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J. Math. Anal. Appl. 338 (2008) 1351–1363

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Common fixed points in best approximation for Banach operator pairs with Ćirić type I-contractions

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Received 16 March 2007

Available online 15 June 2007

Submitted by Richard M. Aron

Abstract

The common fixed point theorems, similar to those of Ćirić [Lj.B. Ćirić, On a common fixed point theorem of a Gregus type, Publ. Inst. Math. (Beograd) (N.S.) 49 (1991) 174–178; Lj.B. Ćirić, On Diviccaro, Fisher and Sessa open questions, Arch. Math. (Brno) 29 (1993) 145–152; Lj.B. Ćirić, On a generalization of Gregus fixed point theorem, Czechoslovak Math. J. 50 (2000) 449–458], Fisher and Sessa [B. Fisher, S. Sessa, On a fixed point theorem of Gregus, Internat. J. Math. Math. Sci. 9 (1986) 23–28], Jungck [G. Jungck, On a fixed point theorem of Fisher and Sessa, Internat. J. Math. Math. Sci. 13 (1990) 497–500] and Mukherjee and Verma [R.N. Mukherjee, V. Verma, A note on fixed point theorem of Gregus, Math. Japon. 33 (1988) 745–749], are proved for a Banach operator pair. As applications, common fixed point and approximation results for Banach operator pair satisfying Ćirić type contractive conditions are obtained without the assumption of linearity or affinity of either T or I. Our results unify and generalize various known results to a more general class of noncommuting mappings.

Keywords: Common fixed points; Compatible maps; Banach operator pair; Contractive condition of Ćirić type; Best approximation

1. Introduction and preliminaries

Let M be a subset of a normed space $(X, \|.\|)$. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called I-Lipschitz if there exists $k \geqslant 0$ such that $d(Tx, Ty) \leqslant kd(Ix, Iy)$ for any $x, y \in M$. If $k < 1$ (respectively $k = 1$), then T is called an I-contraction (respectively I-nonexpansive). A point $x \in M$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ $(x = Ix = Tx)$. The set of fixed points of I is denoted by $F(I)$. The set of coincidence points of I and T is denoted by $C(I, T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$; (2) R-weakly commuting if for all $x \in M$, there exists $R > 0$ such that $d(ITx, TIx) \leqslant Rd(Ix, Tx)$. If $R = 1$, then the maps are called weakly commuting; (3) compatible [18] if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M; (4) weakly compatible if they commute at their coincidence points, i.e., if $ITx = TIx$ whenever $Ix = Tx$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called (5) C_q-commuting if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \bigcup\{C(I, T_k)\colon 0 \leqslant k \leqslant 1\}$ where $T_k x = (1 - k)q + kTx$; (6) R-subweakly commuting on M if for all $x \in M$, there exists a real number $R > 0$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.06.008

such that $\|ITx - TIx\| \leqslant R \operatorname{dist}(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx\colon 0 \leqslant k \leqslant 1\}$. If $R = 1$, then the maps are called 1-subweakly commuting [14]. The set $P_M(u) = \{x \in M\colon \|x - u\| = \operatorname{dist}(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $\operatorname{dist}(u, M) = \inf\{\|y - u\|\colon y \in M\}$. Let $C_C^I(u) = \{x \in C\colon Ix \in P_C(u)\}$. We shall use N to denote the set of positive integers, $\operatorname{cl}(S)$ to denote the closure of a set S and $w\operatorname{cl}(S)$ to denote the weak closure of a set S. We denote by $\Im_0$ (*respectively* $\Im_0^w$) *the class of closed* (*respectively weakly closed*) *convex subsets of X containing* 0 [1,20]. For $M \in \Im_0$, we define $M_u = \{x \in M\colon \|x\| \leqslant 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ whenever $M \in \Im_0$. A Banach space X satisfies Opial's condition if, for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. Every Hilbert space and the space l_p $(1 < p < \infty)$ satisfy Opial's condition. The map $T : M \to X$ is said to be demiclosed at 0 if, for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converges to $0 \in X$, then $0 = Tx$.

In [9], Fisher and Sessa obtained the following generalization of a theorem of Gregus [10].

Theorem 1.1. *Let T and I be two weakly commuting mappings on a closed convex subset C of a Banach space X into itself satisfying the inequality*

$$\|Tx - Ty\| \leqslant a\|Ix - Iy\| + (1-a)\max\bigl\{\|Tx - Ix\|, \|Ty - Iy\|\bigr\}, \tag{1.1}$$

for all $x, y \in C$, where $a \in (0, 1)$. If I is linear and nonexpansive on C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C.

Later, Jungck [19] obtained the following generalization of Theorem 1.1.

Theorem 1.2. *Let T and I be compatible self maps of a closed convex subset C of a Banach space X. Suppose that I is continuous, linear and that $T(C) \subseteq I(C)$. If T and I satisfy inequality* (1.1), *then T and I have a unique common fixed point in C.*

Theorem 1.3. *(See Ćirić [5].) Let C be as in Theorem* 1.2 *and T and I be two compatible self mappings of C satisfying* (1.1). *If I is continuous and $Co[T(C)] \subseteq I(C)$ (Co = convex hull), then T and I have a unique common fixed point.*

Generalizing Theorem 1.1 of Gregus [10], Ćirić [6] proved the following:

Theorem 1.4. *Let T and I be two compatible self mappings of a closed convex subset C of a Banach space X satisfying the inequality*

$$\|Tx - Ty\|^p \leqslant a\|Ix - Iy\|^p + (1-a)\max\bigl\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\bigr\}, \tag{1.2}$$

for all $x, y \in C$, where $0 < a < 1/2^{p-1}$ and $p \geqslant 1$. If I is continuous and $Co[T(C)] \subseteq I(C)$ (Co = convex hull), then T and I have a unique common fixed point.

Many results which are closely related to Gregus's Theorem have appeared in recent years (see [5–7,28]). Recently, Chen and Li [3] introduced the class of Banach operator pairs, as a new class of noncommuting maps. The purpose of this paper is to prove similar results for a newly defined class of Banach operator pairs. We shall prove our results without the assumptions of linearity or affinity of either T or I and nonexpansiveness of I. As applications, common fixed point and invariant approximation results for this class of maps are also derived. Our results extend, unify and compliment the work of Al-Thagafi [1], Chen and Li [3], Habiniak [11], Jungck and Sessa [21], Khan et al. [23] and [24], Khan and Khan [25], Meinardus [27], Sahab, Khan and Sessa [33], Shahzad [35], Singh [36], Smoluk [38] and Subrahmanyam [39].

2. Banach operator pair in Banach spaces

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a Banach operator pair, if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously commuting pair (T, I) is a Banach operator pair but not conversely in general, see [3] and Example 2.3 below. If (T, I) is a Banach operator pair then (I, T) need not be Banach operator pair (cf. Example 1 [3]). If the self-maps T and I of X satisfy

$$d(ITx, Tx) \leqslant kd(Ix, x),$$

for all $x \in X$ and $k \geqslant 0$, then (T, I) is a Banach operator pair. In particular, when $I = T$ and X is a normed space, the above inequality can be rewritten as

$$\|T^2x - Tx\| \leqslant k\|Tx - x\|$$

for all $x \in X$. Such T is called a Banach operator of type k in [39] (see [11] and [25]). In this section we improve and extend the recent results of Chen and Li [3] and as an application, we establish more general approximation results without the condition of linearity or affinity of I which is key assumption in the results of Al-Thagafi [1], Hussain and Khan [14], Hussain and Jungck [13], Jungck and Hussain [20], Jungck and Sessa [21], O'Regan and Hussain [30], Sahab et al. [33] and Shahzad [35].

In 2000, Ćirić [7] introduced the following more general contractive condition and improved the Gregus theorem

$$\|Tx - Ty\| \leqslant a \max\{\|x - y\|, c(\|x - Ty\| + \|y - Tx\|)\} + b \max\{\|x - Tx\|, \|y - Ty\|\},$$

where $a \in (0, 1)$, $a + b = 1$, $0 \leqslant c < \eta$, and $\eta = \min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$.

We begin with the following result, which extends and improves Lemma 3.1 of [3], main theorem in [17] and Theorem 1 in [25].

Lemma 2.1. *Let C be a nonempty closed convex subset of a Banach space X, and (T, I) be a Banach operator pair on C. Assume that T and I satisfy*

$$\|Tx - Ty\| \leqslant a \max\{\|Ix - Iy\|, c(\|Ix - Ty\| + \|Iy - Tx\|)\} + b \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \tag{2.1}$$

where $a \in (0, 1)$, $a + b = 1$, $0 \leqslant c < \eta$, and $\eta = \min\{\frac{2+a}{5+a}, \frac{2-a}{4}, \frac{4}{9+a}\}$. If I is continuous, $F(I)$ is nonempty and convex, then there is a unique common fixed point of T and I.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty closed and convex. Further for all $x, y \in F(I)$, we have by inequality (2.1)

$$\begin{aligned}\|Tx - Ty\| &\leqslant a \max\{\|Ix - Iy\|, c(\|Ix - Ty\| + \|Iy - Tx\|)\} + b \max\{\|Tx - Ix\|, \|Ty - Iy\|\}\\ &= a \max\{\|x - y\|, c(\|x - Ty\| + \|y - Tx\|)\} + b \max\{\|Tx - x\|, \|Ty - y\|\}.\end{aligned}$$

By Theorem 3 of Ćirić [7], T has a unique fixed point y in $F(I)$ and consequently $C \cap F(T) \cap F(I)$ is a singleton. □

Let $X = \mathbb{R}$ with usual norm and $C = [0, 1]$. Let $T(x) = 1$ for $0 \leqslant x \leqslant 1/2$, and $T(x) = 0$ for $1/2 < x \leqslant 1$, $I(x) = 0$ for $0 \leqslant x \leqslant 1/2$, and $I(x) = 1$ for $1/2 < x \leqslant 1$. Then all the assumptions of Lemma 2.1 are satisfied [6] except that $F(I) = \{0, 1\}$ is convex, but T and I have no common fixed point.

The following result generalizes Theorem 2.2 in [1], Theorem 3.3 in [3], Theorem 4 in [11] and corresponding results in [23] and [24] to maps satisfying a more general inequality.

Theorem 2.2. *Let C be a nonempty closed convex subset of a Banach space X and I and T be self maps of C. Suppose that $F(I)$ is nonempty and convex and (T, I) is a continuous Banach operator pair. If $\mathrm{cl}(T(C))$ is compact and satisfies*

$$\begin{aligned}\|Tx - Ty\| \leqslant{}& \max\{\|Ix - Iy\|, c[\operatorname{dist}(Ix, [q, Ty]) + \operatorname{dist}(Iy, [q, Tx])]\}\\ &+ \frac{1-k}{k} \max\{\operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty])\}\end{aligned} \tag{2.2}$$

for some $q \in F(I) \cap C$, for all $x, y \in C$, all $k \in (0, 1)$, and $0 \leqslant c < 0.25$, then $C \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Define $T_n : C \to C$ by $T_n x = (1-k_n)q + k_n Tx$ for some $q \in C \cap F(I)$ and all $x \in C$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. As (T, I) is a Banach operator pair and $F(I)$ is convex, so that for each $x \in F(I)$, $T_n x = (1-k_n)q + k_n Tx \in F(I)$, since $Tx \in F(I)$. Thus (T_n, I) is a Banach operator pair for each n. Also by (2.2),

$$\begin{aligned}
\|T_n x - T_n y\| &= k_n \|Tx - Ty\| \\
&\leqslant k_n \Big\{ \max\big\{\|Ix - Iy\|, c\big[\operatorname{dist}\big(Ix, [q, Ty]\big) + \operatorname{dist}\big(Iy, [q, Tx]\big)\big]\big\} \\
&\quad + \frac{1-k_n}{k_n} \max\big\{\operatorname{dist}\big(Ix, [q, Tx]\big), \operatorname{dist}\big(Iy, [q, Ty]\big)\big\}\Big\} \\
&\leqslant k_n \Big\{ \max\big\{\|Ix - Iy\|, c\big[\|Ix - T_n y\| + \|Iy - T_n x\|\big]\big\} \\
&\quad + \frac{1-k_n}{k_n} \max\big\{\|Ix - T_n x\|, \|Iy - T_n y\|\big\}\Big\} \\
&= k_n \max\big\{\|Ix - Iy\|, c\big[\|Ix - T_n y\| + \|Iy - T_n x\|\big]\big\} \\
&\quad + (1-k_n) \max\big\{\|Ix - T_n x\|, \|Iy - T_n y\|\big\},
\end{aligned}$$

for each $x, y \in C$, $0 < k_n < 1$ and $0 \leqslant c < 0.25$. By Lemma 2.1, for each $n \geqslant 1$, there exists $x_n \in C$ such that $x_n = Ix_n = T_n x_n$. The compactness of $\operatorname{cl}(T(C))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ and $z \in \operatorname{cl}(T(C))$ such that $Tx_m \to z$ as $m \to \infty$. Since $k_m \to 1$, $x_m = T_m x_m = (1-k_m)q + k_m T x_m$ also converges to z. By the continuity of T and I, we obtain that $C \cap F(I) \cap F(T) \neq \emptyset$. $\square$

Example 2.3. Consider $M = \mathbb{R}^2$ with the norm $\|(x, y)\| = |x| + |y|$, $(x, y) \in \mathbb{R}^2$. Define T and I on M as follows:

$$\begin{aligned}
T(x, y) &= \left(\frac{1}{2}(x-2), \frac{1}{2}\big(x^2 + y - 4\big)\right), \\
I(x, y) &= \left(\frac{1}{2}(x-2), x^2 + y - 4\right).
\end{aligned}$$

Obviously, T is I-nonexpansive but I is not linear. Moreover, $F(T) = \{-2, 0\}$, $F(I) = \{(-2, y)\colon\ y \in R\}$ and $C(I, T) = \{(x, y)\colon\ y = 4 - x^2,\ x \in R\}$. Thus (T, I) is a continuous Banach operator pair, which is not compatible pair [3], $F(I)$ is convex and $(-2, 0)$ is a common fixed point of I and T.

The following result extends Theorem 3.2 of Chen and Li [3], Theorem 6 of Jungck and Sessa [21], and corresponding results in [23] and [24].

Theorem 2.4. *Let C be a nonempty convex subset of a Banach space X and I and T be self maps of C. Suppose that $w\operatorname{cl}(T(C))$ is weakly compact, I is strongly and weakly continuous and $F(I)$ is nonempty and convex. If the pair (T, I) is a Banach operator pair and satisfies* (2.2)*, for some $q \in F(I) \cap C$, $0 \leqslant c < 0.25$, for all $x, y \in C$, and all $k \in (0, 1)$, then $C \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following two conditions is satisfied*

(i) *$I - T$ is demiclosed at* 0;
(ii) *X satisfies Opial's condition and $c = 0$ in* (2.2).

Proof. Let $\{k_n\}$ and $\{T_n\}$ be defined as in Theorem 2.2. The analysis in Theorem 2.2, guarantees that there exists $x_n \in C$ such that $x_n = Ix_n = T_n x_n$.

(i) By the weak compactness of $w\operatorname{cl}(T(C))$, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ which converges weakly to $y_0 \in C$ as $m \to \infty$. As I is weakly continuous, $Iy_0 = y_0$. Since $\{x_m\}$ is bounded, $k_m \to 1$, and

$$\big\|(I-T)(x_m)\big\| = \|x_m - Tx_m\| = \big\|\big((1-k_m)q + k_m Tx_m\big) - Tx_m\big\| \leqslant (1-k_m)\big(\|q\| + \|Tx_m\|\big).$$

Thus $Ix_m - Tx_m \to 0$ as $m \to \infty$.

Suppose $(I-T)$ is demiclosed at 0. Then $(I-T)y_0=0$ and hence $Ty_0=Iy_0=y_0$.
(ii) If $Iy_0 \neq Ty_0$, then

$$\liminf_{m\to\infty}\|Ix_m - Iy_0\| < \liminf_{m\to\infty}\|Ix_m - Ty_0\|, \tag{2.3}$$

$$\begin{aligned}\|Ix_m - Ty_0\| &\leqslant \|Ix_m - Tx_m\| + \|Tx_m - Ty_0\| \\ &\leqslant \|Ix_m - Tx_m\| + \|Ix_m - Iy_0\| + \frac{1-k_m}{k_m}\max\bigl\{\|Ix_m - Tx_m\|, \|Iy_0 - Ty_0\|\bigr\} \\ &= \|Ix_m - Iy_0\| + \|Ix_m - Tx_m\| + \frac{1-k_m}{k_m}\|Iy_0 - Ty_0\| \\ &= \|Ix_m - Iy_0\| + r_m,\end{aligned}$$

where

$$r_m = \|Ix_m - Tx_m\| + \frac{1-k_m}{k_m}\|Iy_0 - Ty_0\|.$$

Since $\lim_{m\to\infty} r_m = 0$,

$$\liminf_{m\to\infty}\|Ix_m - Ty_0\| \leqslant \liminf_{m\to\infty}\|Ix_m - Iy_0\|$$

which leads by (2.3) to

$$\liminf_{m\to\infty}\|Ix_m - Iy_0\| < \liminf_{m\to\infty}\|Ix_m - Iy_0\|,$$

which is a contradiction. Thus $Ty_0 = Iy_0 = y_0$ and hence $C \cap F(I) \cap F(T) \neq \emptyset$. $\square$

The following result extends Theorem 3.2 in [1], Theorems 4.1–4.2 of [3], Theorem 7 of [21], Theorem 3 in [33], the corresponding results of Khan et al. [23] and [24], Khan and Khan [25], Singh [36], Smoluk [38] and Subrahmanyam [39].

Theorem 2.5. *Let C be a subset of a Banach space X and $I, T : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(\partial C \cap C) \subseteq C$. Suppose that $P_C(u)$ and $F(I)$ are nonempty and convex, I is continuous on $P_C(u)$, and $I(P_C(u)) \subseteq P_C(u)$. If (T, I) is a Banach operator pair on $P_C(u)$ and satisfies, for some $q \in F(I) \cap P_C(u)$, $0 \leqslant c < 0.25$, all $x \in P_C(u) \cup \{u\}$ and $k \in (0,1)$*

$$\|Tx - Ty\| \leqslant \begin{cases} \|Ix - Iu\|, & \text{if } y = u, \\ \max\{\|Ix - Iy\|, c[\operatorname{dist}(Ix, [q, Ty]) + \operatorname{dist}(Iy, [q, Tx])]\} & \\ \quad + \frac{1-k}{k}\max\{\operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty])\}, & \text{if } y \in P_C(u), \end{cases} \tag{2.4}$$

then $P_C(u) \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions is satisfied

(i) *T is continuous and* $\operatorname{cl}(T(P_C(u)))$ *is compact,*
(ii) $w\operatorname{cl}(T(P_C(u)))$ *is weakly compact, I is weakly continuous and $I - T$ is demiclosed at* 0,
(iii) $w\operatorname{cl}(T(P_C(u)))$ *is weakly compact, I is weakly continuous and X satisfies Opial's condition and $c = 0$ in* (2.4).

Proof. Let $x \in P_C(u)$. Then for any $h \in (0,1)$, $\|hu + (1-h)x - u\| = (1-h)\|x-u\| < \operatorname{dist}(u, C)$. It follows that the line segment $\{hu + (1-h)x\colon\ 0 < h < 1\}$ and the set C are disjoint. Thus x is not in the interior of C and so $x \in \partial C \cap C$. Since $T(\partial C \cap C) \subseteq C$, Tx must be in C. Also since $Ix \in P_C(u)$, $u \in F(I) \cap F(T)$ and I and T satisfy (2.4), we have

$$\|Tx - u\| = \|Tx - Tu\| \leqslant \|Ix - Iu\| = \|Ix - u\| = \operatorname{dist}(u, C).$$

Thus $Tx \in P_C(u)$. Therefore T is a self map of $P_C(u)$. Notice that $P_C(u)$ is closed, the result now follows from Theorems 2.2 and 2.4.

For $h \geqslant 0$, let $D_C^{h,I}(u) = P_C(u) \cap G_C^{h,I}(u)$, where $G_C^{h,I}(u) = \{x \in C\colon\ \|Ix - u\| \leqslant (2h+1)\operatorname{dist}(u, C)\}$. For the relations of the sets $D_C^{h,I}(u)$, $P_C(u)$ and $C_C^I(u)$ we refer the reader to [1,13,20] and references therein. $\square$

Theorem 2.6. *Let C be a subset of a Banach space X and $I, T : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(\partial C \cap C) \subseteq C$. Suppose that I is continuous on the closed convex set $D_C^{h,I}(u)$, $D_C^{h,I}(u) \cap F(I)$ is nonempty convex and $I(D_C^{h,I}(u)) \subseteq D_C^{h,I}(u)$. If the pair (T, I) satisfies*

(a) $\|ITx - Tx\| \leqslant h\|Ix - x\|$ *for all* $x \in D_C^{h,I}(u)$ *and* $h \geqslant 0$;
(b) *for all* $x \in D_C^{h,I}(u) \cup \{u\}$, *and* $k \in (0,1)$,

$$\|Tx - Ty\| \leqslant \begin{cases} \|Ix - Iu\|, & \text{if } y = u, \\ \max\{\|Ix - Iy\|, c[\operatorname{dist}(Ix, [q, Ty]) + \operatorname{dist}(Iy, [q, Tx])\} & \\ \quad + \frac{1-k}{k}\max\{\operatorname{dist}(Ix, [q, Tx]), \operatorname{dist}(Iy, [q, Ty])\}, & \text{if } y \in D_C^{h,I}(u), \end{cases} \tag{2.5}$$

where $q \in F(I) \cap P_C(u)$ and $0 \leqslant c < 0.25$, then $P_C(u) \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions is satisfied

(i) $\operatorname{cl}(T(D_C^{h,I}(u)))$ *is compact, and T is continuous,*
(ii) $w\operatorname{cl}(T(D_C^{h,I}(u)))$ *is weakly compact, I is weakly continuous and $I - T$ is demiclosed at* 0,
(iii) $w\operatorname{cl}(T(D_C^{h,I}(u)))$ *is weakly compact, I is weakly continuous, X satisfies Opial's condition and* $c = 0$.

Proof. Let $x \in D_C^{h,I}(u)$. Then $x \in P_C(u)$ and as in the proof of Theorem 2.5, Tx is in M. Also since $Ix \in P_C(u)$, $u \in F(I) \cap F(T)$ and I and T satisfy (2.4) we have

$$\|Tx - u\| = \|Tx - Tu\| \leqslant \|Ix - Iu\| = \|Ix - u\| = \operatorname{dist}(u, C).$$

Thus $Tx \in P_C(u)$. From inequality in (a) and (2.5), it follows that

$$\begin{aligned} \|ITx - u\| &= \|ITx - Tx + Tx - u\| \\ &\leqslant \|ITx - Tx\| + \|Tx - u\| \\ &\leqslant h\|Ix - x\| + \|Tx - u\| \\ &= h\|Ix - u + u - x\| + \|Tx - u\| \\ &\leqslant h\big(\|Ix - u\| + \|x - u\|\big) + \|Tx - u\| \\ &\leqslant h\big(\operatorname{dist}(u, C) + \operatorname{dist}(u, C)\big) + \operatorname{dist}(u, C) \\ &\leqslant (2h+1)\operatorname{dist}(u, C). \end{aligned}$$

Thus $Tx \in G_C^{h,I}(u)$. Consequently, $T(D_C^{h,I}(u)) \subset D_C^{h,I}(u)$. Inequality in (a) also implies that (T, I) is a Banach operator pair. Now by Theorems 2.2 and 2.4, we obtain $P_C(u) \cap F(I) \cap F(T) \neq \emptyset$ in each of the cases (i)–(iii). $\square$

The following result extends Theorem 4.2 in [1], Theorem 8 in [11], Theorem 2.1 [35] and corresponding results in [38] and [39].

Theorem 2.7. *Let I and T be self maps of a Banach space X with $u \in F(I) \cap F(T)$ and $C \in \Im_0$ such that $T(C_u) \subseteq I(C) \subseteq C$. Suppose that $\operatorname{cl}(I(C_u))$ is compact, $\|Ix - u\| = \|x - u\|$ for all $x \in C_u$, T, I are continuous on C_u, T satisfies $\|Tx - u\| \leqslant \|Ix - u\|$ for all $x \in C_u$. Then*

(i) $P_C(u)$ *is nonempty, closed and convex,*
(ii) $T(P_C(u)) \subseteq I(P_C(u)) \subseteq P_C(u)$, *provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_C^I(u)$, and*
(iii) $P_C(u) \cap F(I) \cap F(T) \neq \emptyset$ *provided that $\|Ix - u\| = \|x - u\|$ for all $x \in C_C^I(u)$, $F(I)$ is nonempty and convex, (T, I) is Banach operator pair on $P_C(u)$ and T satisfies* (2.2) *for all $q \in F(I) \cap P_C(u)$, $x, y \in P_C(u)$, $k \in (0,1)$ and some $0 \leqslant c < 0.25$.*

Proof. (i) We follow the arguments used in [20,30]. We may assume that $u \notin C$. If $x \in C \setminus C_u$, then $\|x\| > 2\|u\|$. Note that

$$\|x-u\| \geqslant \|x\| - \|u\| > \|u\| \geqslant \operatorname{dist}(u, C).$$

Thus, $\operatorname{dist}(u, C_u) = \operatorname{dist}(u, C) \leqslant \|u\|$. Also $\|z-u\| = \operatorname{dist}(u, \operatorname{cl} I(C_u))$ for some $z \in \operatorname{cl} I(C_u)$. This implies that

$$\operatorname{dist}(u, C_u) \leqslant \operatorname{dist}\big(u, \operatorname{cl} I(C_u)\big) \leqslant \operatorname{dist}\big(u, I(C_u)\big) \leqslant \|Ix-u\| \leqslant \|x-u\|,$$

for all $x \in C_u$. Hence $\|z-u\| = \operatorname{dist}(u, C)$ and so $P_C(u)$ is nonempty. Moreover, it is closed and convex.

(ii) Let $z \in P_C(u)$. Then $\|Iz-u\| = \|Iz-Iu\| \leqslant \|z-u\| = \operatorname{dist}(u, C)$. This implies that $Iz \in P_C(u)$ and so $I(P_C(u)) \subseteq P_C(u)$. Let $y \in T(P_C(u))$. Since $T(C_u) \subseteq I(C)$ and $P_C(u) \subseteq C_u$, there exist $z \in P_C(u)$ and $x_0 \in C$ such that $y = Tz = Ix_0$. Further, we have

$$\|Ix_0-u\| = \|Tz-Tu\| \leqslant \|Iz-Iu\| = \|Iz-u\| \leqslant \|z-u\| = \operatorname{dist}(u, C).$$

Thus, $x_0 \in C_C^I(u) = P_C(u)$ and so (ii) holds.

By (ii), the compactness of $\operatorname{cl}(I(C_u))$ implies that $\operatorname{cl} T(P_C(u))$ is compact. The conclusion now follows from Theorem 2.2 applied to $P_C(u)$. □

Theorem 2.8. *Let I and T be self mappings of a Banach space X with $u \in F(I) \cap F(T)$ and $C \in \Im_0$ such that $T(C_u) \subseteq I(C) \subseteq C$. Suppose that $\operatorname{cl}(T(C_u))$ is compact, $\|Ix-u\| = \|x-u\|$ for all $x \in C_u$, T, I are continuous on C_u, T satisfies $\|Tx-u\| \leqslant \|Ix-u\|$ for all $x \in C_u$. Then*

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subseteq I(P_C(u)) \subseteq P_C(u)$, provided that $\|Ix-u\| = \|x-u\|$ for all $x \in C_C^I(u)$, and*
(iii) *$P_C(u) \cap F(I) \cap F(T) \neq \emptyset$, provided that $\|Ix-u\| = \|x-u\|$ for all $x \in C_C^I(u)$, $F(I)$ is nonempty and convex, (T, I) is Banach operator pair on $P_C(u)$ and T satisfies (2.2) for all $q \in F(I) \cap P_C(u)$, $x, y \in P_C(u)$, $k \in (0,1)$ and some $0 \leqslant c < 0.25$.*

Proof. The proof is similar to that of Theorem 2.7. □

Theorem 2.9. *Let I and T be self mappings of a Banach space X with $u \in F(I) \cap F(T)$ and $C \in \Im_0^w$ such that $T(C_u) \subseteq I(C) \subseteq C$. Suppose that $w\operatorname{cl}(I(C_u))$ is weakly compact, $\|Ix-u\| = \|x-u\|$ for all $x \in C_u$, I is weakly continuous on C_u, T satisfies $\|Tx-u\| \leqslant \|Ix-u\|$ for all $x \in C_u$ and either $I-T$ is demiclosed at 0 or X satisfies Opial's condition with $c = 0$ in (2.2). Then*

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subseteq I(P_C(u)) \subseteq P_C(u)$, provided that $\|Ix-u\| = \|x-u\|$ for all $x \in C_C^I(u)$, and*
(iii) *$P_C(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $\|Ix-u\| = \|x-u\|$ for all $x \in C_C^I(u)$, $F(I)$ is nonempty and convex, (T, I) is Banach operator pair on $P_C(u)$ and T satisfies (2.2) for all $q \in F(I) \cap P_C(u)$, $x, y \in P_C(u)$, $k \in (0,1)$ and some $0 \leqslant c < 0.25$.*

Proof. To obtain the result, use an argument similar to that in Theorem 2.7 and apply Theorem 2.4 instead of Theorem 2.2. Use Lemma 5.5 of [37, p. 192] with $f(x) = \|x-u\|$ and $M = w\operatorname{cl}(I(C_u))$ to show that there exists $z \in M$ such that $\operatorname{dist}(u, M) = \|z-u\|$. □

The Banach Contraction Mapping Principle states that if (X, d) is a complete metric space, K is a nonempty closed subset of X and $T : K \to K$ is a self-mapping satisfying $d(Tx, Ty) \leqslant \lambda d(x, y)$ for all $x, y \in K$, where $0 < \lambda < 1$, then T has a unique fixed point, say z in K, and the Picard iterations $\{T^n x\}$ converge to z for all $x \in K$. Ćirić [4] introduced and studied self-mappings on K satisfying

$$d(Tx, Ty) \leqslant \lambda M(x, y),$$

where $0 < \lambda < 1$ and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}.$$

Further investigations were developed by Berinde [2], Jungck [17,18], Hussain and Jungck [13], Hussain and Rhoades [15], Jungck and Hussain [20], O'Regan and Hussain [30] and many other mathematicians (see [8] and references therein). Application of the contraction and generalized contraction principle for self-mappings are well known (cf. [8,31,32]).

Lemma 2.10. *Let C be a nonempty subset of a metric space (X,d), and (T,f) and (T,g) be Banach operator pairs on C. Assume that $\mathrm{cl}(T(C))$ is complete, and T, f and g satisfy for all $x,y\in C$ and $0\leqslant h<1$,*

$$d(Tx,Ty)\leqslant h\max\bigl\{d(fx,gy),d(Tx,fx),d(Ty,gy),d(Tx,gy),d(Ty,fx)\bigr\}. \tag{2.6}$$

If f and g are continuous, $F(f)\cap F(g)$ is nonempty, then there is a unique common fixed point of T, f and g.

Proof. By our assumptions, $T(F(f))\subseteq F(f)$ and $T(F(g))\subseteq F(g)$. Hence $T(F(f)\cap F(g))\subseteq F(f)\cap F(g)$. Also $F(f)\cap F(g)$ is nonempty and closed. Moreover, $\mathrm{cl}\,T(F(f)\cap F(g))$ being a subset of $\mathrm{cl}\,T(C)$ is complete. Further for all $x,y\in F(f)\cap F(g)$, we have by inequality (2.6),

$$\begin{aligned}d(Tx,Ty)&\leqslant h\max\bigl\{d(fx,gy),d(Tx,fx),d(Ty,gy),d(Tx,gy),d(Ty,fx)\bigr\}\\&=h\max\bigl\{d(x,y),d(x,Tx),d(y,Ty),d(Tx,y),d(Ty,x)\bigr\}.\end{aligned}$$

Hence T is a generalized contraction on $F(f)\cap F(g)$ and $\mathrm{cl}\,T(F(f)\cap F(g))\subseteq \mathrm{cl}(F(f)\cap F(g))=F(f)\cap F(g)$. By Theorem 2.1 [20], T has a unique fixed point z in $F(f)\cap F(g)$ and consequently $F(T)\cap F(f)\cap F(g)$ is a singleton. □

The following result properly contains Theorems 3.2–3.3 of [3] and extends and improves Theorem 2.2 of [1], Theorem 4 in [11] and Theorem 6 of [21].

Theorem 2.11. *Let C be a nonempty q-starshaped subset of a normed space X and T, f and g be self-maps of C. Suppose that f and g are continuous and $F(f)$ and $F(g)$ are q-starshaped with $q\in F(f)\cap F(g)$. If (T,f) and (T,g) are Banach operator pairs and satisfy, for all $x,y\in C$,*

$$\|Tx-Ty\|\leqslant\max\bigl\{\|fx-gy\|,\mathrm{dist}\bigl(fx,[q,Tx]\bigr),\mathrm{dist}\bigl(gy,[q,Ty]\bigr),\mathrm{dist}\bigl(gy,[q,Tx]\bigr),\mathrm{dist}\bigl(fx,[q,Ty]\bigr)\bigr\}, \tag{2.7}$$

then $C\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$, provided one of the following conditions holds:

(i) $\mathrm{cl}(T(C))$ *is compact and T is continuous,*
(ii) *X is complete, $w\,\mathrm{cl}(T(C))$ is weakly compact, f,g are weakly continuous and $f-T$ is demiclosed at* 0.

Proof. Since C is q-starshaped with $q\in C$, we can define $T_n: C\to C$ by $T_nx=(1-k_n)q+k_nTx$ for all $x\in C$ and a fixed sequence of real numbers k_n $(0<k_n<1)$ converging to 1. As (T,f) is a Banach operator pair, for $x\in F(f)$ we have $Tx\in F(f)$, and hence $T_nx=(1-k_n)q+k_nTx\in F(f)$ by the fact that $F(f)$ is q-starshaped with $q\in F(f)$. Thus for each $n\geqslant 1$, (T_n,f) is a Banach operator pair on C. Similarly, (T_n,g) is a Banach operator pair on C. Also by (2.7),

$$\begin{aligned}\|T_nx-T_ny\|&=k_n\|Tx-Ty\|\\&\leqslant k_n\max\bigl\{\|fx-gy\|,\mathrm{dist}\bigl(fx,[q,Tx]\bigr),\mathrm{dist}\bigl(gy,[q,Ty]\bigr),\mathrm{dist}\bigl(fx,[q,Ty]\bigr),\mathrm{dist}\bigl(gy,[q,Tx]\bigr)\bigr\}\\&\leqslant k_n\max\bigl\{\|fx-gy\|,\|fx-T_nx\|,\|gy-T_ny\|,\|gy-T_nx\|,\|fx-T_ny\|\bigr\},\end{aligned}$$

for each $x,y\in C$ and $0<k_n<1$.

(i) As $\mathrm{cl}(T(C))$ is compact, for each $n\in\mathbb{N}$, $\mathrm{cl}(T_n(C))$ is compact and hence complete. By Lemma 2.10, for each $n\geqslant 1$, there exists $x_n\in C$ such that $x_n=fx_n=gx_n=T_nx_n$. The compactness of $\mathrm{cl}(T(C))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m\to z\in\mathrm{cl}(T(C))$ as $m\to\infty$. Since $k_m\to 1$, $x_m=T_mx_m=(1-k_m)q+k_mTx_m\to z$. By the continuity of T, f and g, we obtain that $C\cap F(T)\cap F(f)\cap F(g)\neq\emptyset$.

(ii) Proof follows as in Theorem 2.4 (see also Theorem 2.2(ii) [13]). □

Remark 2.12. As an application of Theorem 2.11(i) and (ii), the analogue of all the results (Theorem 2.3–Corollary 2.10) due to Hussain and Jungck [13] can be established for Banach operator pairs (T, f) and (T, g) satisfying more general inequality.

The following result extends Theorem 4.1 in [1], Theorem 8 in [11], Theorem 2.14 in [13] and Theorem 2.1 in [35].

Theorem 2.13. *Let f, g and T be self-mappings of a Banach space X with $u \in F(T) \cap F(f) \cap F(g)$ and $C \in \Im_0$ such that $T(C_u) \subset f(C) \subset C = g(C)$. Suppose that $\|fx - u\| \leqslant \|x - u\|$, $\|gx - u\| = \|x - u\|$ and $\|Tx - u\| \leqslant \|fx - gu\|$ for all $x \in C$, $\mathrm{cl}(f(C_u))$ is compact, then*

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subset f(P_C(u)) \subset P_C(u) = g(P_C(u))$,*
(iii) *$P_C(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$ provided T, f, g are continuous, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g) \cap P_C(u)$, the pairs $\{T, f\}$ and $\{T, g\}$ are Banach operator pairs on $P_C(u)$ and satisfy (2.7) for all $q \in F(f) \cap F(g)$ and for all $x, y \in P_C(u)$.*

Proof. (i) and (ii) follow from Theorem 2.14 [13]. By (ii), the compactness of $\mathrm{cl}\, f(C_u)$ implies that $\mathrm{cl}\, T(P_C(u))$ is compact. The conclusion now follows from Theorem 2.11(i) applied to $P_C(u)$. □

Theorem 2.14. *Let f, g and T be as in Theorem 2.13 and $\mathrm{cl}(f(C_u))$ is compact. Then*

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subset f(P_C(u)) \subset P_C(u) = g(P_C(u))$,*
(iii) *$P_C(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$ provided T, f, g are continuous, $F(g)$ is nonempty and convex, (f, g) is Banach operator pair on $P_C(u)$ and f, g satisfy (2.2) for all $q \in F(g) \cap P_C(u)$, $x, y \in P_C(u)$, $k \in (0, 1)$ and some $0 \leqslant c < 0.25$, $F(f)$ is q-starshaped with $q \in F(f) \cap F(g) \cap P_C(u)$, the pairs $\{T, f\}$ and $\{T, g\}$ are Banach operator pairs on $P_C(u)$ and satisfy (2.7) for all $q \in F(f) \cap F(g)$ and for all $x, y \in P_C(u)$.*

Proof. (i) and (ii) follow from Theorem 2.14 [13]. By (ii), the compactness of $\mathrm{cl}(f(C_u))$ implies that $\mathrm{cl}(f(P_C(u)))$ and $\mathrm{cl}(T(P_C(u)))$ is compact. Theorem 2.2 implies that $F(f) \cap F(g) \cap P_C(u) \neq \emptyset$. Further, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g) \cap P_C(u)$. The conclusion now follows from Theorem 2.11(i) applied to $P_C(u)$. □

Remark 2.15. (i) Theorems 2.7–2.14 represent the strong variants of Theorem 2.4 [1] and Theorem 2.1 [35] in the sense that the commutativity of the maps T and I is replaced by the general hypothesis that (T, I) is a Banach operator pair, I need not be linear or affine and T need not be I-nonexpansive. Further, the comparison of Theorems 2.7–2.14 with Theorems 2.7–2.10 in [16], indicates that the concept about Banach operator pair is more useful for the study of common fixed points in best approximation in the sense that here we are able to prove the results without the linearity or affinity of I and hence it provides positive answer to the question raised in [35].

(ii) Banach operator pairs are different from those of C_q-commuting and R-subweakly commuting maps, so our results are different from those of [13,16]. For this let $X = \mathbb{R}$ with usual norm and $C = [1, \infty)$. Let $T(x) = x^2$ and $I(x) = 2x - 1$, for all $x \in C$. Let $q = 1$. Then C is convex with $q \in F(I)$, $F(I) = \{1\}$ and $C_q(I, T) = [1, \infty)$. Note that the pair (T, I) is a Banach operator but T and I are not C_q-commuting maps and hence not R-subweakly commuting. □

3. Banach operator pair in metrizable topological vector spaces

Metrizable topological vector spaces provide active area of research (see [22,29,34]) and have the following nice characterization (see [26,29,34]).

Theorem 3.1. *A topological vector space X is metrizable if and only if it has a countable base of neighbourhoods of* 0. *The topology of metrizable topological vector space can always be defined by a real-valued function $N: X \to \mathbb{R}$, called F-norm such that for all $x, y \in X$, we have*

(i) $N(x) \geqslant 0$ *and* $N(x) = 0 \Leftrightarrow x = 0$,
(ii) $N(\alpha x) \leqslant N(x)$ *for all* $\alpha \in K$ *with* $|\alpha| \leqslant 1$,
(iii) $N(x + y) \leqslant N(x) + N(y)$,
(iv) *If* $\alpha_n \to 0$ *and* $\alpha_n \in K$, *then* $N(\alpha_n x) \to 0$.

More recently, Olaleru and Akewe [29] have extended Gregus theorem by considering the following contractive condition:

$$d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty), \tag{3.1}$$

for all $x, y \in C$, where $a \in (0, 1)$, $b \geqslant 0$, $c \geqslant 0$, $e \geqslant 0$, $f \geqslant 0$, $b + c > 0$ and $a + b + c + e + f = 1$.

In this section we extend the recent results of Al-Thagafi [1], Chen and Li [3], Habiniak [11], Jungck and Sessa [21], Khan and Khan [25], Khan et al. [23] and [24], Sahab, Khan and Sessa [33], Shahzad [35], Singh [36], Smoluk [38] and Subrahmanyam [39] to the setup of metrizable topological vector space. Complete metrizable topological vector spaces include uniformly convex Banach spaces, Banach spaces and complete metrizable locally convex spaces (cf. [29,34]).

Lemma 3.2. *Let C be a nonempty closed convex subset of a complete metrizable topological vector space X, and (T, I) be a Banach operator pair on C. Assume that T and I satisfy*

$$N(Tx - Ty) \leqslant aN(Ix - Iy) + bN(Ix - Tx) + cN(Iy - Ty) + eN(Iy - Tx) + fN(Ix - Ty), \tag{3.2}$$

for all $x, y \in C$, where $a \in (0, 1)$, $b \geqslant 0$, $c \geqslant 0$, $e \geqslant 0$, $f \geqslant 0$, $b + c > 0$ and $a + b + c + e + f = 1$. If I is continuous, $F(I)$ is nonempty and convex, then there is a unique common fixed point of T and I.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty closed and convex. Further for all $x, y \in F(I)$, we have by inequality (3.2),

$$\begin{aligned} N(Tx - Ty) &\leqslant aN(Ix - Iy) + bN(Ix - Tx) + cN(Iy - Ty) + eN(Iy - Tx) + fN(Ix - Ty) \\ &= aN(x - y) + bN(x - Tx) + cN(y - Ty) + eN(y - Tx) + fN(x - Ty). \end{aligned}$$

By Theorem 3 [29], T has a unique fixed point z in $F(I)$ and consequently $F(T) \cap F(I)$ is singleton. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete metrizable topological vector space X, and (T, I) be a Banach operator pair on C. If I is continuous, $F(I)$ is nonempty and convex and the pair (T, I) satisfies for some $q \in F(I) \cap C$ and for all $x, y \in C$,*

$$\begin{aligned} N(Tx - Ty) \leqslant{}& aN(Ix - Iy) + b\,\mathrm{dist}\big(Ix, [q, Tx]\big) + c\,\mathrm{dist}\big(Iy, [q, Ty]\big) + e\,\mathrm{dist}\big(Iy, [q, Tx]\big) \\ &+ f\,\mathrm{dist}\big(Ix, [q, Ty]\big), \end{aligned} \tag{3.3}$$

where $a \in (0, 1)$, $b \geqslant 0$, $c \geqslant 0$, $e \geqslant 0$, $f \geqslant 0$, $b + c > 0$ and $a + b + c + e + f = 1$, then $C \cap F(I) \cap F(T) \neq \emptyset$ provided one of the following conditions holds:

(i) $\mathrm{cl}(T(C))$ *is compact and T is continuous,*
(ii) $w\,\mathrm{cl}(T(C))$ *is weakly compact, I is weakly continuous and $I - T$ is demiclosed at* 0,
(iii) $w\,\mathrm{cl}(T(C))$ *is weakly compact and T is completely continuous.*

Proof. Let $\{k_n\}$ and $\{T_n\}$ be defined as in Theorem 2.2. As (T, I) is a Banach operator pair and $F(I)$ is convex, so that for each $x \in F(I)$, $T_n x = (1-k_n)q + k_n T x \in F(I)$, since $Tx \in F(I)$. Thus (T_n, I) is a Banach operator pair for each n. Further, by property (ii) of the F-*norm* N and inequality (3.3), we have

$$\begin{aligned} N(T_n x - T_n y) &= N\big(k_n(Tx - Ty)\big) \\ &\leqslant N(Tx - Ty) \\ &\leqslant aN(Ix - Iy) + b\,\mathrm{dist}\big(Ix, [q, Tx]\big) + c\,\mathrm{dist}\big(Iy, [q, Ty]\big) \\ &\quad + e\,\mathrm{dist}\big(Iy, [q, Tx]\big) + f\,\mathrm{dist}\big(Ix, [q, Ty]\big) \\ &\leqslant aN(Ix - Iy) + bN(Ix - T_n x) + cN(Iy - T_n y) \\ &\quad + eN(Iy - T_n x) + fN(Ix - T_n y), \end{aligned}$$

for each $x, y \in C$, where $a \in (0,1)$, $b \geqslant 0$, $c \geqslant 0$, $e \geqslant 0$, $f \geqslant 0$ and $a + b + c + e + f = 1$.

(i) By Lemma 3.2, for each $n \geqslant 1$, there exists $x_n \in C$ such that $x_n = Ix_n = T_n x_n$. Further proceeding as in the proof of Theorem 2.2, we get the required result.

(ii) By the weak compactness of $w\,\mathrm{cl}(T(C))$, there exists subsequence $\{x_m\}$ of $\{x_n\}$ which converges weakly to $y_0 \in C$ as $m \to \infty$. As I is weakly continuous, $Iy_0 = y_0$. Since $\{x_m\}$ is bounded, $k_m \to 1$, and

$$N\big((I - T)(x_m)\big) = N(x_m - Tx_m) = N\big(\big((1-k_m)q + k_m T x_m\big) - Tx_m\big) = N\big((1-k_m)(q - Tx_m)\big).$$

Thus $Ix_m - Tx_m \to 0$ as $m \to \infty$. As $(I - T)$ is demiclosed at 0, $(I - T)y_0 = 0$ and hence $Ty_0 = Iy_0 = y_0$.

(iii) As in (ii), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in $F(I)$ converging weakly to $y \in F(I)$ as $m \to \infty$. Since T is completely continuous, $Tx_m \to Ty$ as $m \to \infty$. Since $k_m \to 1$, $x_m = T_m x_m = k_m T x_m + (1-k_m)q \to Ty$ as $m \to \infty$. Thus $Tx_m \to T^2 y$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Iy$. Also, since $Tx_m = x_m \to Ty = w$, using the continuity of I and the uniqueness of the limit, we have $Iw = w$. Hence $C \cap F(T) \cap F(I) \neq \emptyset$. □

Theorem 3.3 generalizes Theorem 2.2 in [1], Theorems 3.2–3.3 in [3], Theorem 4 in [11] and corresponding results in Khan and Khan [25], Khan et al. [23] and [24] and Subrahmanyam [39] to maps satisfying a more general inequality in the setting of metrizable topological vector space.

Recall that $\Im_0$ denotes the class of closed convex subsets of X containing 0. For $C \in \Im_0$, we define $C_u = \{x \in C\colon N(x) \leqslant 2N(u)\}$. It is clear that $P_C(u) \subset C_u \in \Im_0$.

Theorem 3.4. *Let X be a complete metrizable topological vector space and I and T be self-mappings of X with $u \in F(I) \cap F(T)$ and $C \in \Im_0$ such that $T(C_u) \subset I(C) \subset C$. Suppose that $N(Ix - u) \leqslant N(x - u)$, $N(Tx - u) \leqslant N(Ix - u)$ for all $x \in C$, the pair $\{I, T\}$ is continuous on C and one of the following two conditions is satisfied*:

(a) $\mathrm{cl}(I(C))$ *is compact,*
(b) $\mathrm{cl}(T(C))$ *is compact.*

Then

(i) *$P_C(u)$ is nonempty, closed and convex,*
(ii) *$T(P_C(u)) \subset I(P_C(u)) \subset P_C(u)$ provided that $N(Ix - u) \leqslant N(x - u)$ for all $x \in C_C^I(u)$,*
(iii) *$P_C(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $N(Ix - u) \leqslant N(x - u)$ for all $x \in C_C^I(u)$, $P_C(u) \cap F(I)$ is nonempty and convex, (T, I) is a Banach operator pair on $P_C(u)$ and T satisfies* (3.3) *for all $q \in F(I)$, for all $x, y \in P_C(u)$, $a \in (0,1)$, $b \geqslant 0$, $c \geqslant 0$, $e \geqslant 0$, $f \geqslant 0$, $b + c > 0$ and $a + b + c + e + f = 1$.*

Proof.

(i) Let $r = \operatorname{dist}(u, C)$. Then there is a minimizing sequence $\{y_n\}$ in C such that $\lim_n N(u - y_n) = r$. As $\operatorname{cl} I(C)$ is compact so $\{Iy_n\}$ has a convergent subsequence $\{Iy_m\}$ with $\lim_m Iy_m = x_0$ (say) in C. Now by using $N(Ix - u) \leqslant N(x - u)$ we get

$$r \leqslant N(x_0 - u) = \lim_m N(Iy_m - u) \leqslant \lim_m N(y_m - u) = \lim_n N(y_n - u) = r.$$

Hence $x_0 \in P_C(u)$. Thus $P_C(u)$ is nonempty closed and convex. Similarly, when $\operatorname{cl} T(C)$ is compact we get same conclusion by using inequalities $N(Ix - u) \leqslant N(x - u)$ and $N(Tx - u) \leqslant N(Ix - u)$ for all $x \in C$.

(ii) Let $z \in P_C(u)$. Then $N(Tz - u) \leqslant N(Iz - u) = \operatorname{dist}(u, M)$. This implies that $Tz \in P_C(u)$ and so $T(P_C(u)) \subset P_C(u)$. Also we have $I(P_C(u)) \subset P_C(u)$. Let $y \in T(P_C(u))$. Since $T(C_u) \subset I(C)$ and $P_C(u) \subset C_u$, there exist $z \in P_C(u)$ and $x \in C$ such that $y = Tz = Ix$. Thus, we have

$$N(Ix - u) = N(Tz - u) \leqslant N(Iz - u) \leqslant N(z - u) = \operatorname{dist}(u, C).$$

Hence $x \in C_C^I(u) = P_C(u)$ and so (ii) holds.

(iii) (a) By (i) $P_C(u)$ is closed and by (ii) $P_C(u)$ is I and T-invariant. Further, $P_C(u) \cap F(I) \neq \emptyset$ implies that there exists $q \in P_C(u)$ such that $q \in F(I)$. By (ii), the compactness of $\operatorname{cl} I(C)$ implies that $\operatorname{cl} T(P_C(u))$ is compact. The conclusion now follows from Theorem 3.3(i) applied to $P_C(u)$.

(b) By (ii), the compactness of $\operatorname{cl} T(C)$ implies that $\operatorname{cl}(T(P_C(u)))$ is compact, Theorem 3.3(i) further guarantees that $P_C(u) \cap F(T) \cap F(I) \neq \emptyset$. □

Theorem 3.4 extends Theorem 4.2 in [1], Theorem 8 in [11], Theorem 2.1 in [35] and provides the conclusion, in the setting of complete metrizable topological vector space, of Theorem 3.3 [12], Theorems 2.9–2.10 [20], Theorem 2.6 [30] for Banach operator pair (T, I) where I need not be linear or affine.

For any nonempty subset C of a metric space (X, d), the diameter of C is denoted and defined by $\delta(C) = \sup\{d(x, y)\colon x, y \in C\}$. A subset C of a linear space X is called T-regular if and only if

(i) $T : C \mapsto C$,
(ii) $\frac{x+Tx}{2} \in C$ for each $x \in C$.

Clearly, each convex set, which is T-invariant, is T-regular but not conversely in general. Consider the set $C = [-2, -1] \cup [1, 2]$. Define $T : C \mapsto C$ by $Tx = -1$ for $x \in [-2, -1]$ and $Tx = 1$ for $x \in [1, 2]$. The set C is T-regular but not convex and hence not starshaped (see [22] and references therein for more examples).

In the results to follow, we generalize Theorem 2.2 in [1], Theorem 3.2 in [3], Theorem 4 in [11], Theorem 6 of [21] and corresponding results in Khan et al. [23,24] to the maps defined on nonconvex and nonstarshaped domain.

Theorem 3.5. *Let C be a nonempty weakly compact T-regular subset of a complete metrizable uniformly convex topological vector space X, and (T, I) be a Banach operator pair on C. Assume that I is weakly continuous, $F(I)$ is nonempty and*

either (a) $\frac{x+Tx}{2} \in F(I)$ *for each* $x \in F(I)$ **or** (b) *I is midpoint affine (i.e.* $I(\frac{x+Tx}{2}) = \frac{Ix+ITx}{2}$).

Suppose for each weakly closed T-regular subset K of $F(I)$ with $\delta(K) > 0$, there exists some $\gamma(K)$, $0 < \gamma < 1$, such that for all $x, y \in K$,

$$N(Tx - Ty) \leqslant \max\{N(Ix - Iy), \gamma\delta(K)\}. \tag{3.4}$$

Then $C \cap F(I) \cap F(T) \neq \emptyset$.

Proof. By our assumption, $T(F(I)) \subseteq F(I)$. Thus $F(I)$ is nonempty and T-invariant. Condition (a) implies that $F(I)$ is T-regular. If (b) holds, then for each $x \in F(I)$, $Tx \in F(I)$ and $I(\frac{x+Tx}{2}) = \frac{Ix+ITx}{2} = \frac{x+Tx}{2}$, which implies that $\frac{x+Tx}{2} \in F(I)$. Hence (a) holds. The weak compactness of C and the weak continuity of I imply that $F(I)$ is weakly compact. Further for all $x, y \in K \subset F(I)$, we have by inequality (3.4)

$$N(Tx - Ty) \leqslant \max\{N(Ix - Iy), \gamma\delta(K)\} = \max\{N(x - y), \gamma\delta(K)\}.$$

By Theorem 3.3 [22], T has a unique fixed point y in $F(I)$ and consequently $C \cap F(I) \cap F(T) \neq \emptyset$. □

Corollary 3.6. *Let C be a nonempty weakly compact T-regular subset of a complete metrizable uniformly convex topological vector space X, and (T, I) be a Banach operator pair on C. Assume that T is I-nonexpansive, I is weakly continuous, $F(I)$ is nonempty and $\frac{x+Tx}{2} \in F(I)$ for each $x \in F(I)$. Then $C \cap F(I) \cap F(T) \neq \emptyset$.*

Acknowledgment

The author would like to thank the referee(s) for their valuable suggestions to improve presentation of the paper.

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