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New comparison results for impulsive integro-differential equations and applications ☆

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Abstract

We prove some new maximum principles for ordinary integro-differential equations. This allows us to introduce a new definition of lower and upper solutions which leads to the development of the monotone iterative technique for a periodic boundary value problem related to a nonlinear first-order impulsive integro-differential equation.

Keywords: Maximum principle; Impulsive integro-differential equation; Periodic boundary value problem; Upper and lower solution; Monotone method

1. Introduction

Impulsive differential equations are recognized as adequate models to study the evolution of processes that are subject to sudden changes in their states. The interest of researchers on this field has grown very fast due to applications to real world phenomena and impulsive functional equations have been analyzed by many authors in the literature [1–8,11–25,27–36] and several references therein. See [9] for the basic theory of monotone iterative technique and [10,26] for the foundations of the theory of impulsive differential equations.

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Recently, Z. He and X. He have considered in [6,7] the following impulsive integro-differential equation with periodic boundary value conditions

$$\begin{cases} u'(t) = f\big(t, u(t), [\mathcal{T}u](t), [\mathcal{S}u](t)\big), & t \in J_0, \\ \Delta u(t_k) = I_k\big(u(t_k)\big), & k = 1, \dots, p, \\ u(0) = u(T), \end{cases} \tag{1}$$

where $T = 2\pi$, $J = [0, T]$,

$$0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T,$$

$J_0 = J \setminus \{t_1, t_2, \dots, t_p\}$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, p$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$,

$$[\mathcal{T}x](t) = \int_0^t K(t,s)x(s)\,ds, \qquad [\mathcal{S}x](t) = \int_0^T H(t,s)x(s)\,ds,$$

$K \in C(D, \mathbb{R}^+)$, $D = \{(t,s) \in J \times J\colon\ t \geqslant s\}$, $H \in C(J \times J, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$. In those references, a comparison result and the existence and uniqueness of solutions for a linear periodic boundary value problem related to an impulsive integro-differential equation are presented, and the monotone iterative technique is used to obtain two sequences which approximate the extremal solutions of (1) between a lower and an upper solution. Here, we take any $T > 0$ and consider the spaces

$$\begin{aligned} PC(J) = \big\{&u : J \to \mathbb{R}\colon\ u \text{ is continuous in } J_0; \\ &\text{and } \exists u(0^+), u(T^-), u\big(t_k^+\big), u\big(t_k^-\big) = u(t_k),\ k = 1, \dots, p\big\} \end{aligned}$$

and

$$\begin{aligned} PC^1(J) = \big\{&u \in PC(J)\colon\ u \text{ is } C^1 \text{ in } J \setminus \{t_1, \dots, t_p\}; \\ &\text{and } \exists u'(0^+), u'(T^-), u'\big(t_k^+\big), u'\big(t_k^-\big),\ k = 1, \dots, p\big\} \end{aligned}$$

which are complete spaces under the norms

$$\|u\|_{PC(J)} = \sup\big\{\big|u(t)\big|\colon\ t \in J\big\}$$

and

$$\|u\|_{PC^1(J)} = \|u\|_{PC(J)} + \|u'\|_{PC(J)}.$$

The nonlinearity f is assumed to be continuous in $J_0 \times \mathbb{R}^2$, and there exist the lateral limits

$$f\big(t_k^+, x, y\big) \quad \text{and} \quad f\big(t_k^-, x, y\big) = f(t_k, x, y) \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

In Section 2, we present new comparison results. In Section 3, we introduce a new more general concept of upper and lower solution relative to problem (1). These results are an important tool to develop the monotone iterative technique for (1) and to obtain two sequences approximating the extremal solutions of this problem between appropriate lower and upper solutions (see Section 4). Finally, in Section 5, we present some examples to illustrate the applicability of the new results.

2. Maximum principles

In relation to problem (1), let $m \in PC^1(J)$ and suppose that the following inequalities hold, where $M > 0$, $N_1 > 0$, $N_2 > 0$, $L_k \geqslant 0$, $k = 1, 2, \dots, p$:

$$\begin{cases} m'(t) \geqslant Mm(t) + N_1[\mathcal{T}m](t) + N_2[\mathcal{S}m](t), & t \in J_0, \\ \Delta m(t_k) \geqslant L_k m(t_k), & k = 1, \dots, p, \\ m(0) \geqslant m(T). \end{cases} \tag{2}$$

In Lemma 2.2 [7], for $T = 2\pi$ and $m \in PC(J) \cap C^1(J_0)$ satisfying inequalities (2) and

$$M^{-1}(N_1 k_0 + N_2 h_0)\left(e^{4\pi M} - 1\right) \leqslant \frac{\{\prod_{0<t_k<2\pi}(1 + L_k)^{-1}\}^2}{\int_0^{2\pi} \prod_{0<t_k<s}(1 + L_k)^{-1}\, ds}, \tag{3}$$

where $k_0 = \max\{K(t, s)\colon (t, s) \in D\}$, $h_0 = \max\{H(t, s)\colon (t, s) \in J \times J\}$, it is proved that $m(t) \leqslant 0$, for all $t \in J$.

Estimate (3) is trivially true if $N_1 = N_2 = 0$, in this case, we are dealing with an impulsive ordinary inequality.

We present new estimates on the constants M, N_1, and N_2, improving the aforementioned result. We use the following lemma which can be found in [7,13].

Lemma 1. *Let $s \in [0, T)$. Assume that the sequence $\{t_k\}$ satisfies $0 \leqslant t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ with $\{t_k\} \to +\infty$, $m \in PC^1(\mathbb{R}^+)$, $c_k \geqslant 0$, α_k, $k = 1, \dots, p$, are constants and $p, q \in PC(\mathbb{R}^+)$ with*

$$\begin{cases} m'(t) \geqslant p(t)m(t) + q(t), & t \geqslant s,\ t \neq t_k, \\ m\left(t_k^+\right) \geqslant c_k m(t_k) + \alpha_k, & t_k \geqslant s. \end{cases}$$

Then, for $t \geqslant s$,

$$\begin{aligned} m(t) \geqslant{}& m(s^+)\left(\prod_{s<t_k<t} c_k\right)\exp\left(\int_s^t p(u)\, du\right) \\ &+ \int_s^t \left(\prod_{u<t_k<t} c_k\right)\exp\left(\int_u^t p(\tau)\, d\tau\right) q(u)\, du \\ &+ \sum_{s<t_k<t}\left(\prod_{t_k<t_i<t} c_i\right)\exp\left(\int_{t_k}^t p(\tau)\, d\tau\right)\alpha_k. \end{aligned}$$

Theorem 1. *Let $M > 0$, $N_1, N_2 \geqslant 0$. Suppose that $m \in PC^1(J)$ satisfies* (2) *with $L_k \geqslant 0$, $k = 1, 2, \dots, p$, and*

$$e^{MT}\int_0^T \prod_{0<t_k\leqslant s}(1 + L_k)^{-1} q(s)\, ds \leqslant \left\{\prod_{k=1}^{p}(1 + L_k)^{-1}\right\}^2, \tag{4}$$

where

$$q(t) = N_1 \int_0^t K(t,s)e^{-M(t-s)}\,ds + N_2 \int_0^T H(t,s)e^{-M(t-s)}\,ds, \quad t \in J.$$

Then $m(t) \leqslant 0$, *for* $t \in J$.

Proof. Suppose that there exists $t_1^* \in [0,T]$ such that $m(t_1^*) > 0$ and distinguish two cases.

Case 1. $m(t) \geqslant 0$, $t \in J$, $m \not\equiv 0$, then

$$m'(t) \geqslant Mm(t) + N_1 \int_0^t K(t,s)m(s)\,ds + N_2 \int_0^T H(t,s)m(s)\,ds \geqslant 0, \quad t \in J_0,$$

and

$$m\big(t_k^+\big) \geqslant L_k m(t_k) + m(t_k) \geqslant m(t_k), \quad k = 1, 2, \ldots, p,$$

so that m is nondecreasing in J, then

$$m(T) \geqslant m(0) \geqslant m(T)$$

and m is a constant function $m(t) = R > 0$, which implies that

$$m'(t) = 0 \geqslant MR + N_1 \int_0^t K(t,s)R\,ds + N_2 \int_0^T H(t,s)R\,ds \geqslant MR > 0,$$

getting a contradiction.

Case 2. $m(t) < 0$, for some $t \in [0,T]$. Take $u(t) = m(t)e^{-Mt}$, $t \in J$, then we have

$$u'(t) \geqslant N_1 \int_0^t K(t,s)e^{-M(t-s)}u(s)\,ds + N_2 \int_0^T H(t,s)e^{-M(t-s)}u(s)\,ds, \quad t \in J_0,$$

$$\Delta u(t_k) \geqslant L_k u(t_k), \quad k = 1, 2, \ldots, p,$$

$$u(0) \geqslant u(T)e^{MT}.$$

Let $\inf_J u = -\lambda < 0$, then there exists $t_0^* \in (t_i, t_{i+1}]$, for some i such that $u(t_0^*) = -\lambda$ or $u(t_i^+) = -\lambda$. We suppose that $u(t_0^*) = -\lambda$, since the proof in the other case is similar. Then, for every $t \in J_0$,

$$u'(t) \geqslant (-\lambda)\left(N_1 \int_0^t K(t,s)e^{-M(t-s)}\,ds + N_2 \int_0^T H(t,s)e^{-M(t-s)}\,ds\right).$$

Taking into account the definition of $q(t)$, last inequality is written as

$$u'(t) \geqslant (-\lambda)q(t), \quad t \in J_0,$$

and we also obtain that function u satisfies

$$u\big(t_k^+\big) \geqslant (1 + L_k)u(t_k), \quad k = 1, 2, \ldots, p.$$

Using Lemma 1, we get

$$u(t) \geqslant u(0) \prod_{0<t_k<t} (1+L_k) + \int_0^t \prod_{s<t_k<t} (1+L_k)(-\lambda)q(s)\,ds, \quad t \in [0,T],$$

in particular, for $t = t_0^*$,

$$u\big(t_0^*\big) \geqslant u(0) \prod_{0<t_k<t_0^*} (1+L_k) + \int_0^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)(-\lambda)q(s)\,ds, \tag{5}$$

so that

$$u(0) \prod_{0<t_k<t_0^*} (1+L_k) \leqslant -\lambda + \lambda \int_0^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)q(s)\,ds.$$

If $u(0) > 0$, then the above inequality implies that

$$1 < \int_0^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)q(s)\,ds \leqslant \int_0^{T} \prod_{s<t_k<T} (1+L_k)q(s)\,ds.$$

Therefore,

$$\prod_{k=1}^{p} (1+L_k)^{-1} < \int_0^T \prod_{0<t_k\leqslant s} (1+L_k)^{-1} q(s)\,ds,$$

contradicting condition (4).

Suppose that $u(0) \leqslant 0$.

If $t_1^* < t_0^*$, then Lemma 1 provides that

$$u(t) \geqslant u(t_1^*) \prod_{t_1^*<t_k<t} (1+L_k) + \int_{t_1^*}^t \prod_{s<t_k<t} (1+L_k)(-\lambda)q(s)\,ds, \quad t \geqslant t_1^*,$$

in particular, for $t = t_0^*$,

$$-\lambda = u\big(t_0^*\big) \geqslant u\big(t_1^*\big) \prod_{t_1^*<t_k<t_0^*} (1+L_k) + \int_{t_1^*}^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)(-\lambda)q(s)\,ds,$$

and, therefore,

$$0 < u\big(t_1^*\big) \prod_{t_1^*<t_k<t_0^*} (1+L_k) \leqslant -\lambda + \lambda \int_{t_1^*}^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)q(s)\,ds.$$

Hence,

$$1 < \int_{t_1^*}^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k)q(s)\,ds \leqslant \int_0^{T} \prod_{s<t_k<T} (1+L_k)q(s)\,ds,$$

since $(1+L_k)\geqslant 1$, for all k and $q\geqslant 0$, obtaining again a contradiction with condition (4).

Now, assume that $t_1^*>t_0^*$. Since $0\geqslant u(0)\geqslant u(T)e^{MT}$, then $u(T)\leqslant 0$ so that $t_1^*<T$. By Lemma 1,

$$u(t)\geqslant u\bigl(t_1^*\bigr)\prod_{t_1^*<t_k<t}(1+L_k)+\int_{t_1^*}^{t}\prod_{s<t_k<t}(1+L_k)(-\lambda)q(s)\,ds,\quad t\geqslant t_1^*,$$

in particular, for $t=T$,

$$u(T)\geqslant u\bigl(t_1^*\bigr)\prod_{t_1^*<t_k<T}(1+L_k)+\int_{t_1^*}^{T}\prod_{s<t_k<T}(1+L_k)(-\lambda)q(s)\,ds,$$

and

$$u(0)\geqslant u(T)e^{MT}.$$

In consequence,

$$u(0)\geqslant u(T)e^{MT}>-\lambda\int_{t_1^*}^{T}\prod_{s<t_k<T}(1+L_k)q(s)\,ds\,e^{MT}. \tag{6}$$

If $t_0^*=0$, then

$$-\lambda>-e^{MT}\lambda\int_{t_1^*}^{T}\prod_{s<t_k<T}(1+L_k)q(s)\,ds$$

and

$$1<e^{MT}\int_{t_1^*}^{T}\prod_{s<t_k<T}(1+L_k)q(s)\,ds\leqslant e^{MT}\int_{0}^{T}\prod_{s<t_k<T}(1+L_k)q(s)\,ds,$$

which is absurd.

If $t_0^*>0$, we obtain, from (5),

$$u(0)\prod_{0<t_k<t_0^*}(1+L_k)\leqslant-\lambda+\lambda\int_{0}^{t_0^*}\prod_{s<t_k<t_0^*}(1+L_k)q(s)\,ds.$$

This joint to (6) yields

$$\begin{aligned}&-\lambda\int_{t_1^*}^{T}\prod_{s<t_k<T}(1+L_k)q(s)\,ds\,e^{MT}\prod_{0<t_k<t_0^*}(1+L_k)\\&\quad<u(0)\prod_{0<t_k<t_0^*}(1+L_k)\leqslant-\lambda+\lambda\int_{0}^{t_0^*}\prod_{s<t_k<t_0^*}(1+L_k)q(s)\,ds.\end{aligned}$$

Therefore

$$\begin{aligned}
1 &< \int_0^{t_0^*} \prod_{s<t_k<t_0^*} (1+L_k) q(s)\, ds + \int_{t_1^*}^{T} \prod_{s<t_k<T} (1+L_k) q(s)\, ds\, e^{MT} \prod_{0<t_k<t_0^*} (1+L_k) \\
&\leqslant \int_0^{t_0^*} \prod_{s<t_k<T} (1+L_k) q(s)\, ds + \int_{t_1^*}^{T} \prod_{s<t_k<T} (1+L_k) q(s)\, ds\, e^{MT} \prod_{k=1}^{p} (1+L_k) \\
&\leqslant e^{MT} \prod_{k=1}^{p} (1+L_k) \int_0^T \prod_{s<t_k<T} (1+L_k) q(s)\, ds \\
&= e^{MT} \left\{ \prod_{k=1}^{p} (1+L_k) \right\}^2 \int_0^T \prod_{0<t_k\leqslant s} (1+L_k)^{-1} q(s)\, ds,
\end{aligned}$$

contradicting (4). □

Remark 1. Note that

$$\prod_{0<t_k\leqslant s} (1+L_k)^{-1} = \prod_{0<t_k<s} (1+L_k)^{-1}, \quad \text{for } s \neq t_j,\ \forall j,$$

then condition (4) can be replaced by

$$e^{MT} \int_0^T \prod_{0<t_k<s} (1+L_k)^{-1} q(s)\, ds \leqslant \left\{ \prod_{k=1}^{p} (1+L_k)^{-1} \right\}^2.$$

We present some particular cases, considering an upper bound for functions K and H, which show that our estimate (4) improves estimate (5) in Lemma 2.2 [7] for the case $T = 2\pi$.

Corollary 1. *Let $M > 0$, $N_1, N_2 \geqslant 0$, $J = [0, T]$ and $T > 0$. Suppose that $m \in PC^1(J)$ satisfies* (2) *with $L_k \geqslant 0$, $k = 1, 2, \ldots, p$, and*

$$M^{-1} e^{MT} (N_1 k_0 + N_2 h_0)\big(e^{MT} - 1\big) \int_0^T \prod_{0<t_k\leqslant s} (1+L_k)^{-1}\, ds \leqslant \left\{ \prod_{k=1}^{p} (1+L_k)^{-1} \right\}^2, \tag{7}$$

where $k_0 = \max\{K(t,s)\colon (t,s) \in D\}$, $h_0 = \max\{H(t,s)\colon (t,s) \in J \times J\}$. Then $m(t) \leqslant 0$, for $t \in J$.

Proof. Estimate (4) holds, since

$$\begin{aligned}
q(s) &= N_1 \int_0^s K(s,u)\, e^{-M(s-u)}\, du + N_2 \int_0^T H(s,u) e^{-M(s-u)}\, du \\
&\leqslant N_1 k_0 e^{-Ms} \int_0^s e^{Mu}\, du + N_2 h_0 e^{-Ms} \int_0^T e^{Mu}\, du
\end{aligned}$$

$$\leqslant (N_1k_0 + N_2h_0)\int_0^T e^{Mu}\,du = (N_1k_0 + N_2h_0)M^{-1}\big(e^{MT} - 1\big). \qquad \square$$

Note that (7) is more general than condition (3) (Condition (5) in [7]), since

$$e^{MT}\big(e^{MT} - 1\big) < \big(e^{MT} + 1\big)\big(e^{MT} - 1\big) = e^{2MT} - 1, \quad \text{for } T > 0,$$

in particular, for $T = 2\pi$, $e^{2MT} - 1 = e^{4\pi M} - 1$.

Another particular case is established in the following result.

Corollary 2. *Let $M > 0$, $N_1, N_2 \geqslant 0$, $J = [0, T]$ and $T > 0$. Suppose that $m \in PC^1(J)$ satisfies* (2) *with $L_k \geqslant 0$, $k = 1, 2, \ldots, p$, and*

$$M^{-1}e^{MT}\int_0^T \prod_{0<t_k\leqslant s}(1 + L_k)^{-1}\big[N_1k_0\big(1 - e^{-Ms}\big) + N_2h_0e^{-Ms}\big(e^{MT} - 1\big)\big]\,ds$$
$$\leqslant \left\{\prod_{k=1}^{p}(1 + L_k)^{-1}\right\}^2, \tag{8}$$

where $k_0 = \max\{K(t,s)\colon (t,s) \in D\}$, $h_0 = \max\{H(t,s)\colon (t,s) \in J \times J\}$. Then $m(t) \leqslant 0$, for $t \in J$.

Proof. Estimate (4) is true again, and the proof is similar to the proof of the previous corollary:

$$q(s) = N_1\int_0^s K(s,u)\,e^{-M(s-u)}\,du + N_2\int_0^T H(s,u)e^{-M(s-u)}\,du$$
$$\leqslant N_1k_0e^{-Ms}M^{-1}\big(e^{Ms} - 1\big) + N_2h_0e^{-Ms}M^{-1}\big(e^{MT} - 1\big). \qquad \square$$

Remark 2. We study function $\varphi(t) = N_1k_0(1 - e^{-Mt}) + N_2h_0e^{-Mt}(e^{MT} - 1)$, $t \in [0, T]$, which is a factor in the integrand of (8). The following conditions are verified:

$$\varphi(0) = N_2h_0\big(e^{MT} - 1\big),$$
$$\varphi(T) = N_1k_0\big(1 - e^{-MT}\big) + N_2h_0e^{-MT}\big(e^{MT} - 1\big) = (N_1k_0 + N_2h_0)\big(1 - e^{-MT}\big),$$
$$\varphi'(t) = M\big(N_1k_0 - N_2h_0\big(e^{MT} - 1\big)\big)e^{-Mt}, \quad t \in [0, T].$$

Take

$$B = \frac{\{\prod_{k=1}^{p}(1 + L_k)^{-1}\}^2}{\int_0^T \prod_{0<t_k\leqslant s}(1 + L_k)^{-1}\,ds},$$

and distinguish three cases:

(i) If $N_1k_0 = N_2h_0(e^{MT} - 1)$, then φ is constant,

$$\varphi(t) = N_2h_0\big(e^{MT} - 1\big) = (N_1k_0 + N_2h_0)\big(1 - e^{-MT}\big),$$

and the following estimate implies the validity of (8):

$$M^{-1}e^{MT}N_2h_0\big(e^{MT}-1\big)\leqslant B,$$

or, equivalently,

$$M^{-1}e^{MT}(N_1k_0+N_2h_0)\big(1-e^{-MT}\big)\leqslant B.$$

(ii) If $N_1k_0 > N_2h_0(e^{MT}-1)$, then φ is nondecreasing and

$$M^{-1}e^{MT}(N_1k_0+N_2h_0)\big(1-e^{-MT}\big)\leqslant B$$

implies the validity of (8).

(iii) If $N_1k_0 < N_2h_0(e^{MT}-1)$, then φ is nonincreasing and

$$M^{-1}e^{MT}N_2h_0\big(e^{MT}-1\big)\leqslant B$$

implies (8).

Remark 3. If

$$\mu=\max\left\{\int_{t_{k-1}}^{t_k} q(s)\,ds\colon\ k=1,2,\ldots,p+1\right\},$$

then the following condition

$$e^{MT}\left\{1+\sum_{n=1}^{p}\prod_{k=1}^{n}(1+L_k)^{-1}\right\}\mu\leqslant\left\{\prod_{k=1}^{p}(1+L_k)^{-1}\right\}^2$$

implies the validity of (4). Indeed,

$$\begin{aligned}
&\int_0^T\prod_{0<t_k\leqslant s}(1+L_k)^{-1}q(s)\,ds\\
&\quad=\sum_{n=0}^{p}\int_{t_n}^{t_{n+1}}\prod_{0<t_k\leqslant s}(1+L_k)^{-1}q(s)\,ds\\
&\quad=\int_{t_0}^{t_1}q(s)\,ds+\sum_{n=1}^{p}\int_{t_n}^{t_{n+1}}\prod_{k=1}^{n}(1+L_k)^{-1}q(s)\,ds\\
&\quad=\int_{t_0}^{t_1}q(s)\,ds+\sum_{n=1}^{p}\prod_{k=1}^{n}(1+L_k)^{-1}\int_{t_n}^{t_{n+1}}q(s)\,ds\\
&\quad\leqslant\mu\left\{1+\sum_{n=1}^{p}\prod_{k=1}^{n}(1+L_k)^{-1}\right\}.
\end{aligned}$$

Taking

$$k_0=\max\big\{K(t,s)\colon\ (t,s)\in D\big\}\quad\text{and}\quad h_0=\max\big\{H(t,s)\colon\ (t,s)\in J\times J\big\},$$

and using that $q(t)\leqslant(N_1k_0+N_2h_0)M^{-1}(e^{MT}-1)$ (see Corollary 1), then

$$\int_{t_{k-1}}^{t_k} q(s)\,ds \leqslant (N_1k_0+N_2h_0)M^{-1}\bigl(e^{MT}-1\bigr)(t_k-t_{k-1})$$
$$\leqslant (N_1k_0+N_2h_0)M^{-1}\bigl(e^{MT}-1\bigr)\delta,$$

choosing $\delta=\max\{t_k-t_{k-1}\colon\ k=1,2,\dots,p+1\}$. In consequence, for this value of $\mu=(N_1k_0+N_2h_0)M^{-1}(e^{MT}-1)\delta$, we obtain the particular estimate

$$e^{MT}(N_1k_0+N_2h_0)M^{-1}\bigl(e^{MT}-1\bigr)\delta\left\{1+\sum_{n=1}^{p}\prod_{k=1}^{n}(1+L_k)^{-1}\right\}$$
$$\leqslant\left\{\prod_{k=1}^{p}(1+L_k)^{-1}\right\}^2,$$

which improves Condition (17) in [7].

Remark 4. If $K(t,s)=H(t,s)=1$, for all t and s, then

$$[\mathcal{T}x](t)=\int_0^t x(s)\,ds,\qquad [\mathcal{S}x](t)=\int_0^T x(s)\,ds,$$

and

$$q(t)=M^{-1}\bigl[N_1-(N_1+N_2)e^{-Mt}+N_2e^{M(T-t)}\bigr],$$

in consequence, condition (4) is reduced to

$$M^{-1}e^{MT}\int_0^T\prod_{0<t_k\leqslant s}(1+L_k)^{-1}\bigl[N_1-(N_1+N_2)e^{-Ms}+N_2e^{M(T-s)}\bigr]\,ds$$
$$\leqslant\left\{\prod_{k=1}^{p}(1+L_k)^{-1}\right\}^2.$$

Next, we consider the case $m(0)<m(T)$, for which we also present a comparison result improving the one given in [7].

Theorem 2. *Let $M>0$, $N_1,N_2\geqslant 0$, $L_k\geqslant 0$, $k=1,2,\dots,p$. Suppose that $m\in PC^1(J)$ satisfies*

$$\begin{cases} m'(t)\geqslant Mm(t)+N_1[\mathcal{T}m](t)+N_2[\mathcal{S}m](t)+\sigma_m(t), & t\in J_0,\\ \Delta m(t_k)\geqslant L_km(t_k)+\dfrac{L_k(T-t_k)}{T}\bigl(m(T)-m(0)\bigr), & k=1,\dots,p,\\ m(0)<m(T), \end{cases}\tag{9}$$

where

$$\sigma_m(t)=\bigl(m(T)-m(0)\bigr)$$
$$\times\left(\frac{M(T-t)+1}{T}+\frac{N_1\int_0^t K(t,s)(T-s)\,ds}{T}+\frac{N_2\int_0^T H(t,s)(T-s)\,ds}{T}\right),$$

and condition (4) *holds. Then $m(t)\leqslant 0$, for $t\in J$.*

Proof. Take $g(t) = \frac{T-t}{T}(m(T) - m(0))$, $t \in J$, and define

$$\bar{m}(t) = m(t) + g(t) = m(t) + \frac{T-t}{T}\big(m(T) - m(0)\big), \quad t \in J.$$

Note that $g(0) = m(T) - m(0)$, $g(T) = 0$, $g \geqslant 0$ on $[0, T]$. If we prove that $\bar{m} \leqslant 0$, then $m \leqslant m + g \leqslant 0$ and the proof is complete. Function $\bar{m}$ is under the hypotheses of the comparison result Theorem 1. Indeed,

$$\bar{m}(0) = m(0) + g(0) = m(T) = m(T) + g(T) = \bar{m}(T),$$

$$\begin{aligned}\Delta\bar{m}(t_k) &= \Delta m(t_k) + \Delta g(t_k)\\ &= \Delta m(t_k) \geqslant L_k m(t_k) + L_k\frac{(T-t_k)}{T}\big(m(T) - m(0)\big) = L_k\bar{m}(t_k),\end{aligned}$$

for $k = 1, 2, \ldots, p$, and

$$\begin{aligned}\bar{m}'(t) &= m'(t) + g'(t)\\ &\geqslant Mm(t) + N_1[\mathcal{T}m](t) + N_2[\mathcal{S}m](t) + \sigma_m(t) - \frac{1}{T}\big(m(T) - m(0)\big)\\ &= M\bar{m}(t) + N_1[\mathcal{T}\bar{m}](t) + N_2[\mathcal{S}\bar{m}](t) + \sigma_m(t) - \frac{1}{T}\big(m(T) - m(0)\big)\\ &\quad - Mg(t) - N_1[\mathcal{T}g](t) - N_2[\mathcal{S}g](t)\\ &= M\bar{m}(t) + N_1[\mathcal{T}\bar{m}](t) + N_2[\mathcal{S}\bar{m}](t) + \sigma_m(t) - \big(m(T) - m(0)\big)\\ &\quad\times\frac{M(T-t) + 1 + N_1\int_0^t K(t,s)(T-s)\,ds + N_2\int_0^T H(t,s)(T-s)\,ds}{T}\\ &= M\bar{m}(t) + N_1[\mathcal{T}\bar{m}](t) + N_2[\mathcal{S}\bar{m}](t), \quad t \in J_0.\end{aligned}$$

Using Theorem 1, we get $\bar{m} \leqslant 0$ on J and, therefore, $m \leqslant 0$ on J. □

The following result extends, to any $T > 0$, Lemma 2.2 [7], for the case where $m(0) < m(T)$.

Corollary 3. *Let $M > 0$, $N_1, N_2 \geqslant 0$, $L_k \geqslant 0$, $k = 1, 2, \ldots, p$. Suppose that $m \in PC^1(J)$ satisfies*

$$\begin{cases} m'(t) \geqslant Mm(t) + N_1[\mathcal{T}m](t) + N_2[\mathcal{S}m](t) + \gamma_m(t), & t \in J_0,\\ \Delta m(t_k) \geqslant L_k m(t_k) + \dfrac{L_k(T-t_k)}{T}\big(m(T) - m(0)\big), & k = 1, \ldots, p,\\ m(0) < m(T), \end{cases} \tag{10}$$

where

$$\gamma_m(t) = \left(\frac{M(T-t)+1}{T} + \frac{N_1k_0(Tt - \frac{t^2}{2})}{T} + N_2h_0\frac{T}{2}\right)\big(m(T) - m(0)\big),$$

and condition (4) *holds. Then $m(t) \leqslant 0$, for $t \in J$.*

Proof. This result comes from Theorem 2, taking into account that

$$\sigma_m(t) \leqslant \gamma_m(t), \quad t \in J.$$

Indeed, for $t \in J$,

$$\begin{aligned}\sigma_m(t) &\leqslant \big(m(T)-m(0)\big)\\ &\quad\times\left(\frac{M(T-t)+1}{T}+\frac{N_1k_0}{T}\int_0^t (T-s)\,ds+\frac{N_2}{T}h_0\int_0^T (T-s)\,ds\right)\\ &=\frac{M(T-t)+1+N_1k_0(Tt-\frac{t^2}{2})+N_2h_0\frac{T^2}{2}}{T}\big(m(T)-m(0)\big)=\gamma_m(t). \qquad \square\end{aligned}$$

In particular, for $T=2\pi$, function γ_m is equal to function with the same name given in [7]:

$$\gamma_m(t)=\left(\frac{M(2\pi-t)+1}{2\pi}+\frac{N_1k_0(4\pi t-t^2)}{4\pi}+N_2h_0\pi\right)\big(m(2\pi)-m(0)\big).$$

Function σ_m in Theorem 2 provides a more general estimate and, therefore, Theorem 2 improves Lemma 2.2 [7], for $m(0)<m(T)$. Now, we present a more general result.

Theorem 3. *Let $M>0$, $N_1, N_2 \geqslant 0$, $L_k \geqslant 0$, $k=1,2,\dots,p$. Suppose that $m\in PC^1(J)$ satisfies*

$$\begin{cases} m'(t)\geqslant Mm(t)+N_1[\mathcal{T}m](t)+N_2[\mathcal{S}m](t)+\sigma_m(t), & t\in J_0,\\ \Delta m(t_k)\geqslant L_k m(t_k)+\vartheta_{mk}, & k=1,\dots,p,\\ m(0)<m(T), \end{cases} \tag{11}$$

where

$$\begin{aligned}\sigma_m(t)&=-g'(t)+Mg(t)+N_1[\mathcal{T}g](t)+N_2[\mathcal{S}g](t)\\ &=-g'(t)+Mg(t)+N_1\int_0^t K(t,s)g(s)\,ds+N_2\int_0^T H(t,s)g(s)\,ds,\\ \vartheta_{mk}&=L_kg(t_k)-\Delta g(t_k),\end{aligned}$$

for some $g\in PC^1(J)$, with $g\geqslant 0$ in $[0,T]$,

$$g(0)-g(T)\geqslant m(T)-m(0)>0.$$

Suppose also that condition (4) *holds. Then $m(t)\leqslant 0$, for $t\in J$.*

Proof. Define

$$\bar m(t)=m(t)+g(t), \quad t\in J=[0,T],$$

and prove that $\bar m\leqslant 0$ on J, which implies that $m\leqslant m+g\leqslant 0$ on J. Function $\bar m$ is under the assumptions of Theorem 1. Indeed, by hypotheses,

$$\bar m(0)=m(0)+g(0)\geqslant m(T)+g(T)=\bar m(T),$$

$$\Delta\bar m(t_k)=\Delta m(t_k)+\Delta g(t_k)\geqslant L_km(t_k)+\vartheta_{mk}+\Delta g(t_k)=L_k\bar m(t_k),$$

for $k=1,2,\dots,p$, and

$$\begin{aligned}\bar m'(t)&=m'(t)+g'(t)\geqslant Mm(t)+N_1[\mathcal{T}m](t)+N_2[\mathcal{S}m](t)+\sigma_m(t)+g'(t)\\ &=M\bar m(t)+N_1[\mathcal{T}\bar m](t)+N_2[\mathcal{S}\bar m](t)+\sigma_m(t)+g'(t)-Mg(t)\\ &\quad-N_1[\mathcal{T}g](t)-N_2[\mathcal{S}g](t)=M\bar m(t)+N_1[\mathcal{T}\bar m](t)+N_2[\mathcal{S}\bar m](t), \quad t\in J_0.\end{aligned}$$

Using Theorem 1, we achieve $\bar m\leqslant 0$ on J and, consequently, $m\leqslant 0$ on J. $\square$

Remark 5. Taking $g(t) = \frac{T-t}{T}(m(T) - m(0))$, $t \in J$, in Theorem 3, we obtain Theorem 2. Indeed, $g \in C^1(J, \mathbb{R})$, $g \geqslant 0$ in $[0, T]$, and

$$g(0) - g(T) = m(T) - m(0).$$

Expressions σ_m and ϑ_{mk} are given by

$$\begin{aligned}\sigma_m(t) &= -g'(t) + Mg(t) + N_1 \int_0^t K(t,s)g(s)\,ds + N_2 \int_0^T H(t,s)g(s)\,ds \\ &= \frac{1 + M(T-t) + N_1 \int_0^t K(t,s)(T-s)\,ds + N_2 \int_0^T H(t,s)(T-s)\,ds}{T} \\ &\quad \times \big(m(T) - m(0)\big)\end{aligned}$$

and

$$\vartheta_{mk} = L_k \frac{T - t_k}{T}\big(m(T) - m(0)\big) - \Delta g(t_k) = \frac{L_k(T - t_k)}{T}\big(m(T) - m(0)\big).$$

If $m(0) < m(T)$, we can choose any function g with the afore-mentioned properties. For example, for

$$g(t) = \int_t^T e^{M(T-s)}a(s)\,ds\,\big(m(T) - m(0)\big), \quad t \in J,$$

with $a \in PC(J)$, $a \geqslant 0$, and

$$\int_0^T e^{M(T-s)}a(s)\,ds \geqslant 1,$$

then $g \in PC^1(J)$ is continuous, $g \geqslant 0$ in $[0, T]$,

$$g(0) - g(T) = \int_0^T e^{M(T-s)}a(s)\,ds\,\big(m(T) - m(0)\big) \geqslant m(T) - m(0),$$

and

$$\begin{aligned}\sigma_m(t) = \Bigg(& e^{M(T-t)}a(t) + M \int_t^T e^{M(T-s)}a(s)\,ds \\ &+ N_1 \int_0^t K(t,s) \int_s^T e^{M(T-\tau)}a(\tau)\,d\tau\,ds \\ &+ N_2 \int_0^T H(t,s) \int_s^T e^{M(T-\tau)}a(\tau)\,d\tau\,ds\Bigg)\big(m(T) - m(0)\big),\end{aligned}$$

$$\vartheta_{mk} = L_k \int_{t_k}^{T} e^{M(T-s)} a(s)\, ds\, \big(m(T) - m(0)\big).$$

Note that, if $a(t) = e^{-M(T-t)} \frac{1}{T} \in C(J)$, then $a \geqslant 0$,

$$\int_0^T e^{M(T-s)} a(s)\, ds = \int_0^T \frac{1}{T}\, ds = 1,$$

and we obtain precisely

$$g(t) = \int_t^T e^{M(T-s)} e^{-M(T-s)} \frac{1}{T}\, ds\, \big(m(T) - m(0)\big) = \frac{T-t}{T} \big(m(T) - m(0)\big), \quad t \in J.$$

We can take different expressions for

$$\tilde{g}(t) = \int_t^T e^{M(T-s)} \tilde{a}(s)\, ds\, \big(m(T) - m(0)\big), \tag{12}$$

where $\tilde{a}$ satisfies the appropriate conditions and is different from $a(t) = e^{-M(T-t)} \frac{1}{T}$. It is possible to take functions $\tilde{a}$ which are not comparable to a, in such a way that $\tilde{g}$ is not comparable to g and

$$-\tilde{g}'(t) + M\tilde{g}(t) + N_1 \int_0^t K(t,s)\tilde{g}(s)\, ds + N_2 \int_0^T H(t,s)\tilde{g}(s)\, ds$$

is not comparable to $\phi(t)(m(T) - m(0))$, where

$$\phi(t) := \frac{1}{T}\left(M(T-t) + 1 + N_1 \int_0^t K(t,s)(T-s)\, ds + N_2 \int_0^T H(t,s)(T-s)\, ds \right).$$

This suggests that the use of a general function $\tilde{g}$, instead of $g(t) = \frac{T-t}{T}(m(T) - m(0))$, is a considerable improvement in comparison with previous results, taking into account that, even restricting our attention to very particular expressions such as (12), we obtain situations not comparable to the cases previously studied. See Section 5, for details.

3. Lower and upper solutions

To develop the method of upper and lower solutions and the monotone method, the usual condition on functions f and I_k is a one-sided Lipschitz condition. Using the new maximum principles presented in this paper, the results in [7] which provide existence of solution to (1) [7, Theorem 3.1] and existence of monotone sequences which converge uniformly to the extremal solutions of problem (1) between a lower and an upper solution [7, Theorem 3.2] can be extended to more general hypotheses. In fact, Condition (5) in [7] can be replaced by our estimate (4), and definitions of lower and upper solutions can be replaced by a more general one, as follows.

Definition 1. Suppose that there exist functions $\alpha, \beta \in PC^1(J)$, with $\beta \leqslant \alpha$ on J, such that

$$\alpha'(t) \leqslant f\big(t, \alpha(t), [\mathcal{T}\alpha](t), [\mathcal{S}\alpha](t)\big) - \sigma_\alpha(t), \quad t \in J_0,$$
$$\Delta\alpha(t_k) \leqslant I_k\big(\alpha(t_k)\big) - \vartheta_{\alpha k}, \quad k = 1, \ldots, p,$$
$$\beta'(t) \geqslant f\big(t, \beta(t), [\mathcal{T}\beta](t), [\mathcal{S}\beta](t)\big) + \sigma_\beta(t), \quad t \in J_0,$$
$$\Delta\beta(t_k) \geqslant I_k\big(\beta(t_k)\big) + \vartheta_{\beta k}, \quad k = 1, \ldots, p,$$

where, for $M > 0$, $N_1, N_2 \geqslant 0$, $L_k \geqslant 0$, $k = 1, 2, \ldots, p$, expressions σ_α, σ_β, $\vartheta_{\alpha k}$, $\vartheta_{\beta k}$ are given by

$$\sigma_\alpha(t) = \begin{cases} 0, \quad \text{if } \alpha(0) \leqslant \alpha(T), \\ -g'(t) + Mg(t) + N_1 \int_0^t K(t,s)g(s)\,ds + N_2 \int_0^T H(t,s)g(s)\,ds, \\ \qquad \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$\vartheta_{\alpha k} = \begin{cases} 0, & \text{if } \alpha(0) \leqslant \alpha(T), \\ L_k g(t_k) - \Delta g(t_k), & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

for some $g \in PC^1(J)$, with $g \geqslant 0$ in $[0, T]$,

$$g(0) - g(T) \geqslant \alpha(0) - \alpha(T) > 0,$$

$$\sigma_\beta(t) = \begin{cases} 0, \quad \text{if } \beta(0) \geqslant \beta(T), \\ -\tilde{g}'(t) + M\tilde{g}(t) + N_1 \int_0^t K(t,s)\tilde{g}(s)\,ds + N_2 \int_0^T H(t,s)\tilde{g}(s)\,ds, \\ \qquad \text{if } \beta(0) < \beta(T), \end{cases}$$

$$\vartheta_{\beta k} = \begin{cases} 0, & \text{if } \beta(0) \geqslant \beta(T), \\ L_k \tilde{g}(t_k) - \Delta\tilde{g}(t_k), & \text{if } \beta(0) < \beta(T), \end{cases}$$

for some $\tilde{g} \in PC^1(J)$, with $\tilde{g} \geqslant 0$ in $[0, T]$,

$$\tilde{g}(0) - \tilde{g}(T) \geqslant \beta(T) - \beta(0) > 0.$$

In this case, we say that α, β are, respectively, lower and upper solutions to (1).

Remark 6. Note that functions g and $\tilde{g}$ are not necessarily the same.

As a particular case, we extend Condition (A_0) in [7], in the sense that the new concept of upper (respectively lower) solution is more general.

Remark 7. Functions $\alpha, \beta \in PC^1(J)$, with $\beta \leqslant \alpha$ on J, satisfying the assumptions below are, respectively, admissible lower and upper solutions for (1), according to Definition 1:

$$\alpha'(t) \leqslant f\big(t, \alpha(t), [\mathcal{T}\alpha](t), [\mathcal{S}\alpha](t)\big) - \sigma_\alpha(t), \quad t \in J_0,$$
$$\Delta\alpha(t_k) \leqslant I_k\big(\alpha(t_k)\big) - \vartheta_{\alpha k}, \quad k = 1, \ldots, p,$$
$$\beta'(t) \geqslant f\big(t, \beta(t), [\mathcal{T}\beta](t), [\mathcal{S}\beta](t)\big) + \sigma_\beta(t), \quad t \in J_0,$$
$$\Delta\beta(t_k) \geqslant I_k\big(\beta(t_k)\big) + \vartheta_{\beta k}, \quad k = 1, \ldots, p,$$

where, for $M > 0$, $N_1, N_2 \geqslant 0$, $L_k \geqslant 0$, $k = 1, 2, \dots, p$, expressions σ_α, σ_β, $\vartheta_{\alpha k}$, $\vartheta_{\beta k}$ are given by

$$\sigma_\alpha(t) = \begin{cases} 0, \quad \text{if } \alpha(0) \leqslant \alpha(T), \\ \left(\frac{M(T-t)+1}{T} + \frac{N_1 \int_0^t K(t,s)(T-s)\,ds}{T} + \frac{N_2 \int_0^T H(t,s)(T-s)\,ds}{T}\right)(\alpha(0) - \alpha(T)), \\ \quad \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$\vartheta_{\alpha k} = \begin{cases} 0, & \text{if } \alpha(0) \leqslant \alpha(T), \\ \frac{L_k(T-t_k)}{T}(\alpha(0) - \alpha(T)), & \text{if } \alpha(0) > \alpha(T), \end{cases}$$

$$\sigma_\beta(t) = \begin{cases} 0, \quad \text{if } \beta(0) \geqslant \beta(T), \\ \left(\frac{M(T-t)+1}{T} + \frac{N_1 \int_0^t K(t,s)(T-s)\,ds}{T} + \frac{N_2 \int_0^T H(t,s)(T-s)\,ds}{T}\right)(\beta(T) - \beta(0)), \\ \quad \text{if } \beta(0) < \beta(T), \end{cases}$$

$$\vartheta_{\beta k} = \begin{cases} 0, & \text{if } \beta(0) \geqslant \beta(T), \\ \frac{L_k(T-t_k)}{T}(\beta(T) - \beta(0)), & \text{if } \beta(0) < \beta(T). \end{cases}$$

4. Existence results

Using fixed point theory, it is possible to prove existence of a solution to problem (1) between an upper and a lower solution, obtaining an analogue of Theorem 3.1 [7], in which statement we can replace Condition (3) (estimate (5) in [7]) by our condition (4) and the definition of lower and upper solutions by our Definition 1. Existence of appropriate lower and upper solutions as well as one-sided Lipschitz conditions for functions f and I_k between β and α are required in order to prove existence of a solution. Extending some well-known results about existence of a unique solution for impulsive linear integro-differential problems, an analogue of [7, Theorem 3.2] can be established and existence of monotone sequences starting at α, β and converging uniformly to the extremal solutions to (1) in $[\beta, \alpha]$ can be proved. Thus, the monotone iterative technique can be developed allowing a more general concept of lower and upper solutions and considering a more general estimate on the constants. Of course, one-sided Lipschitz conditions are assumed for functions f and I_k between the upper and the lower solutions.

Lemma 2. [18] *Let $M > 0$, $N_1 > 0$, $N_2 > 0$, $L_k \geqslant 0$, $k = 1, 2, \dots, p$, $I_k \in C(J, \mathbb{R})$, $\sigma \in PC(J)$, and $\eta \in PC^1(J)$.*

A function $u \in PC^1(J)$ is a solution of the periodic boundary value problem

$$\begin{cases} u' - Mu = N_1 \mathcal{T}u + N_2 \mathcal{S}u + \sigma(t), & t \in J_0, \\ \Delta u(t_k) = L_k u(t_k) + I_k\big(\eta(t_k)\big) - L_k \eta(t_k), & k = 1, 2, \dots, p, \\ u(0) = u(T), \end{cases} \tag{13}$$

if and only if $u \in PC(J)$ is a solution of the following impulsive integral equation

$$u(t) = -\int_0^T G(t,s)\big\{N_1[\mathcal{T}u](s) + N_2[\mathcal{S}u](s) + \sigma(s)\big\}\,ds - \sum_{0<t_k<T} G(t,t_k)\big(L_k u(t_k) + I_k\big(\eta(t_k)\big) - L_k \eta(t_k)\big), \quad t \in J, \tag{14}$$

where

$$G(t,s)=\frac{1}{e^{MT}-1}\begin{cases} e^{M(t-s)}, & 0\leqslant s<t\leqslant T,\\ e^{M(T+t-s)}, & 0\leqslant t\leqslant s\leqslant T.\end{cases}$$

Lemma 3. *Let $M>0$, $N_1>0$, $N_2>0$, $L_k\geqslant 0$, $k=1,2,\dots,p$, $I_k\in C(J,\mathbb{R})$, $\sigma\in PC(J)$, $\eta\in PC^1(J)$, and assume that*

$$\sup_{t\in J}\int_0^T G(t,s)\left[N_1\int_0^s K(s,r)\,dr+N_2\int_0^T H(s,r)\,dr\right]ds+\frac{1}{1-e^{-MT}}\sum_{k=1}^{p}L_k<1. \tag{15}$$

Then problem (13) *has a unique solution in $PC^1(J)$.*

Proof. Define the mapping $F:PC(J)\to PC(J)$, where Fu is given by the right-hand term in (14). Then

$$\begin{aligned}
&\|Tu-Tv\|\\
&\quad=\sup_{t\in J}\left|-\int_0^T G(t,s)\big(N_1\big\{[\mathcal{T}u](s)-[\mathcal{T}v](s)\big\}+N_2\big\{[\mathcal{S}u](s)-[\mathcal{S}v](s)\big\}\,ds\big)\right.\\
&\qquad\left.-\sum_{0<t_k<T}G(t,t_k)L_k\big(u(t_k)-v(t_k)\big)\right|\\
&\quad\leqslant\sup_{t\in J}\left\{\int_0^T G(t,s)\left[N_1\int_0^s K(s,r)\big|u(r)-v(r)\big|\,dr+N_2\int_0^T H(s,r)\big|u(r)-v(r)\big|\,dr\right]ds\right.\\
&\qquad\left.+\sum_{0<t_k<T}G(t,t_k)L_k\big|u(t_k)-v(t_k)\big|\right\}\\
&\quad\leqslant\|u-v\|\left(\sup_{t\in J}\int_0^T G(t,s)\left[N_1\int_0^s K(s,r)\,dr+N_2\int_0^T H(s,r)\,dr\right]ds\right.\\
&\qquad\left.+\frac{1}{1-e^{-MT}}\sum_{k=1}^{p}L_k\right),
\end{aligned}$$

and condition (15) guarantees that F is a contractive mapping, which completes the proof. □

The previous lemma extends Lemma 2.4 in [7], since (15) is more general than condition (22) in [7]: if the following inequality holds

$$\frac{T}{M}(N_1k_0+N_2h_0)+\frac{1}{1-e^{-MT}}\sum_{k=1}^{p}L_k<1,$$

where

$$k_0=\max\big\{K(t,s):\ (t,s)\in D\big\}\quad\text{and}\quad h_0=\max\big\{H(t,s):\ (t,s)\in J\times J\big\}$$

(take $T = 2\pi$ to obtain (22) in [7]), then condition (15) is valid. Indeed, take into account that

$$\sup_{t\in J}\int_0^T G(t,s)\left[N_1\int_0^s K(s,r)\,dr + N_2\int_0^T H(s,r)\,dr\right]ds$$
$$\leqslant (N_1k_0 + N_2h_0)T\sup_{t\in J}\int_0^T G(t,s)\,ds = \frac{T}{M}(N_1k_0 + N_2h_0).$$

Condition (15) is also valid if

$$\sup_{t\in J}\int_0^T G(t,s)[N_1k_0s + N_2h_0T]\,ds + \frac{1}{1-e^{-MT}}\sum_{k=1}^p L_k < 1.$$

Theorem 4. *Assume the existence of upper and lower solutions for* (1) (*see Definition* 1) *and also suppose that the following conditions hold*:

- *The function* $f \in C(J \times \mathbb{R}^3, \mathbb{R})$ *satisfies*
$$f(t,\alpha,\mathcal{T}\alpha,\mathcal{S}\alpha) - f(t,u,\mathcal{T}u,\mathcal{S}u) \leqslant M(\alpha - u) + N_1(\mathcal{T}\alpha - \mathcal{T}u) + N_2(\mathcal{S}\alpha - \mathcal{S}u)$$
and
$$f(t,u,\mathcal{T}u,\mathcal{S}u) - f(t,\beta,\mathcal{T}\beta,\mathcal{S}\beta) \leqslant M(u-\beta) + N_1(\mathcal{T}u - \mathcal{T}\beta) + N_2(\mathcal{S}u - \mathcal{S}\beta),$$
whenever $\beta(t) \leqslant u(t) \leqslant \alpha(t)$, $[\mathcal{T}\beta](t) \leqslant [\mathcal{T}u](t) \leqslant [\mathcal{T}\alpha](t)$, $[\mathcal{S}\beta](t) \leqslant [\mathcal{S}u](t) \leqslant [\mathcal{S}\alpha](t)$, $t \in J$, *where* $M > 0$, $N_1 > 0$, $N_2 > 0$.
- *The functions* $I_k \in C(\mathbb{R}, \mathbb{R})$ *satisfy*
$$I_k\big(\alpha(t_k)\big) - I_k(u) \leqslant L_k\big(\alpha(t_k) - u\big)$$
and
$$I_k(u) - I_k\big(\beta(t_k)\big) \leqslant L_k\big(u - \beta(t_k)\big),$$
whenever $\beta(t_k) \leqslant u \leqslant \alpha(t_k)$, $k = 1,2,\ldots,p$, *where* $L_k \geqslant 0$, $k = 1,2,\ldots,p$.

If inequalities (4) *and* (15) *hold, then there exists a solution* x *of the periodic boundary value problem* (1) *such that* $\beta(t) \leqslant x(t) \leqslant \alpha(t)$, *for* $t \in J$.

Proof. Analogous to the proof of Theorem 3.1 [7], using Lemma 3. □

Theorem 5. *Assume that there exist upper and lower solutions for* (1) *and that*

- *The function* $f \in C(J \times \mathbb{R}^3, \mathbb{R})$ *satisfies*
$$f(t,u,v,w) - f(t,\bar{u},\bar{v},\overline{w}) \leqslant M(u-\bar{u}) + N_1(v-\bar{v}) + N_2(w-\overline{w}),$$
whenever $\beta(t) \leqslant \bar{u} \leqslant u \leqslant \alpha(t)$, $[\mathcal{T}\beta](t) \leqslant \bar{v} \leqslant v \leqslant [\mathcal{T}\alpha](t)$, $[\mathcal{S}\beta](t) \leqslant \overline{w} \leqslant w \leqslant [\mathcal{S}\alpha](t)$, $t \in J$, *where* $M > 0$, $N_1 > 0$, $N_2 > 0$.

- *The functions $I_k \in C(\mathbb{R},\mathbb{R})$ satisfy*

$$I_k(x) - I_k(y) \leqslant L_k(x-y),$$

whenever $\beta(t_k) \leqslant y \leqslant x \leqslant \alpha(t_k)$, $k = 1, 2, \ldots, p$, and $L_k \geqslant 0$, $k = 1, 2, \ldots, p$.

Suppose that inequalities (4) *and* (15) *hold. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, which converge uniformly on J to the extremal solutions of the periodic boundary value problem* (1) *in* $[\beta, \alpha]$ (*the functional interval delimited by α and β*).

Proof. Analogous to the proof of Theorem 3.2 [7], using Lemma 3. □

An analogous study can be made in relation with reference [6], where the case $\alpha \leqslant \beta$ is dealt with. Results in [6] can be extended using a procedure similar to the one exposed in this paper. The comparison result of that reference can be improved and, in consequence, more general definitions of upper and lower solutions can be considered, making it possible to extend the applicability of the monotone iterative technique.

5. Examples

Following the ideas in Remark 5, we show some examples to illustrate the achievements of the new results.

Example 1. Take $T = 1$ and $M = 1$. For the quadratic function

$$\tilde{a}(t) = -\lambda t^2 + \lambda T t + \frac{1}{2T} e^{-MT},$$

where $\lambda > 0$, we get $\tilde{a} \in C(J)$, $\tilde{a} \geqslant 0$, and

$$\int_0^T e^{M(T-s)} \tilde{a}(s)\, ds = \frac{1}{2}(1 - 2\lambda e) + \frac{-1 + 6\lambda e}{2e}.$$

If we choose $\lambda = 2.42775$, then $\int_0^T e^{M(T-s)} \tilde{a}(s)\, ds \geqslant 1$. Function $\tilde{a}$ is not comparable with $a(t) = e^{-M(T-t)} \frac{1}{T}$, as we can deduce from Fig. 1.

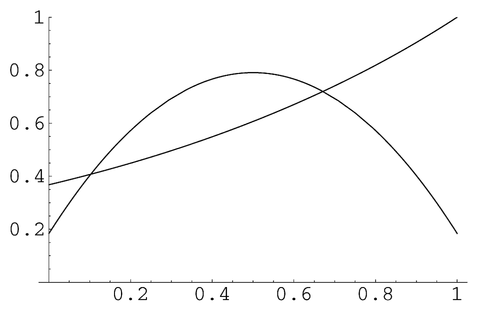

Fig. 1. a and $\tilde{a}$.

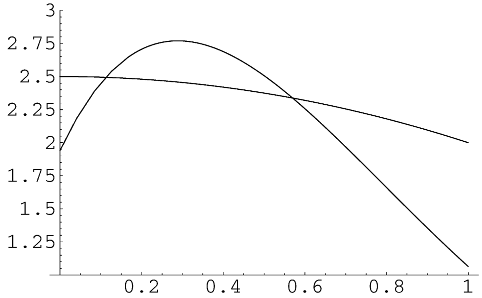

Fig. 2. ϕ and φ.

Taking $K(t,s)=1$, $H(t,s)=1$, $N_1=1$, $N_2=1$, and

$$\tilde{g}(t)=\int_t^T e^{M(T-s)}\tilde{a}(s)\,ds,$$

we obtain that

$$\varphi(t)=-\tilde{g}'(t)+M\tilde{g}(t)+N_1\int_0^t K(t,s)\tilde{g}(s)\,ds+N_2\int_0^T H(t,s)\tilde{g}(s)\,ds$$

is not comparable to function $\phi(t)=\frac{5}{2}-\frac{t^2}{2}$ defined in Remark 5 (see Fig. 2).

Moreover, if $t_0=0<t_1=\frac{1}{2}<t_2=1$ and $L_1\geqslant 0$, then

$$L_1\int_{1/2}^1 e^{M(T-s)}\tilde{a}(s)\,ds=0.39788L_1<\frac{1}{2}L_1=L_1\frac{T-t_1}{T}.$$

Example 2. Take $T=2$, $M=1$, and $\tilde{a}$ as in Example 1. In this case,

$$\int_0^T e^{M(T-s)}\tilde{a}(s)\,ds=\frac{1}{4}+\frac{-1+16\lambda e^2}{4e^2}.$$

If we choose $\lambda=0.195959$, then $\int_0^T e^{M(T-s)}\tilde{a}(s)\,ds\geqslant 1$. Function $\tilde{a}$ is not comparable with $a(t)=e^{-M(T-t)}\frac{1}{T}$, as we can see in Fig. 3.

Take $K(t,s)=1$, $H(t,s)=1$, $N_1=1$, $N_2=1$, and $\tilde{g}$ according to Example 1. In this case, φ is not comparable to $\phi(t)=\frac{5}{2}+\frac{t}{2}-\frac{t^2}{4}$ (see Fig. 4).

Besides, if $t_0=0<t_1=1<t_2=2$ and $L_1\geqslant 0$, then

$$L_1\int_1^2 e^{M(T-s)}\tilde{a}(s)\,ds=0.3093L_1<\frac{1}{2}L_1=L_1\frac{T-t_1}{T}.$$

Example 3. Take $T=2$, $M=\frac{1}{2}$, and $\tilde{a}$ as in the previous examples. If we choose $\lambda=0.303469$, then $\int_0^T e^{M(T-s)}\tilde{a}(s)\,ds\geqslant 1$. Function $\tilde{a}$ is not comparable with $a(t)=e^{-M(T-t)}\frac{1}{T}$, as we can see in Fig. 5.

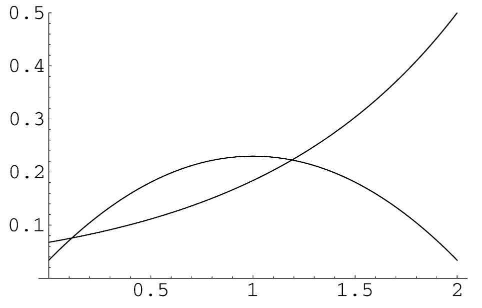

Fig. 3. a and $\tilde{a}$.

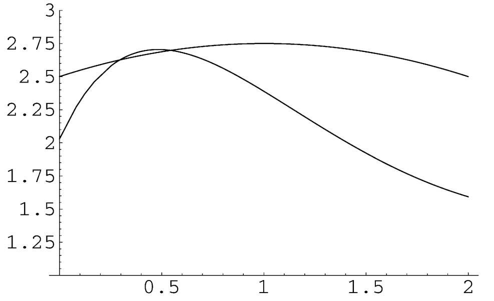

Fig. 4. ϕ and φ.

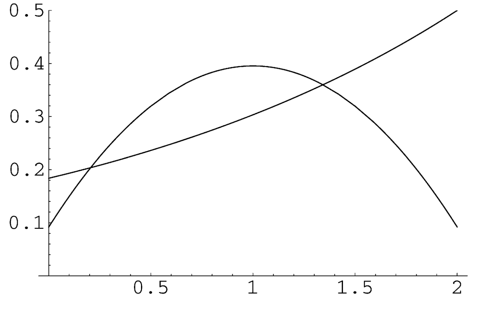

Fig. 5. a and $\tilde{a}$.

For $K(t,s) = (t-s)^2$, $H(t,s) = (t-s)$, $N_1 = 1.2$, $N_2 = 0.3$, and $\tilde{g}$ as in Example 1, φ is not comparable to ϕ (see Fig. 6).

Besides, if $t_0 = 0 < t_1 = 1 < t_2 = 2$ and $L_1 \geqslant 0$, then

$$L_1 \int_1^2 e^{M(T-s)} \tilde{a}(s)\, ds = 0.39788 L_1 < \frac{1}{2} L_1 = L_1 \frac{T - t_1}{T}.$$

However, in these examples, we have not used the potential of the new comparison results. Function g should satisfy $g \in PC^1(J)$, $g \geqslant 0$ in $[0, T]$,

$$g(0) - g(T) \geqslant m(T) - m(0) > 0.$$

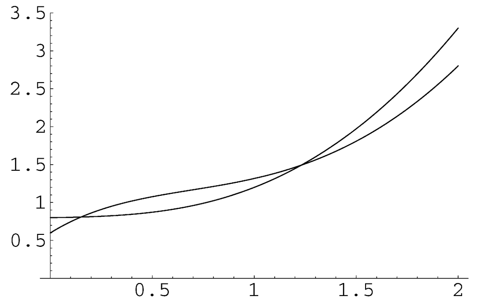

Fig. 6. ϕ and φ.

There is no need to consider just the identity in last inequality and g is not obliged to be continuous. For $g(t)=\frac{T-t}{T}(m(T)-m(0))$, these two restrictions are satisfied simultaneously, thus this choice is very restrictive. We can take other options for function g which are discontinuous in J, and continuous on each J_k, which allows to adapt the study to each subinterval J_k, obtaining conclusions which justify that the results in this paper improve substantially previous results.

Example 4. Take $T=4$, $t_0=0<t_1=1<t_2=3<t_3=4$, $M=\frac{1}{2}$, $K(t,s)=\frac{1}{2}e^{-M(t-s)}$, $H(t,s)=\frac{1}{2}$, $N_1=0.2$, $N_2=0.3$, and $g(t)=\gamma(t)(m(T)-m(0))$, where

$$\gamma(t)=\begin{cases} 1-\frac{t}{4}, & \text{if } t\in[0,1],\\ \frac{3}{4}-\frac{t}{4}, & \text{if } t\in(1,3],\\ 0, & \text{if } t\in(3,4]. \end{cases}$$

Note that $g\in PC^1(J)$, $g\geqslant 0$ in $[0,T]$, and

$$g(0)-g(T)=\big(\gamma(0)-\gamma(T)\big)\big(m(T)-m(0)\big)=m(T)-m(0).$$

Here

$$\phi(t)=\frac{1}{T}\left(M(T-t)+1+N_1\int_0^t K(t,s)(T-s)\,ds+N_2\int_0^T H(t,s)(T-s)\,ds\right)$$

is greater than the piecewise continuous function

$$\frac{1}{m(T)-m(0)}\left\{-g'(t)+Mg(t)+N_1\int_0^t K(t,s)g(s)\,ds+N_2\int_0^T H(t,s)g(s)\,ds\right\}$$
$$=-\gamma'(t)+M\gamma(t)+N_1\int_0^t K(t,s)\gamma(s)\,ds+N_2\int_0^T H(t,s)\gamma(s)\,ds,$$

see Fig. 7.

Example 5. In the context of Example 4, consider problem (1), for

$$f(t,u,x,y)=\frac{1}{4}\left(e^{-t}-\frac{1}{2}\cos(u)-\frac{1}{5}x+y\right),$$

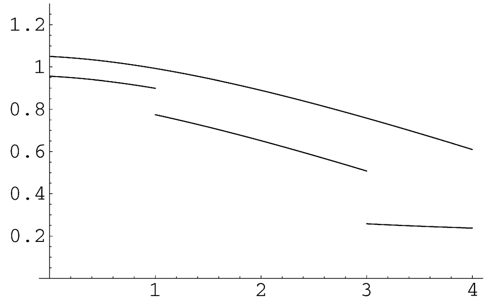

Fig. 7. ϕ is the continuous function.

that is,

$$\begin{cases} u'(t) = f\big(t, u(t), [\mathcal{T}u](t), [\mathcal{S}u](t)\big), & t \in J_0, \\ \Delta u(t_k) = I_k\big(u(t_k)\big), & k = 1, 2, \\ u(0) = u(T), \end{cases} \tag{16}$$

where $T = 4$, $J = [0, 4]$, $0 = t_0 < t_1 = 1 < t_2 = 3 < t_3 = 4$, $J_0 = J \setminus \{1, 3\}$, $I_1(x) = \frac{1}{3}x$, $L_1 = \frac{1}{3}$, $I_2(x) = \frac{1}{2}x$, $L_2 = \frac{1}{2}$, $K(t, s) = \frac{1}{2}e^{-M(t-s)}$, $H(t, s) = \frac{1}{2}$, $M = \frac{1}{2}$, $N_1 = 0.2$, $N_2 = 0.3$, and consider functions $\alpha, \beta : J \to \mathbb{R}$ given by

$$\alpha(t) = \begin{cases} 2.2, & \text{if } t \in [0, 1], \\ 1.6, & \text{if } t \in (1, 3], \\ 1.2, & \text{if } t \in (3, 4], \end{cases} \qquad \beta(t) = \begin{cases} -3.3, & \text{if } t \in [0, 1], \\ -2.6, & \text{if } t \in (1, 3], \\ -2.3, & \text{if } t \in (3, 4], \end{cases}$$

which satisfy $\alpha(0) > \alpha(4)$, $\beta(0) < \beta(4)$, and $\beta \leqslant \alpha$ on J. Taking $g(t) = \gamma(t)(\alpha(0) - \alpha(4)) = \gamma(t)$, $t \in J$, and $\tilde{g}(t) = \gamma(t)(\beta(4) - \beta(0)) = \gamma(t)$, $t \in J$, where γ is given in Example 4, it is easy to check that α and β are, respectively, lower and upper solutions to problem (16), according to Definition 1, but they do not satisfy properties in Remark 7. Indeed, for α to be a lower solution, it is necessary that

$$\begin{aligned} &-\gamma'(t) + M\gamma(t) + N_1 \int_0^t K(t, s)\gamma(s)\,ds + N_2 \int_0^T H(t, s)\gamma(s)\,ds \\ &\quad \leqslant f\left(t, \alpha(t), \int_0^t K(t, s)\alpha(s)\,ds, \int_0^T H(t, s)\alpha(s)\,ds\right), \end{aligned}$$

which is satisfied. However, it is not valid that

$$\begin{aligned} \phi(t) &= \frac{1}{T}\left(M(T - t) + 1 + N_1 \int_0^t K(t, s)(T - s)\,ds + N_2 \int_0^T H(t, s)(T - s)\,ds\right) \\ &\leqslant f\left(t, \alpha(t), \int_0^t K(t, s)\alpha(s)\,ds, \int_0^T H(t, s)\alpha(s)\,ds\right). \end{aligned}$$

See Fig. 8.

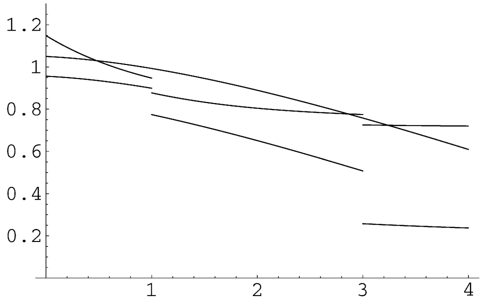

Fig. 8. ϕ is the continuous function.

Besides,

$$\begin{aligned}
&I_1\big(\alpha(t_1)\big) - \big(L_1 g(t_1) - \Delta g(t_1)\big)\\
&\quad= \frac{1}{3}\alpha(1) - \big(L_1 g(1) - \Delta g(1)\big)\\
&\quad= \frac{1}{3}2.2 - \left(\frac{1}{3}\gamma(1) - \gamma(1^+) + \gamma(1^-)\right) = \frac{1}{3}2.2 - \frac{1}{4} + \frac{1}{2} - \frac{3}{4} = \frac{1}{3}2.2 - \frac{1}{2}\\
&\quad\geqslant \Delta\alpha(t_1) = \alpha(1^+) - \alpha(1^-) = 1.6 - 2.2 = -0.6
\end{aligned}$$

and

$$\begin{aligned}
I_2\big(\alpha(t_2)\big) - \big(L_2 g(t_2) - \Delta g(t_2)\big) &= \frac{1}{2}\alpha(3) - \big(L_2 g(3) - \Delta g(3)\big)\\
&= \frac{1}{2}1.6 - \left(\frac{1}{2}\gamma(3) - \gamma(3^+) + \gamma(3^-)\right) = \frac{1}{2}1.6\\
&\geqslant \Delta\alpha(t_2) = \alpha(3^+) - \alpha(3^-) = 1.2 - 1.6 = -0.4.
\end{aligned}$$

Now, for β to be an upper solution, we check that the following inequality is satisfied

$$\begin{aligned}
&-\gamma'(t) + M\gamma(t) + N_1\int_0^t K(t,s)\gamma(s)\,ds + N_2\int_0^T H(t,s)\gamma(s)\,ds\\
&\quad\leqslant -f\left(t, \beta(t), \int_0^t K(t,s)\beta(s)\,ds, \int_0^T H(t,s)\beta(s)\,ds\right).
\end{aligned}$$

However, it is not valid that

$$\begin{aligned}
\phi(t) &= \frac{1}{T}\left(M(T-t) + 1 + N_1\int_0^t K(t,s)(T-s)\,ds + N_2\int_0^T H(t,s)(T-s)\,ds\right)\\
&\leqslant -f\left(t, \beta(t), \int_0^t K(t,s)\beta(s)\,ds, \int_0^T H(t,s)\beta(s)\,ds\right).
\end{aligned}$$

See Fig. 9.

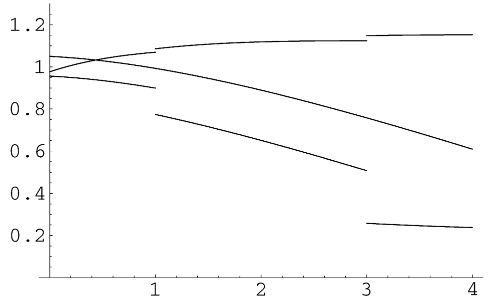

Fig. 9. ϕ is the continuous function.

Moreover,

$$\begin{aligned}
&I_1\big(\beta(t_1)\big)+\big(L_1\tilde g(t_1)-\Delta\tilde g(t_1)\big)\\
&\quad=\frac{1}{3}\beta(1)+\big(L_1\tilde g(1)-\Delta\tilde g(1)\big)\\
&\quad=\frac{1}{3}(-3.3)+\left(\frac{1}{3}\gamma(1)-\gamma(1^+)+\gamma(1^-)\right)=\frac{1}{3}(-3.3)+\frac{1}{4}-\frac{1}{2}+\frac{3}{4}=\frac{1}{3}(-3.3)+\frac{1}{2}\\
&\quad\leqslant\Delta\beta(t_1)=\beta(1^+)-\beta(1^-)=-2.6+3.3=0.7,
\end{aligned}$$

and

$$\begin{aligned}
&I_2\big(\beta(t_2)\big)+\big(L_2\tilde g(t_2)-\Delta\tilde g(t_2)\big)\\
&\quad=\frac{1}{2}\beta(3)+\big(L_2\tilde g(3)-\Delta\tilde g(3)\big)\\
&\quad=\frac{1}{2}(-2.6)+\left(\frac{1}{2}\gamma(3)-\gamma(3^+)+\gamma(3^-)\right)=\frac{1}{2}(-2.6)\\
&\quad\leqslant\Delta\beta(t_2)=\beta(3^+)-\beta(3^-)=-2.3+2.6=0.3.
\end{aligned}$$

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