\chi_i(t)| + d)^p\,dt \leqslant (\|\chi_i\|_p + \omega^{1/p}d)^p.$$

So from Minkovski's inequality, we have

$$\begin{aligned}\|x\|_p &= \left(\sum_{i=1}^n \|x_i\|_p^p\right)^{1/p} \leqslant \left(\sum_{i=1}^n (\|\chi_i\|_p + \omega^{1/p}d)^p\right)^{1/p} \\ &\leqslant \|\chi\|_p + (n\omega)^{1/p}d \leqslant \frac{\omega}{\pi_p}\|\chi'\|_p + (n\omega)^{1/p}d = \frac{\omega}{\pi_p}\|x'\|_p + (n\omega)^{1/p}d.\end{aligned} \tag{3.12}$$

In view of (3.10), we get

$$\|x'\|_p^p \leqslant b\left(\frac{\omega}{\pi_p}\|x'\|_p + (n\omega)^{1/p}d\right)^p + \|e\|_q\left(\frac{\omega}{\pi_p}\|x'\|_p + (n\omega)^{1/p}d\right) + c\omega. \tag{3.13}$$

Since $b(\frac{\omega}{\pi_p})^p < 1$, from (3.13), there exists a constant $R_9 > 0$, such that

$$\|x'\|_p \leqslant R_9. \tag{3.14}$$

Therefore,

$$\|x\|_p \leqslant \frac{\omega}{\pi_p} R_9 + (n\omega)^{1/p} d := R_{10}. \tag{3.15}$$

From (3.14) and (3.15), we know that the rest of the proof of the theorem is similar to that of Theorem 3.1. □

Theorem 3.3. *Suppose that there exist constants $b > 0$, $c \geqslant 0$ and $\alpha > 1$, such that*

$$\langle x, \operatorname{grad} G(x) \rangle \leqslant -b|x|_\alpha^\alpha + c, \quad \forall x \in \mathbf{R}^n.$$

Then (1.1) *has at least one ω-periodic solution.*

Proof. Consider the parameter equation (3.1), suppose that $z(t) = \binom{x(t)}{y(t)}$ is a possible ω-periodic solution of (3.1). From the condition of this theorem and (3.2), we have

$$-\|x'\|_p^p - b\|x\|_\alpha^\alpha + c\omega \geqslant \lambda \int_0^\omega \langle x(t), e(t) \rangle \, dt \geqslant -\|e\|_\beta \|x\|_\alpha,$$

i.e.,

$$0 \leqslant \|x'\|_p^p \leqslant -b\|x\|_\alpha^\alpha + \|e\|_\beta \|x\|_\alpha + c\omega, \tag{3.16}$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

It follows that there exist two constants $R_{11} > 0$ and $R_{12} > 0$ such that

$$\|x\|_\alpha \leqslant R_{11}, \qquad \|x'\|_p \leqslant R_{12}. \tag{3.17}$$

From the proof of Theorem 3.1, we know that (1.1) has at least one ω-periodic solution. □

As applications, we list the following examples.

Example 3.1. Consider the following system:

$$\left(\phi_p\left(u'(t)\right)\right)' + \frac{d}{dt} \operatorname{grad} F\left(u(t)\right) + \operatorname{grad} G\left(u(t)\right) = \begin{pmatrix} 1 + \sin(2t) \\ 2 - \cos(2t) \end{pmatrix}, \tag{3.18}$$

where $F \in C^2(\mathbf{R}^2, \mathbf{R})$, $G \in C^1(\mathbf{R}^2, \mathbf{R})$.

Let

$$\begin{aligned} &x = (x_1, x_2)^{\mathrm{T}}, \\ &F(x_1, x_2) = x_1^2 + x_2^2 - \frac{x_1 x_2}{2} - \sqrt{1 + x_1^2}, \\ &G(x_1, x_2) = x_1^4 + x_1^3 - \frac{1}{4} x_1^2 x_2^2 + x_2^4, \end{aligned}$$

then

$$y^{\mathrm{T}} \frac{\partial^2 F(x)}{\partial x^2} y \geqslant \frac{3 - \sqrt{2}}{2} |y|_2^2, \qquad y = (y_1, y_2)^{\mathrm{T}}, \qquad \langle x, \operatorname{grad} G(x) \rangle \geqslant \frac{1}{2} |x|_4^4 - 3, \quad x \in \mathbf{R}^2.$$

By Theorem 3.1, (3.18) has at least one π-periodic solution when $1 < p \leqslant 2$.

Example 3.2. We also consider (3.18). Let $p = 4$, $\omega = \pi$, $G(x_1, x_2) = \frac{1}{8}x_1^4 - \frac{1}{2}x_2^2$, then

$$\langle x, \operatorname{grad} G(x) \rangle \leqslant \frac{1}{2}|x|_4^4 + \frac{1}{2}, \qquad \left(\frac{\pi_4}{\omega}\right)^4 = \frac{3}{4},$$

$$\lim_{|x_1| \to \infty} x_1 \left(\frac{\partial G(x)}{\partial x_1} - \bar{e}_1 \right) = \lim_{|x_1| \to \infty} x_1 \left(\frac{1}{2}x_1^3 - 1 \right) = +\infty,$$

$$\lim_{|x_2| \to \infty} x_2 \left(\frac{\partial G(x)}{\partial x_2} - \bar{e}_2 \right) = \lim_{|x_2| \to \infty} \left(-x_2^2 - 2x_2 \right) = -\infty,$$

so there exists $d > 0$, such that $x_1(\frac{\partial G(x)}{\partial x_1} - \bar{e}_1) > 0$ for $|x_1| > d$, $x_2(\frac{\partial G(x)}{\partial x_2} - \bar{e}_2) < 0$ for $|x_2| > d$. By Theorem 3.2, (3.18) has at least one π-periodic solution for any $F \in C^2(\mathbf{R}^2, \mathbf{R})$.

If we set $G(x_1, x_2) = -x_1^4 - x_1^3 + \frac{1}{4}x_1^2 x_2^2 - x_2^4$, then $\langle x, \operatorname{grad} G(x) \rangle \leqslant -\frac{1}{2}|x|_4^4 + 3$, from Theorem 3.3, one can see that (3.18) has at least one π-periodic solution for any $F \in C^2(\mathbf{R}^2, \mathbf{R})$ and $p > 1$.

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