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Two realizations of Schrödinger operators on Riemannian manifolds

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ABSTRACT

We consider a Schrödinger differential expression $P = \Delta_M + V$ on a complete Riemannian manifold (M, g) with metric g, where Δ_M is the scalar Laplacian on M and V is a real-valued locally integrable function on M. We study two self-adjoint realizations of P in $L^2(M)$ and show their equality. This is an extension of a result of S. Agmon.

1. Introduction and the main result

1.1. The setting

Let (M, g) be a C^∞-Riemannian manifold without boundary, with metric $g = (g_{jk})$ and $\dim M = n$. We will assume that M is connected and oriented. By $d\nu$ we will denote the Riemannian volume element of M. In any local coordinates $x^1, \dots, x^n$, we have $d\nu = \sqrt{\det(g_{jk})}\, dx^1\, dx^2 \dots dx^n$.

By $L^2(M)$ we denote the space of complex-valued square integrable functions on M with the inner product

$$(u, v) = \int_M (u\bar{v})\, d\nu. \tag{1}$$

We will denote the norm in $L^2(M)$ by $\|\cdot\|$.

In what follows, by $C^\infty(M)$ we denote the space of smooth functions on M, by $C_c^\infty(M)$—the space of smooth compactly supported functions on M, by $\Omega^1(M)$—the space of smooth 1-forms on M, and by $\mathbb{Z}_+$—the set of positive integers.

By $d : C^\infty(M) \to \Omega^1(M)$ we denote the standard differential, and by $d^* : \Omega^1(M) \to C^\infty(M)$ we denote the formal adjoint of d with respect to the inner product (1). By $\Delta_M := d^*d$ we will denote the scalar Laplacian on M.

We consider a Schrödinger-type differential expression

$$P = \Delta_M + V, \tag{2}$$

where $V \in L^1_{\mathrm{loc}}(M)$ is a real-valued function.

Defining $V_+ := \max(V, 0)$ and $V_- := \max(-V, 0)$, we can write $V = V_+ - V_-$.

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Assumption (A). Assume that $V \in L^1_{\mathrm{loc}}(M)$. Additionally, assume that there exist constants $0 \leqslant a < 1$ and $b \geqslant 0$ such that

$$\int_M V_-|u|^2\,d\nu \leqslant a \int_M |du|^2\,d\nu + b\|u\|^2, \quad \text{for all } u \in C_c^\infty(M). \tag{3}$$

1.2. Sobolev space $W^{1,2}(M)$

By $W^{1,2}(M)$ we will denote the completion of the space $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u,v)_1 := (u,v) + (du,dv), \quad u,v \in C_c^\infty(M). \tag{4}$$

Remark 1. It is well known that for a complete Riemannian manifold (M,g), we have $W^{1,2}(M) = \{u \in L^2\colon du \in L^2(\Lambda^1 T^*M)\}$.

We will consider two realizations of the expression P in $L^2(M)$.

1.3. Operator T

Let V be as in Assumption (A). Define an operator T in $L^2(M)$ by the formula $Tu = Pu$, with the domain

$$\mathrm{Dom}(T) = \{u \in W^{1,2}(M)\colon |V|^{1/2}u \in L^2(M) \text{ and } Pu \in L^2(M)\}.$$

1.4. Operator S

Let V be as in Assumption (A). Define an operator S in $L^2(M)$ by the formula $Su = Pu$, with the domain

$$\mathrm{Dom}(S) = \{u \in L^2(M) \cap W^{1,2}_{\mathrm{loc}}(M)\colon Vu \in L^1_{\mathrm{loc}}(M) \text{ and } Pu \in L^2(M)\}.$$

We now state the main result.

Theorem 2. *Assume that (M,g) is a C^∞-Riemannian manifold without boundary. Assume that M is connected, oriented and complete. Assume that V satisfies Assumption* (A). *Then the operator T is self-adjoint in $L^2(M)$. Furthermore, the following equality holds*: $T = S$.

Remark 3. Theorem 2 extends the result of S. Agmon (see [2, Theorem 1.2]) concerning the Schrödinger differential expression $P = -\Delta + V$, where Δ is the standard Laplacian on $\mathbb{R}^n$ and V is as in the hypotheses of Theorem 2. The self-adjointness of the realization S of P, where $P = -\Delta + V$ is an operator in $L^2(\mathbb{R}^n)$ and S is as in Section 1.4 with $M = \mathbb{R}^n$, follows from the result of Kato [6]. In the context of a manifold of bounded geometry, the paper [9] proved the self-adjointness of the realization T of P, where T and P are as in Section 1.3 and V is as in Assumption (A) above. In the same bounded geometry context, the paper [11] proved the self-adjointness of the realization S of P, where S and P are as in Section 1.4, and $V \in L^1_{\mathrm{loc}}(M)$ with $V \geqslant -C$ (here $C > 0$ is a constant). The works [9] and [11] used Kato inequality technique, leading to a certain distributional inequality which is well understood only on a manifold of bounded geometry; see [3, Appendix B]. By adopting Agmon's wighted L^2 estimates [2, Section 4] to our setting, we were able to study realizations T and S on an arbitrary complete Riemannian manifold. Moreover, this approach enabled us to include V_- in the study of the realization S of P in $L^2(M)$, which is not present in [11].

Remark 4. If (M,g) is a manifold of bounded geometry, Assumption (A) holds if $V_- \in L^p(M)$, where $p = n/2$ for $n \geqslant 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$; see [9, Remark 2.1].

2. Weighted L^2-estimate

In this section, we will prove an L^2-estimate for solutions of the differential equation used in the proof of Theorem 2. Throughout this section, we assume that all hypotheses of Theorem 2 are satisfied.

Weak solution. Let P be as in (2), let $f \in L^1_{\mathrm{loc}}(M)$ be a complex valued function, and let $\lambda \in \mathbb{C}$. A (complex valued) function u is called a weak solution of the equation

$$(P - \lambda)u = f \tag{5}$$

if $u \in W^{1,2}_{\mathrm{loc}}(M)$, $Vu \in L^1_{\mathrm{loc}}(M)$ and

$$\int_M \langle du, d\bar{\phi}\rangle\, d\nu + \int_M (V-\lambda)u\bar{\phi}\, d\nu = \int_M f\bar{\phi}\, d\nu, \quad \text{for all } \phi \in C_c^\infty(M), \tag{6}$$

where $\langle\cdot,\cdot\rangle$ is the pointwise inner product in $\Lambda^1 T_x^* M$.

The following proposition provides a key weighted L^2-estimate.

Proposition 5. *Assume that (M, g) is a complete Riemannian manifold. Let $\lambda \in \mathbb{R}$ and let $c(x)$ be a positive continuous function on M. Assume that*

$$\int_M \big(|d\phi|^2 + (V-\lambda)|\phi|^2\big)\, d\nu \geqslant \int_M c(x)|\phi|^2\, d\nu, \quad \textit{for all } \phi \in C_c^\infty(M). \tag{7}$$

Let $f \in L^2_{\mathrm{loc}}(M)$ and assume that u is a weak solution of (5)*. Additionally, assume that $u \in L^2(M)$.*

Let $h(x)$ be a non-negative Lipschitz function on M such that

$$\big|dh(x)\big|^2 \leqslant c(x), \quad \textit{a.e. on } M, \tag{8}$$

where $|\cdot|$ denotes the length of the cotangent vector $dh(x) \in T_x^* M$.

Then the following inequality holds:

$$\int_M \big(c(x) - \big|dh(x)\big|^2\big)e^{2h(x)}|u|^2\, d\nu \leqslant \int_M \big(c(x) - \big|dh(x)\big|^2\big)^{-1} e^{2h(x)}|f|^2\, d\nu. \tag{9}$$

The proof adopts the technique of Agmon [2] and Agmon [1, Chapters 1 and 3] to our setting with the help of cut off functions described in Section 2.1 below. We will first prove a few preliminary lemmas.

We will assume that the right-hand side of (9) is finite. (Otherwise, there is nothing to prove.) By the definition of Δ_M and since V and λ are real by hypotheses, without loss of generality, we may (and we will) assume that f and u are real valued.

Let $\epsilon > 0$, let u be as in hypotheses of Proposition 5, and define

$$u_\epsilon := \frac{u}{1+\epsilon u^2}. \tag{10}$$

Clearly, $u_\epsilon \in L^\infty(M) \cap W^{1,2}_{\mathrm{loc}}(M)$.

Lemma 6. *Let ψ be a real valued compactly supported Lipschitz function on M. Let u be as in hypothesis of Proposition* 5*, and let u_ϵ be as in* (10)*. Then*

$$\int_M \big\langle du, d\big(\psi^2 u_\epsilon\big)\big\rangle\, d\nu + \int_M (V-\lambda)u\psi^2 u_\epsilon\, d\nu = \int_M f\psi^2 u_\epsilon\, d\nu. \tag{11}$$

Proof. By definition of the weak solution, we have $u \in W^{1,2}_{\mathrm{loc}}(M)$, $Vu \in L^1_{\mathrm{loc}}(M)$ and u satisfies Eq. (6).

Since ψ be a compactly supported Lipschitz function on M and since $u_\epsilon \in L^\infty(M) \cap W^{1,2}_{\mathrm{loc}}(M)$, it follows that $\psi^2 u_\epsilon \in W^{1,2}_{\mathrm{comp}}(M) \cap L^\infty(M)$.

By using a partition of unity we may assume that ψ is supported in a coordinate neighborhood U on M. Hence, we can use Friedrichs mollifiers. Let $\rho > 0$ and let $(\psi^2 u_\epsilon)^\rho := J^\rho(\psi^2 u_\epsilon)$, where J^ρ denotes Friedrichs mollifying operator as in [3, Section 5.12]. We have $(\psi^2 u_\epsilon)^\rho \in C_c^\infty(M)$, and $(\psi^2 u_\epsilon)^\rho \to \psi^2 u_\epsilon$ in $W^{1,2}(M)$, as $\rho \to 0+$; see, for instance, [3, Lemma 5.13].

Thus, (6) holds with $\phi = (\psi^2 u_\epsilon)^\rho$:

$$\int_M \big\langle du, d\big(\big(\psi^2 u_\epsilon\big)^\rho\big)\big\rangle d\nu + \int_M (V-\lambda)u\big(\psi^2 u_\epsilon\big)^\rho\, d\nu = \int_M f\big(\psi^2 u_\epsilon\big)^\rho\, d\nu. \tag{12}$$

Moreover, we have

$$\int_M \big\langle du, d\big(\psi^2 u_\epsilon\big)^\rho\big\rangle d\nu \to \int_M \big\langle du, d\big(\psi^2 u_\epsilon\big)\big\rangle d\nu, \quad \text{as } \rho \to 0+, \tag{13}$$

and since $f \in L^2_{\mathrm{loc}}(M)$, we have

$$\int_M f\big(\psi^2 u_\epsilon\big)^\rho\, d\nu \to \int_M f\psi^2 u_\epsilon\, d\nu. \tag{14}$$

We now consider the term

$$\int_M (V-\lambda)u\big(\psi^2 u_\epsilon\big)^\rho \, d\nu.$$

Since $\psi^2 u_\epsilon \in L^\infty(M)$ is compactly supported, by properties of Friedrichs mollifiers (see, for example, the proof of [4, Theorem 1.2.1]) it follows that

(1) there exists a compact set K containing the supports of $\psi^2 u_\epsilon$ and $(\psi^2 u_\epsilon)^\rho$ for all $0 < \rho < 1$, and
(2) the following inequality holds for all $\rho > 0$:

$$\big\|\big(\psi^2 u_\epsilon\big)^\rho\big\|_{L^\infty} \leqslant \big\|\psi^2 u_\epsilon\big\|_{L^\infty}. \tag{15}$$

Since $(\psi^2 u_\epsilon)^\rho \to \psi^2 u_\epsilon$ in $L^2(M)$ as $\rho \to 0+$, after passing to a subsequence we have

$$\big(\psi^2 u_\epsilon\big)^\rho \to \psi^2 u_\epsilon \quad \text{a.e. on } M, \text{ as } \rho \to 0+. \tag{16}$$

By (15) we have a.e. on M:

$$\big|\big(V(x)-\lambda\big)u(x)\big(\psi^2 u_\epsilon\big)^\rho(x)\big| \leqslant \big|\big(V(x)-\lambda\big)u(x)\big|\big\|\psi^2 u_\epsilon\big\|_{L^\infty}. \tag{17}$$

Since, by definition of the weak solution, $(V-\lambda)u \in L^1_{\mathrm{loc}}(M)$, it follows that $(V-\lambda)u \in L^1(K)$.

By (16), (17) and since $(V-\lambda)u \in L^1(K)$, using dominated convergence theorem, we have

$$\int_M (V-\lambda)u\big(\psi^2 u_\epsilon\big)^\rho \, d\nu \to \int_M (V-\lambda)u\big(\psi^2 u_\epsilon\big)\, d\nu, \quad \text{as } \rho \to 0+. \tag{18}$$

From (12)–(14) and (18) we get (11), and the lemma is proven. □

Lemma 7. *Let c be as in hypotheses of Proposition* 5 *and let u and ψ be as in hypotheses of Lemma* 6. *Then*

$$\int_M \big(c(x)(u\psi)^2 - |d\psi|^2 u^2\big)\, d\nu \leqslant \int_M f u \psi^2 \, d\nu. \tag{19}$$

Proof. Since ψ is compactly supported, by using a partition of unity we may assume that ψ is supported on a coordinate neighborhood U on M. Note that in (19) the function u is relevant only on a neighborhood of support of ψ. Hence, we may assume that u is supported in the same coordinate neighborhood U.

Now by definition of u_ϵ it follows that

$$u_\epsilon \to u \quad \text{in } W^{1,2}_{\mathrm{loc}}(U), \text{ as } \epsilon \to 0+. \tag{20}$$

Since the hypotheses of Lemma 6 are satisfied, we can use (11). We will begin by rewriting (11) as follows:

$$\int_M \big\langle d(u_\epsilon), d\big(\psi^2 u_\epsilon\big)\big\rangle \, d\nu + \int_M (V-\lambda)\psi^2 (u_\epsilon)^2 \, d\nu = \int_M f\psi^2 u_\epsilon \, d\nu + I_\epsilon, \tag{21}$$

where

$$\begin{aligned} I_\epsilon &:= \int_M \big\langle d(u_\epsilon - u), d\big(\psi^2 u_\epsilon\big)\big\rangle \, d\nu + \int_M (V_+ - V_- - \lambda)(u_\epsilon - u)u_\epsilon \psi^2 \, d\nu \\ &\leqslant \int_{\operatorname{supp}\psi} \big\langle d(u_\epsilon - u), d\big(\psi^2 u_\epsilon\big)\big\rangle \, d\nu + \int_{\operatorname{supp}\psi} (V_- + \lambda)(u - u_\epsilon)u_\epsilon \psi^2 \, d\nu. \end{aligned}$$

In the last inequality we used the hypothesis $V_+ \geqslant 0$ and the definition of u_ϵ.

Using Cauchy–Schwarz inequality, we have

$$\int_{\operatorname{supp}\psi} \big\langle d(u_\epsilon - u), d\big(\psi^2 u_\epsilon\big)\big\rangle \, d\nu \leqslant \bigg(\int_{\operatorname{supp}\psi} \big|d(u_\epsilon - u)\big|^2 \, d\nu\bigg)^{1/2} \cdot \bigg(\int_{\operatorname{supp}\psi} \big|d\big(\psi^2 u_\epsilon\big)\big|^2 \, d\nu\bigg)^{1/2}. \tag{22}$$

By (20) and since $\operatorname{supp}\psi$ is compact, it follows that, as $\epsilon \to 0+$, the first term in the product on the right-hand side of (22) converges to 0, while the second term in the product remains bounded.

Using Cauchy–Schwarz inequality, we also have

$$\int_{\operatorname{supp}\psi} (V_- + \lambda)(u - u_\epsilon)u_\epsilon \psi^2 \, d\nu \leqslant \left(\int_{\operatorname{supp}\psi} V_- |u - u_\epsilon|^2 \, d\nu \right)^{1/2} \cdot \left(\int_{\operatorname{supp}\psi} V_- \big(u_\epsilon \psi^2\big)^2 \, d\nu \right)^{1/2}$$
$$+ |\lambda| \left(\int_{\operatorname{supp}\psi} \big|(u - u_\epsilon)\big|^2 \, d\nu \right)^{1/2} \cdot \left(\int_{\operatorname{supp}\psi} \big(u_\epsilon \psi^2\big)^2 \, d\nu \right)^{1/2}. \tag{23}$$

By (20) and since $\operatorname{supp}\psi$ is compact, it follows that, as $\epsilon \to 0+$, the first term in second product on the right-hand side of (23) converges to 0, while the second term in the second product remains bounded.

Using an approximation argument as in the proof of Lemma 6, from Assumption (A) we get

$$\int_{\operatorname{supp}\psi} V_- |u - u_\epsilon|^2 \, d\nu \leqslant a \int_{\operatorname{supp}\psi} \big|d(u - u_\epsilon)\big|^2 \, d\nu + b \int_{\operatorname{supp}\psi} |u - u_\epsilon|^2 \, d\nu. \tag{24}$$

By (20) it follows that the right-hand side of (24) converges to 0 as $\epsilon \to 0+$.

Likewise, by Assumption (A) we have

$$\int_{\operatorname{supp}\psi} V_- \big(u_\epsilon \psi^2\big)^2 \, d\nu \leqslant a \int_{\operatorname{supp}\psi} \big|d\big(u_\epsilon \psi^2\big)\big|^2 \, d\nu + b \int_{\operatorname{supp}\psi} \big(u_\epsilon \psi^2\big)^2 \, d\nu. \tag{25}$$

It follows that, as $\epsilon \to 0+$, the left-hand side of (25) remains bounded.

Using (22)–(25) and the remarks after those estimates, we get

$$\limsup_{\epsilon \to 0+} I_\epsilon \leqslant 0. \tag{26}$$

Using Leibniz rule, we have the following equality:

$$\big\langle du_\epsilon, d\big(\psi^2 u_\epsilon\big)\big\rangle = \big|d(\psi u_\epsilon)\big|^2 - (u_\epsilon)^2 |d\psi|^2, \tag{27}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\Lambda^1 T_x^* M$ and $|\cdot|$ is the length of the cotangent vector in $T_x^* M$.

Using (27) we can rewrite (21) as follows:

$$\int_M \big(\big|d(\psi u_\epsilon)\big|^2 + (V - \lambda)\psi^2 (u_\epsilon)^2 - (u_\epsilon)^2 |d\psi|^2 \big) \, d\nu = \int_M f \psi^2 u_\epsilon \, d\nu + I_\epsilon. \tag{28}$$

Using the hypothesis (7) and the same approximation argument as in the proof of Lemma 6 we get

$$\int_M \big(\big|d(\psi u_\epsilon)\big|^2 + (V - \lambda)|\psi u_\epsilon|^2 \big) \, d\nu \geqslant \int_M c(x)(\psi u_\epsilon)^2 \, d\nu. \tag{29}$$

From (28) and (29) we get

$$\int_M \big(c(x)(\psi u_\epsilon)^2 - (u_\epsilon)^2 |d\psi|^2 \big) \, d\nu \leqslant \int_M f \psi^2 u_\epsilon \, d\nu + I_\epsilon. \tag{30}$$

We now let $\epsilon \to 0+$ in (30). Using (26), hypotheses on f, ψ, c, and (20), we obtain (19). This concludes the proof of the lemma. □

In the sequel, we will use a sequence of cut off functions due to Karcher [5]. (The construction of this sequence is also explained in Shubin [12].)

2.1. Cut off functions

Let (M, g) be a complete Riemannian manifold. Then there exists a sequence of functions $\phi_j : M \to \mathbb{R}$, $j = 1, 2, \ldots$ such that

(a) $\phi_j \in C_c^\infty(M)$.
(b) $0 \leqslant \phi_j(x) \leqslant 1$, $x \in M$, $j = 1, 2, \ldots$.
(c) For every compact set $K \subset M$, there exists j_0 such that $\phi_j = 1$ on K, for $j \geqslant j_0$.
(d) $\epsilon_j := \sup_{x \in M} |d\phi_j| \to 0$, as $j \to \infty$.

Proof of Proposition 5. Let h be as in hypotheses of the proposition, let $N \in \mathbb{Z}_+$, and define

$$h_N(x) := \begin{cases} (\frac{N-1}{N})h(x), & \text{if } h(x) \leqslant N, \\ N-1, & \text{if } h(x) > N. \end{cases}$$

Clearly, h_N is as non-negative bounded Lipschitz function on M. By the well-known Rademacher's theorem $h_N(x)$ is differentiable almost everywhere. By the definition of h_N and by the hypothesis (8) it follows that

$$\left|dh_N(x)\right|^2 \leqslant \left(\frac{N-1}{N}\right)^2 \left|dh(x)\right|^2 < \left(\frac{N-1}{N}\right)^2 c(x). \tag{31}$$

Let ϕ_j be as in Section 2.1, and define $\psi_{j,N}(x) := \phi_j(x)e^{h_N(x)}$.

Using Leibniz rule, the inequality $2ab \leqslant a^2/\beta + \beta b^2$, where $\beta > 0$, and the estimate (31), we get

$$\begin{aligned} |d\psi_{j,N}|^2 &= \left(\phi_j^2|dh_N|^2 + 2\phi_j\langle d\phi_j, dh_N\rangle + |d\phi_j|^2\right)e^{2h_N} \leqslant \left(\phi_j^2|dh_N|^2\left(1+N^{-1}\right) + (1+N)|d\phi_j|^2\right)e^{2h_N} \\ &\leqslant (\psi_{j,N})^2|dh|^2 + (1+N)|d\phi_j|^2 e^{2h_N}. \end{aligned} \tag{32}$$

Using (19) with $\psi = \psi_{j,N}$, the estimate (32), and Cauchy–Schwarz inequality, we get

$$\begin{aligned} \int_M (u\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right) d\nu &\leqslant \int_M fu\psi_{j,N}^2\, d\nu + (1+N)\int_M u^2|d\phi_j|^2 e^{2h_N}\, d\nu \\ &\leqslant \left(\int_M (u\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right) d\nu\right)^{1/2}\left(\int_M (f\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right)^{-1} d\nu\right)^{1/2} \\ &\quad + (1+N)\int_M u^2|d\phi_j|^2 e^{2h_N}\, d\nu. \end{aligned} \tag{33}$$

The leftmost side and rightmost side in (33) give an inequality of the form $A \leqslant A^{1/2}B^{1/2} + C$, where A, B and C are positive numbers. Using an elementary calculation, we have the following inequality: $A \leqslant B + 2C$.

Therefore,

$$\int_M (u\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right) d\nu \leqslant \int_M (f\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right)^{-1} d\nu + 2(1+N)\int_M u^2|d\phi_j|^2 e^{2h_N}\, d\nu. \tag{34}$$

Using the conditions $u \in L^2(M)$ and $h_N \in L^\infty(M)$ and the property (d) of Section 2.1, it follows that

$$\int_M u^2|d\phi_j|^2 e^{2h_N}\, d\nu \to 0, \quad \text{as } j \to \infty. \tag{35}$$

By definition of h_N it follows that $h_N \leqslant h$; hence,

$$\int_M f^2\left(c - |dh|^2\right)^{-1} e^{2h_N}\, d\nu \leqslant \int_M f^2\left(c - |dh|^2\right)^{-1} e^{2h}\, d\nu < \infty. \tag{36}$$

Let $j \to \infty$ on the right-hand side of (34). Using (35), (36) and the dominated convergence theorem, the right-hand side of (34) converges to

$$\int_M f^2\left(c(x) - \left|dh(x)\right|^2\right)^{-1} e^{2h_N}\, d\nu.$$

Therefore,

$$\limsup_{j\to\infty} \int_M (u\psi_{j,N})^2\left(c(x) - \left|dh(x)\right|^2\right) d\nu \leqslant \int_M f^2\left(c(x) - \left|dh(x)\right|^2\right)^{-1} e^{2h_N}\, d\nu. \tag{37}$$

Using the definition of $\psi_{j,N}$ and Fatou's lemma on the left-hand side of (37), we obtain

$$\int_M u^2\left(c(x) - \left|dh(x)\right|^2\right)e^{2h_N}\, d\nu \leqslant \int_M f^2\left(c(x) - \left|dh(x)\right|^2\right)^{-1} e^{2h_N}\, d\nu. \tag{38}$$

We now let $N \to \infty$ in (38). Using (36) and a dominated convergence and Fatou's lemma argument, we get (9). This concludes the proof of the proposition. □

3. Proof of Theorem 2

We will first define appropriate quadratic forms.

3.1. Quadratic forms

In what follows, all quadratic forms are considered in the Hilbert space $L^2(M)$.

By h_0 we denote the quadratic form

$$h_0(u) = \int |du|^2 \, d\nu \tag{39}$$

with the domain $\mathrm{D}(h_0) = W^{1,2}(M) \subset L^2(M)$. Clearly, h_0 is a non-negative, densely defined and closed form.

By h_1 we denote the quadratic form

$$h_1(u) = \int_M V_+|u|^2 \, d\nu \tag{40}$$

with the domain

$$\mathrm{D}(h_1) = \left\{ u \in L^2(M): \int V_+|u|^2 \, d\nu < +\infty \right\}. \tag{41}$$

Clearly, h_1 is a non-negative, densely defined (since $C_c^\infty(M) \subset \mathrm{D}(h_0)$), and closed form (see [7, Example VI.1.15]).

By h_2 we denote the quadratic form

$$h_2(u) = -\int_M V_-|u|^2 \, d\nu \tag{42}$$

with the domain

$$\mathrm{D}(h_2) = \left\{ u \in L^2(M): \int_M V_-|u|^2 \, d\nu < +\infty \right\}. \tag{43}$$

Clearly, h_2 is a densely defined form. Moreover, h_2 is symmetric (but not semi-bounded below).

Lemma 8. *Let h_0 and h_1 be as in Section* 3.1. *Then the following properties hold*:

(1) *the form $h_0 + h_1$ is non-negative and closed*;
(2) *$C_c^\infty(M)$ is a form core of $h_0 + h_1$.*

Proof. Since h_0 and h_1 are non-negative and closed, it follows by [7, Theorem VI.1.31] that $h_0 + h_1$ is non-negative and closed, and (i) is proven. For the proof of property (ii), see, for instance, [8, Lemma 2.1] or [10, Lemma 2.2]. □

Lemma 9. *Let h_0, h_1 and h_2 be as in Section* 3.1. *Then the following properties hold*:

(1) *the form $h := (h_0 + h_1) + h_2$ is densely-defined, semi-bounded below and closed*;
(2) *$C_c^\infty(M)$ is a form core of h.*

Proof. By Assumption (A) and by property (ii) of Lemma 8 above, it follows that h_2 is h_0-bounded with relative bound $b < 1$. Since h_1 is non-negative, it follows that h_2 is $(h_0 + h_1)$-bounded with relative bound $b < 1$. Since $h_0 + h_1$ is a closed, non-negative form, by [7, Theorem VI.1.33], it follows that $h = (h_0 + h_1) + h_2$ is a closed semi-bounded below form. Since $C_c^\infty(M) \subset \mathrm{D}(h_0) \cap \mathrm{D}(h_1) \subset \mathrm{D}(h_2)$, it follows that h is densely defined. This proves (i). The property (ii) holds since h_2 is $(h_0 + h_1)$-bounded with relative bound $b < 1$ and since $C_c^\infty(M)$ is a form core of $h_0 + h_1$. The lemma is proven. □

Lemma 10. *Let $0 \leqslant F \in L^1_{\mathrm{loc}}(M)$. Assume that $u \in L^2(M)$ satisfies $F|u|^2 \in L^1(M)$. Then $Fu \in L^1_{\mathrm{loc}}(M)$.*

Proof. Since $F \in L^1_{\mathrm{loc}}(M)$, the lemma follows directly from the inequality

$$|Fu| = F|u| \leqslant F|u|^2 + F. \qquad \square$$

Corollary 11. *If $u \in \mathrm{D}(h)$, then $Vu \in L^1_{\mathrm{loc}}(M)$.*

Proof. Let $u \in \mathrm{D}(h) = \mathrm{D}(h_0) \cap \mathrm{D}(h_1)$. By Lemma 10 it follows that $V_+u \in L^1_{\mathrm{loc}}(M)$. Since $\mathrm{D}(h) \subset \mathrm{D}(h_2)$, by Lemma 10 we have $V_-u \in L^1_{\mathrm{loc}}(M)$. Thus $Vu \in L^1_{\mathrm{loc}}(M)$, and the corollary is proven. □

In what follows, $h(\cdot,\cdot)$ will denote the corresponding sesquilinear form obtained from h via polarization identity.

3.2. Self-adjoint operator H associated to h

Since h is densely defined, closed and semi-bounded below form in $L^2(M)$, by [7, Theorem VI.2.1] there exists a semi-bounded below self-adjoint operator H in $L^2(M)$ such that

(i) $\mathrm{Dom}(H) \subset \mathrm{D}(h)$ and

$$h(u, v) = (Hu, v) \quad \text{for all } u \in \mathrm{Dom}(H), \text{ and } v \in \mathrm{D}(h).$$

(ii) $\mathrm{Dom}(H)$ is a core of h.

(iii) If $u \in \mathrm{D}(h)$, $w \in L^2(M)$ and $h(u, v) = (w, v)$ holds for every v belonging to a core of h, then $u \in \mathrm{Dom}(H)$ and $Hu = w$.
The semi-bounded below self-adjoint operator H is uniquely determined by the condition (i).

Lemma 12. *Let T be as in Section* 1.3 *and let H be as in Section* 3.2*. Then the following operator relation holds*: $H \subset T$.

Proof. We will show that for all $u \in \mathrm{Dom}(H)$, we have $Hu = Pu$. Let $u \in \mathrm{Dom}(H)$. By property (i) of operator H we have $u \in \mathrm{D}(h)$; hence, by Corollary 11 we get $Vu \in L^1_{\mathrm{loc}}(M)$. Then, for any $v \in C_c^\infty(M)$, we have

$$(Hu, v) = h(u, v) = (du, dv) + (Vu, v), \tag{44}$$

where $(\cdot,\cdot)$ denotes the L^2-inner product.

The first equality in (44) holds by property (i) of operator H, and the second equality holds by definition of h.

Hence, using integration by parts in the first term on the right-hand side of the second equality in (44) (see, for example, [3, Lemma 8.8]), we get

$$(u, d^*dv) = \int_M (Hu - Vu)\bar{v}\, d\nu, \quad \text{for all } v \in C_c^\infty(M). \tag{45}$$

Since $Vu \in L^1_{\mathrm{loc}}(M)$ and $Hu \in L^2(M)$, it follows that $(Hu - Vu) \in L^1_{\mathrm{loc}}(M)$, and (45) implies $\Delta_M u = Hu - Vu$ (as distributions on M). Therefore, $\Delta_M u + Vu = Hu$, and this shows that $Hu = Pu$ for all $u \in \mathrm{Dom}(H)$.

Now by definition of T it follows that $\mathrm{Dom}(H) \subset \mathrm{Dom}(T)$ and $Hu = Tu$ for all $u \in \mathrm{Dom}(H)$. Therefore $H \subset T$, and the lemma is proven. □

Lemma 13. *Let T be as in Section* 1.3 *and let H be as in Section* 3.2*. Then the following operator relation holds*: $H = T$.

Proof. By Lemma 12 we have $H \subset T$, so it is enough to show that $\mathrm{Dom}(T) \subset \mathrm{Dom}(H)$.

Let $u \in \mathrm{Dom}(T)$. By definition of $\mathrm{Dom}(T)$, we have $u \in \mathrm{D}(h_0) \subset \mathrm{D}(h_2)$ and $u \in \mathrm{D}(h_1)$. Hence, $u \in \mathrm{D}(h)$. Furthermore, for all $v \in C_c^\infty(M)$ we have

$$h(u, v) = h_0(u, v) + h_1(u, v) + h_2(u, v) = (u, \Delta_M v) + \int_M Vu\bar{v}\, d\nu = (Pu, v).$$

The last equality holds since $Pu = Tu \in L^2(M)$. By Lemma 9 we know that $C_c^\infty(M)$ is a form core of h. Thus, from property (iii) of Section 3.2 we have $u \in \mathrm{Dom}(H)$ with $Hu = Pu$. This concludes the proof of the lemma. □

Proof of Theorem 2. In Lemma 13 we showed that $H = T$. Therefore, T is a self-adjoint operator in $L^2(M)$. To prove the theorem, we need to show that $T = S$. By definitions of S and T and by Corollary 11 it follows that $T \subset S$. Thus, it remains to show that $\mathrm{Dom}(S) \subset \mathrm{Dom}(T)$. Since $T = H$, it is enough to show that $\mathrm{Dom}(S) \subset \mathrm{Dom}(H)$.

By Section 3.2 the operator H is self-adjoint and semi-bounded below. By adding the form $\alpha\|u\|^2$ to the form h, where $u \in L^2(M)$ and α is a sufficiently large constant, we may assume that H is a strictly positive self-adjoint operator. Then H^{-1} is a bounded linear operator in $L^2(M)$; see [7, Section V.3.10].

Let $u \in \mathrm{Dom}(S)$, and define

$$w := u - H^{-1}(Su).$$

Since $H^{-1}(Su) \in \mathrm{Dom}(H)$ and $\mathrm{Dom}(H) \subset \mathrm{Dom}(S)$, we get $w \in \mathrm{Dom}(S)$.

Since $H = T$ and by definition of T, it follows that

$$PH^{-1}(Su) = TH^{-1}(Su) = HH^{-1}(Su) = Su.$$

Hence, since $u \in \mathrm{Dom}(S)$, by definition of S we get $Pw = Pu - Su = Su - Su = 0$.

Since $w \in \mathrm{Dom}(S)$, it follows that $w \in L^2(M) \cap L^2_{\mathrm{loc}}(M)$ and $Vw \in L^1_{\mathrm{loc}}(M)$. Moreover, $Pw = 0$. We now use Proposition 5 with $\lambda = 0$, $f(x) \equiv 0$, $c(x) = \beta$, where β is a positive constant, and $h(x) \equiv 0$. The estimate (9) gives us $w = 0$. This shows that $u = H^{-1}(Su)$. Hence, $u \in \mathrm{Dom}(H)$, and the theorem is proven. $\square$

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