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A unified Pietsch domination theorem ☆

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ABSTRACT

In this paper we prove an abstract version of Pietsch's domination theorem which unify a number of known Pietsch-type domination theorems for classes of mappings that generalize the ideal of absolutely p-summing linear operators. A final result shows that Pietsch-type dominations are totally free from algebraic conditions, such as linearity, multilinearity, etc.

1. Introduction

Pietsch's domination theorem (PDT) is a cornerstone in the theory of absolutely summing linear operators. In this paper we prove an abstract version of PDT which has a twofold purpose.

On the one hand, as expected, a Pietsch-type domination theorem turned out to be a basic result in the several (linear and non-linear) theories which generalize and extend the linear theory of absolutely summing operators. The canonical approach is as follows: (i) a class of (linear or non-linear) mappings between (normed, metric) spaces is defined, (ii) a Pietsch-type domination theorem is proved, (iii) when restricted to the class of linear operators the class defined in (i) is proved to coincide with the ideal of absolutely p-summing linear operators and the theorem proved in (ii) recovers the classical PDT, (iv) the new class is studied on its own and often compared with other related classes. Our point is that these several PDTs are always proved using the same argument. In Section 2 we construct an abstract setting where this canonical argument yields a unified version of PDT which comprises, as we show in Section 3, all the known (as far as we know) domination theorems as particular cases.

The idea behind this first purpose is to avoid the repetition of the canonical argument whenever a new class is introduced. Among the results that are unified by our main theorem (Theorem 2.2) we mention: the classic PDT for absolutely summing linear operators [3, Theorem 2.12], the Farmer and Johnson domination theorem for Lipschitz summing mappings between metric spaces [5, Theorem 1(2)], the Pietsch and Geiss domination theorem for dominated multilinear mappings ([12, Theorem 14], [6, Satz 3.2.3]), the Dimant domination theorem for strongly summing multilinear mappings and homogeneous polynomials [4, Proposition 1.2(ii) and Proposition 3.2(ii)], the domination theorem for α-subhomogeneous mappings

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[1, Theorem 2.4], the Martínez-Giménez and Sánchez-Pérez domination theorem for (D, p)-summing operators [7, Theorem 3.11], etc.

On the other hand, we note that several of the known domination theorems rely on certain algebraic conditions of the mappings in question, such as linear operators, multilinear mappings, homogeneous polynomials, etc. Our second purpose is to show that the Pietsch-type domination does not depend on any algebraic condition. In fact, in Section 4 we introduce the notion of absolutely summing arbitrary mapping just mimicking the linear definition and we apply our unified PDT proved in Section 2 to show that, even for arbitrary mappings, being absolutely summing is equivalent to satisfy a PDT. In summary, this second purpose is to show that Pietsch-type dominations are algebra-free.

2. The main result

Let X, Y and E be (arbitrary) sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Let

$$R : K \times E \times G \to [0, \infty) \quad \text{and} \quad S : \mathcal{H} \times E \times G \to [0, \infty)$$

be mappings so that:

- For each $f \in \mathcal{H}$, there is $x_0 \in E$ such that
$$R(\varphi, x_0, b) = S(f, x_0, b) = 0$$
for every $\varphi \in K$ and $b \in G$.
- The mapping
$$R_{x,b} : K \to [0, \infty), \quad R_{x,b}(\varphi) = R(\varphi, x, b)$$
is continuous for every $x \in E$ and $b \in G$.
- It holds that
$$R(\varphi, x, \eta b) \leqslant \eta R(\varphi, x, b) \quad \text{and} \quad \eta S(f, x, b) \leqslant S(f, x, \eta b)$$
for every $\varphi \in K$, $x \in E$, $0 \leqslant \eta \leqslant 1$, $b \in G$ and $f \in \mathcal{H}$.

Definition 2.1. Let R and S be as above and $0 < p < \infty$. A mapping $f \in \mathcal{H}$ is said to be *R–S-abstract p-summing* is there is a constant $C_1 > 0$ so that

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^p\right)^{\frac{1}{p}} \leqslant C_1 \sup_{\varphi \in K}\left(\sum_{j=1}^{m} R(\varphi, x_j, b_j)^p\right)^{\frac{1}{p}}, \tag{2.1}$$

for all $x_1, \dots, x_m \in E$, $b_1, \dots, b_m \in G$ and $m \in \mathbb{N}$. The infimum of such constants C_1 is denoted by $\pi_{RS,p}(f)$.

It is not difficult to show that the infimum of the constants above is attained, i.e., $\pi_{RS,p}(f)$ satisfies (2.1).

Theorem 2.2. *Let R and S be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then f is R–S-abstract p-summing if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on K such that*

$$S(f, x, b) \leqslant C\left(\int_K R(\varphi, x, b)^p \, d\mu(\varphi)\right)^{\frac{1}{p}} \tag{2.2}$$

for all $x \in E$ and $b \in G$. Moreover, the infimum of such constants C equals $\pi_{RS,p}(f)$.

Proof. Assume the existence of such a measure μ. Then, given $m \in \mathbb{N}$, $x_1, \dots, x_m \in E$ and $b_1, \dots, b_m \in G$,

$$\begin{aligned}
\sum_{j=1}^{m} S(f, x_j, b_j)^p &\leqslant C^p \sum_{j=1}^{m} \int_K R(\varphi, x_j, b_j)^p \, d\mu(\varphi) \\
&= C^p \int_K \sum_{j=1}^{m} R(\varphi, x_j, b_j)^p \, d\mu(\varphi) \\
&\leqslant C^p \sup_{\varphi \in K} \sum_{j=1}^{m} R(\varphi, x_j, b_j)^p.
\end{aligned}$$

Hence f is R–S-abstract p-summing with $\pi_{RS,p}(f) \leqslant C$.

Conversely, suppose that $f : E \to F$ is R–S-abstract p-summing. Consider the Banach space $C(K)$ of continuous real-valued functions on K. For every finite set $M = \{(x_1, b_1), \ldots, (x_k, b_k)\} \subset E \times G$, let

$$\Psi_M : K \to \mathbb{R}, \quad \Psi_M(\varphi) = \sum_{(x,b)\in M} \big(S(f, x, b)^p - \pi_{RS,p}(f)^p R(\varphi, x, b)^p\big).$$

It is convenient to regard M as a finite sequence of elements of $E \times G$ rather than a finite set (that is, repetitions are allowed). Since the functions $R_{x,b} : K \to [0, \infty)$, $R_{x,b}(\varphi) = R(\varphi, x, b)$, are continuous, it is plain that $\Psi_M \in C(K)$.

Let $\mathcal{G}$ be the set of all such Ψ_M and $\mathcal{F}$ be its convex hull. Let us show that for every $\Psi \in \mathcal{F}$ there is $\varphi_\Psi \in K$ such that $\Psi(\varphi_\Psi) \leqslant 0$: given $\Psi \in \mathcal{F}$, there are $k \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_k \in [0, 1]$, $\sum_{j=1}^k \lambda_j = 1$, and $\Psi_{M_1}, \ldots, \Psi_{M_k} \in \mathcal{G}$ so that $\Psi = \sum_{j=1}^k \lambda_j \Psi_{M_j}$.

Define

$$M_\Psi = \bigcup_{j=1}^k \big\{\big(x, \lambda_j^{\frac{1}{p}} b\big);\ (x, b) \in M_j\big\}$$

and choose $\varphi_\Psi \in K$ so that

$$\sup_{\varphi\in K} \sum_{(x,b)\in M_\Psi} R(\varphi, x, b)^p = \sum_{(x,b)\in M_\Psi} R(\varphi_\Psi, x, b)^p.$$

Notice that such a φ_Ψ exists since K is compact and $R_{x,b}$ is continuous. Then

$$\begin{aligned}
\Psi(\varphi_\Psi) &= \sum_{j=1}^k \lambda_j \Psi_{M_j}(\varphi_\Psi) \\
&= \sum_{j=1}^k \lambda_j \sum_{(x,b)\in M_j} \big(S(f, x, b)^p - \pi_{RS,p}(f)^p R(\varphi_\Psi, x, b)^p\big) \\
&\leqslant \sum_{j=1}^k \sum_{(x,b)\in M_j} \Big(\big(S\big(f, x, \lambda_j^{\frac{1}{p}} b\big)\big)^p - \pi_{RS,p}(f)^p R\big(\varphi_\Psi, x, \lambda_j^{\frac{1}{p}} b\big)^p\Big) \\
&\overset{(1)}{=} \sum_{(x,w)\in M_\Psi} \big(S(f, x, w)^p - \pi_{RS,p}(f)^p R(\varphi_\Psi, x, w)^p\big) \\
&= \Psi_{M_\Psi}(\varphi_\Psi).
\end{aligned}$$

We have been considering finite sequences instead of finite sets precisely for equality (1) to hold true. Using (2.1) we obtain $\Psi_{M_\Psi}(\varphi_\Psi) \leqslant 0$ and therefore

$$\Psi(\varphi_\Psi) \leqslant 0. \tag{2.3}$$

Let

$$\mathcal{P} = \big\{f \in C(K);\ f(\varphi) > 0 \text{ for all } \varphi \in K\big\}.$$

It is clear that $\mathcal{P}$ is non-void (every constant positive map belongs to $\mathcal{P}$), open and convex. From the definition of $\mathcal{P}$ and (2.3) it follows that $\mathcal{P} \cap \mathcal{F} = \emptyset$. So, by Hahn–Banach theorem there are $\mu_1 \in C(K)^*$ and $L > 0$ such that

$$\mu_1(g) \leqslant L < \mu_1(h) \tag{2.4}$$

for all $g \in \mathcal{F}$ and $h \in \mathcal{P}$.

If $x_0 \in E$ is such that $R(\varphi, x_0, b) = S(f, x_0, b) = 0$ for every $\varphi \in K$ and $b \in G$, then

$$0 = S(f, x_0, b)^p - \pi_{RS,p}(f)^p R(\varphi, x_0, b)^p = \Psi_{\{(x_0,b)\}} \in \mathcal{F}.$$

Hence $0 = \mu_1(0) \leqslant L$. As the constant functions $\frac{1}{k}$ belong to $\mathcal{P}$ for all $k \in \mathbb{N}$, it follows from (2.4) that $0 \leqslant L < \mu_1(\frac{1}{k})$. Making $k \to +\infty$ we get $L = 0$. Therefore (2.4) becomes

$$\mu_1(g) \leqslant 0 < \mu_1(h) \tag{2.5}$$

for all $g \in \mathcal{F}$ and all $h \in \mathcal{P}$.

Using the continuity of μ_1, we conclude that $\mu_1(h) \geqslant 0$ whenever $h \geqslant 0$ and then we can consider μ_1 a positive regular Borel measure on K.

Defining

$$\mu := \frac{1}{\mu_1(K)}\mu_1$$

it is plain that μ is a regular probability measure on K, and from (2.5) we get

$$\int_K g(\varphi)\,d\mu(\varphi) \leqslant 0$$

for all $g \in \mathcal{F}$. In particular, for each x, b we have $\Psi_{\{(x,b)\}} \in \mathcal{F}$, and

$$\int_K \Psi_{\{(x,b)\}}(\varphi)\,d\mu(\varphi) \leqslant 0.$$

So

$$\int_K \big(S(f,x,b)^p - \pi_{RS,p}(f)^p R(\varphi,x,b)^p\big)\,d\mu(\varphi) \leqslant 0,$$

and then

$$S(f,x,b)^p \leqslant \pi_{RS,p}(f)^p \int_K R(\varphi,x,b)^p\,d\mu(\varphi).$$

Taking p-roots the result follows. □

3. Recovering the known domination theorems

In this section we show how Theorem 2.2 can be easily invoked in order to obtain, as simple corollaries, all known domination theorems (to the best of our knowledge) that have appeared in the several different generalizations of the concept of absolutely p-summing linear operator. Given one of such classes of *absolutely summing* mappings, it is easy to see that for convenient choices of X, Y, E, G, $\mathcal{H}$, K, R and S, for a mapping to belong to the class is equivalent to be R–S abstract summing mapping and that, in this case, the corresponding domination theorem that holds for this class is nothing but Theorem 2.2. Whenever X is a Banach space, $B_{X'}$ denotes the closed unit ball of the dual X' of X endowed with the weak* topology.

3.1. Pietsch's domination theorem for absolutely p-summing linear operators

Let X and Y be Banach spaces. A linear operator $T : X \to Y$ is absolutely p-summing if $(T(x_n))_{n=1}^{\infty}$ is absolutely p-summable in Y whenever $(x_n)_{n=1}^{\infty}$ is weakly p-summable in X. Consider $E = X$, $x_0 = 0$, $K = B_{X'}$ and $G = \mathbb{K}$ (the scalar field). Take $\mathcal{H} = L(X;Y)$ the space of all linear operators from X into Y and define R and S by:

$$R : B_{X'} \times X \times \mathbb{K} \to [0,\infty), \quad R(\varphi,x,\lambda) = |\lambda|\big|\varphi(x)\big|,$$
$$S : L(X;Y) \times X \times \mathbb{K} \to [0,\infty), \quad S(T,x,\lambda) = |\lambda|\big\|T(x)\big\|.$$

With R and S so defined and any $0 < p < \infty$, a linear operator $T : X \to Y$ is R–S abstract p-summing if and only if it is absolutely p-summing and Theorem 2.2 becomes the classical and well-known Pietsch domination theorem [3, Theorem 2.12].

3.2. The domination theorem for Lipschitz p-summing mappings

Let X and Y be metric spaces. According to Farmer and Johnson [5], a mapping $T : X \to Y$ is Lipschitz p-summing if there is a constant C such that, for all natural n, positive real numbers $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in X$,

$$\sum_{i=1}^n a_i d_Y\big(T(x_i), T(y_i)\big)^p \leqslant C \sup_{f \in B_{X^\#}} \sum_{i=1}^n a_i \big|f(x_i) - f(y_i)\big|^p,$$

where d_Y is the metric on Y and $B_{X^\#}$ is the unit ball of the Lipschitz dual $X^\#$ of X, which is the space of all real valued Lipschitz functions on X. Then T is Lipschitz p-summing if and only if it is R–S abstract p-summing where $E = X \times X \times \mathbb{R}$, $x_0 = (x, x, 0)$ (where x is any point in X), $G = \mathbb{R}$, $K = B_{X^\#}$, which is a compact Hausdorff space in the topology of pointwise convergence on X, $\mathcal{H}$ is the set of all mappings from X to Y and R and S are defined as follows:

$$R : B_{X^\#} \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0,\infty), \quad R\big(f,(x,y,a),\lambda\big) = |a|^{1/p}|\lambda|\big|f(x) - f(y)\big|,$$
$$S : \mathcal{H} \times (X \times X \times \mathbb{R}) \times \mathbb{R} \to [0,\infty), \quad S\big(T,(x,y,a),\lambda\big) = |a|^{1/p}|\lambda| d_Y\big(T(x),T(y)\big).$$

In this context Theorem 2.2 coincides with the domination theorem for Lipschitz p-summing mappings [5, Theorem 1(2)].

3.3. The domination theorem for dominated multilinear mappings and polynomials

Dominated multilinear mappings were introduced by Pietsch [12] and dominated polynomials by Matos [8]. Let $X_1,\dots,X_n$, Y be Banach spaces. A continuous n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is p-dominated if $(T(x_j^1,\dots,x_j^n))_{j=1}^\infty$ is $\frac{p}{n}$-summable in Y whenever $(x_j^k)_{j=1}^\infty$ is weakly p-summable in X_k, $k = 1,\dots,n$. Consider $X = E = X_1 \times \cdots \times X_n$, $\mathcal{H} = L(^nX_1,\dots,X_n;Y)$ the space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y, $x_0 = (0,\dots,0)$, $K = B_{X_1'} \times \cdots \times B_{X_n'}$, $G = \mathbb{K}$ and R and S defined by:

$$R : (B_{X_1'} \times \cdots \times B_{X_n'}) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0,\infty),$$
$$R\big((\varphi_1,\dots,\varphi_n),(x_1,\dots,x_n),\lambda\big) = |\lambda|\big|\varphi_1(x_1) + \cdots + \varphi_n(x_n)\big|,$$
$$S : L(^nX_1,\dots,X_n;Y) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0,\infty),$$
$$S\big(T,(x_1,\dots,x_n),\lambda\big) = |\lambda|\big\|T(x_1,\dots,x_n)\big\|^{1/n}.$$

Then, by [1, Theorem 3.2, $(d) \Leftrightarrow (f)$], an n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is p-dominated if and only if it is R–S abstract p-summing. Moreover, in this setting Theorem 2.2 recovers the domination as it appears in [1, Theorem 3.2(D)]. In this same reference one can learn how the latter domination theorem leads to the standard domination theorem for dominated multilinear mappings that first appeared in [12, Theorem 14] (a direct proof can be found in [6, Satz 3.2.3]).

The polynomial case is easier. Let X and Y be Banach spaces. A continuous n-homogeneous polynomial $P : X \to Y$ is p-dominated if its associated symmetric n-linear mapping is p-dominated, or, equivalently, if there is a constant C such that

$$\left(\sum_{j=1}^k \big\|P(x_j)\big\|^{\frac{p}{n}}\right)^{\frac{1}{p}} \leqslant C \sup_{\varphi \in B_{X'}} \left(\sum_{j=1}^k \big|\varphi(x_j)\big|^p\right)^{\frac{1}{p}},$$

for every natural k and vectors $x_1,\dots,x_k \in X$. Then P is p-dominated if and only if it is R–S abstract p-summing, where $E = X$, $\mathcal{H}$ is the space of all n-homogeneous polynomials from X to Y, $K = B_{X'}$, $x_0 = 0$, $G = \mathbb{K}$ and R and S are defined by

$$R(\varphi,x,\lambda) = |\lambda|\big|\varphi(x)\big|, \qquad S(Q,x,\lambda) = |\lambda|\big\|Q(x)\big\|^{\frac{1}{n}}.$$

In this case, Theorem 2.2 recovers the standard domination theorem for dominated polynomials [8, Proposition 3.1].

3.4. The domination theorem for strongly p-summing multilinear mappings and homogeneous polynomials

Strongly p-summing mappings were introduced by Dimant [4]. Let $X_1,\dots,X_n$, Y be Banach spaces. A continuous n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is strongly p-summing if there is a constant C such that

$$\left(\sum_{j=1}^k \big\|T\big(x_j^1,\dots,x_j^n\big)\big\|^p\right)^{\frac{1}{p}} \leqslant C \sup_{A \in B_{\mathcal{L}(X_1,\dots,X_n)}} \left(\sum_{j=1}^k \big|A\big(x_j^1,\dots,x_j^n\big)\big|^p\right)^{\frac{1}{p}},$$

for every natural k and vectors $x_j^m \in X_m$, $j = 1,\dots,k$, $m = 1,\dots,n$, where $\mathcal{L}(X_1,\dots,X_n)$ is the space of all continuous n-linear forms on $X_1 \times \cdots \times X_n$. Then T is strongly p-summing if and only if it is R–S abstract p-summing considering $X = E = X_1 \times \cdots \times X_n$, $\mathcal{H} = L(^nX_1,\dots,X_n;Y)$ the space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y, $x_0 = (0,\dots,0)$, $K = B_{(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)'}$, $G = \mathbb{K}$ and R and S defined by:

$$R : B_{(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)'} \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0,\infty),$$
$$R\big(\varphi,(x_1,\dots,x_n),\lambda\big) = |\lambda|\big|\varphi(x_1 \otimes \cdots \otimes x_n)\big|,$$
$$S : L(^nX_1,\dots,X_n;Y) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0,\infty),$$
$$S\big(T,(x_1,\dots,x_n),\lambda\big) = |\lambda|\big\|T(x_1,\dots,x_n)\big\|.$$

In that case, Theorem 2.2 recovers the corresponding domination theorem [4, Proposition 1.2(ii)].

On the other hand, a continuous n-homogeneous polynomial $P : X \to Y$ is strongly p-summing if there is a constant C such that

$$\left(\sum_{j=1}^{k} \left\| P(x_j) \right\|^p \right)^{\frac{1}{p}} \leqslant C \sup_{Q \in B_{\mathcal{P}(^n X)}} \left(\sum_{j=1}^{k} \left| Q(x_j) \right|^p \right)^{\frac{1}{p}},$$

for every natural k and vectors $x_1, \ldots, x_k \in X$, where $\mathcal{P}(^n X)$ is the space of all continuous scalar-valued n-homogeneous polynomials on X. These polynomials are also particular cases of R–S abstract p-summing mappings. Just take $E = X$, $\mathcal{H} = P(^n X; Y)$ the space of all n-homogeneous polynomials from X to Y, $x_0 = 0$, $K = B_{(\hat{\otimes}^s_{n,\pi} X)'} = B_{\mathcal{P}(^n X)}$, $G = \mathbb{K}$ and R and S defined by:

$R : B_{\mathcal{P}(^n X)} \times X \times \mathbb{K} \to [0, \infty)$,

$$R(Q, x, \lambda) = |\lambda| \left| Q(x) \right|,$$

$S : P(^n X; Y) \times X \times \mathbb{K} \to [0, \infty)$,

$$S(P, x, \lambda) = |\lambda| \left\| P(x) \right\|.$$

As expected, Theorem 2.2 also recovers the corresponding domination theorem for strongly p-summing n-homogeneous polynomials [4, Theorem 3.2(ii)].

3.5. The domination theorem for p-semi-integral multilinear mappings

The class of p-semi-integral multilinear mappings was introduced in [2,11]. Let $X_1, \ldots, X_n, Y$ be Banach spaces. A continuous n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is p-semi-integral if there is a constant C such that

$$\left(\sum_{j=1}^{k} \left\| T\left(x_j^1, \ldots, x_j^n\right) \right\|^p \right)^{\frac{1}{p}} \leqslant C \sup_{\varphi_i \in B_{X_i'},\, i=1,\ldots,n} \left(\sum_{j=1}^{k} \left| \varphi_1\left(x_j^1\right) \cdots \varphi_n\left(x_j^n\right) \right|^p \right)^{\frac{1}{p}},$$

for every natural k and vectors $x_j^m \in X_m$, $j = 1, \ldots, k$, $m = 1, \ldots, n$. Then T is p-semi-integral if and only if it is R–S abstract p-summing considering $E = X_1 \times \cdots \times X_n$, $\mathcal{H} = L(^n X_1, \ldots, X_n; Y)$ the space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y, $x_0 = (0, \ldots, 0)$, $K = B_{X_1'} \times \cdots \times B_{X_n'}$, $G = \mathbb{K}$ and R and S defined by:

$R : (B_{X_1'} \times \cdots \times B_{X_n'}) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0, \infty)$,

$$R\big((\varphi_1, \ldots, \varphi_n), (x_1, \ldots, x_n), \lambda\big) = |\lambda| \left| \varphi_1(x_1) \cdots \varphi_n(x_n) \right|,$$

$S : L(^n X_1, \ldots, X_n; Y) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0, \infty)$,

$$S\big(T, (x_1, \ldots, x_n), \lambda\big) = |\lambda| \left\| T(x_1, \ldots, x_n) \right\|.$$

In that case, Theorem 2.2 recovers the corresponding domination theorem [2, Theorem 1].

3.6. The domination theorem for $\tau(p)$-summing multilinear mappings

The class of $\tau(p)$-summing multilinear mappings was introduced by X. Mujica [10]. Let $X_1, \ldots, X_n, Y$ be Banach spaces. A continuous n-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is $\tau(p)$-summing if there is a constant C such that

$$\left(\sum_{j=1}^{k} \left| b_j\left(T\left(x_j^1, \ldots, x_j^n\right)\right) \right|^p \right)^{\frac{1}{p}} \leqslant C \sup_{\varphi_i \in B_{X_i'},\, i=1,\ldots,n,\, y \in B_Y} \left(\sum_{j=1}^{k} \left| \varphi_1\left(x_j^1\right) \cdots \varphi_j\left(x_j^n\right) b_j(y) \right|^p \right)^{\frac{1}{p}},$$

for every natural k, functionals $b_1, \ldots, b_k \in Y'$ and vectors $x_j^m \in X_m$, $j = 1, \ldots, k$, $m = 1, \ldots, n$. Then T is $\tau(p)$-summing if and only if it is R–S abstract p-summing considering $E = X_1 \times \cdots \times X_n$, $\mathcal{H} = L(^n X_1, \ldots, X_n; Y)$ the space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to Y, $x_0 = (0, \ldots, 0)$, $K = B_{X_1'} \times \cdots \times B_{X_n'} \times B_{Y''}$, $G = Y'$ and R and S defined by:

$R : (B_{X_1'} \times \cdots \times B_{X_n'} \times B_{Y''}) \times (X_1 \times \cdots \times X_n) \times Y' \to [0, \infty)$,

$$R\big((\varphi_1, \ldots, \varphi_n, \varphi), (x_1, \ldots, x_n), b\big) = \left| \varphi_1(x_1) \cdots \varphi_n(x_n) \right| \left| \varphi(b) \right|,$$

$S : L(^n X_1, \ldots, X_n; Y) \times (X_1 \times \cdots \times X_n) \times Y' \to [0, \infty)$,

$$S\big(T, (x_1, \ldots, x_n), b\big) = \left| \langle b, T(x_1, \ldots, x_n) \rangle \right|.$$

In that case, Theorem 2.2 recovers the corresponding domination theorem [10, Theorem 3.5].

3.7. The domination theorem for subhomogeneous mappings

The class of α-subhomogeneous (p,q)-summing mappings was introduced in [1]. Let X and Y be Banach spaces. A mapping $f : X \to Y$ is α-subhomogeneous (p,q)-summing, $\alpha > 0$, if $\|f(\lambda x)\| \geqslant \lambda^{\alpha}\|f(x)\|$ for every $x \in X$ and $0 < \lambda < 1$, and $(f(x_n))_{n=1}^{\infty}$ is absolutely p-summable in Y whenever $(x_n)_{n=1}^{\infty}$ is weakly q-summable in X. Then f is α-subhomogeneous $(\frac{p}{\alpha}, p)$-summing if and only if it is R–S abstract p-summing considering $E = X$,

$$\mathcal{H} = \big\{ f : E \to Y;\ f \text{ is } \alpha\text{-subhomogeneous and } f(0) = 0 \big\},$$

$x_0 = 0$, $K = B_{X'}$, $G = \mathbb{K}$ and R and S defined by:

$R : B_{X'} \times E \times \mathbb{K} \to [0,\infty)$,

$$R(\varphi, x, \eta) = |\eta|\big|\varphi(x)\big|,$$

$S : \mathcal{H} \times E \times \mathbb{K} \to [0,\infty)$,

$$S(f, x, \eta) = |\eta|\big\|f(x)\big\|^{1/\alpha}.$$

In that case, Theorem 2.2 recovers the corresponding domination theorem [1, Theorem 2.4].

3.8. The domination theorem for (D,p)-summing linear operators

The class of (D,p)-summing linear operators was introduced by F. Martínez-Giménez and E.A. Sánchez-Pérez [7]. Let Y be a Banach space and X be a Banach function space compatible with the countably additive vector measure λ of the range dual pair $D = (\lambda, \lambda')$. A linear operator $T : X \to Y$ is (D,p)-summing if there is a constant C such that

$$\left(\sum_{j=1}^{k} \big\|T(f_j)\big\|^p \right)^{\frac{1}{p}} \leqslant C \sup_{g \in B_{L_1(\lambda')}} \left(\sum_{j=1}^{k} \left| \left\langle \int f_i \, d\lambda, \int g \, d\lambda' \right\rangle \right|^p \right)^{\frac{1}{p}},$$

for every natural k and functions $f_1, \ldots, f_k \in X$. Then T is (D,p)-summing if and only if it is R–S abstract p-summing considering $E = X$,

$$\mathcal{H} = L(E;Y)$$

the space of all linear mappings from E to Y, $x_0 = 0$, $K = \overline{B}_{L_1(\lambda')}$, $G = \mathbb{R}$ and R and S defined by:

$R : \overline{B}_{L_1(\lambda')} \times E \times \mathbb{R} \to [0,\infty)$,

$$R(\phi, f, \eta) = |\eta|\big|(f,\phi)\big|,$$

$S : \mathcal{H} \times E \times \mathbb{R} \to [0,\infty)$,

$$S(T, f, \eta) = |\eta|\big\|T(f)\big\|.$$

In that case, Theorem 2.2 recovers the corresponding domination theorem [7, Theorem 3.11].

4. Absolutely summing arbitrary mappings

According to the usual definition of absolutely summing linear operators by means of inequalities, the following definition is quite natural:

Definition 4.1. Let E and F be Banach spaces. An arbitrary mapping $f : E \to F$ is *absolutely p-summing at $a \in E$* if there is a $C \geqslant 0$ so that

$$\sum_{j=1}^{m} \big\|f(a + x_j) - f(a)\big\|^p \leqslant C \sup_{\varphi \in B_{E'}} \sum_{j=1}^{m} \big|\varphi(x_j)\big|^p$$

for every natural number m and every $x_1, \ldots, x_m \in E$.

As [9, Theorem 3.5] makes clear, the above definition is actually an adaptation of [9, Definition 3.1].

We finish the paper applying Theorem 2.2 once more to show that, even in the absence of algebraic conditions, absolutely p-summing mappings are exactly those which enjoy a Pietsch-type domination:

Theorem 4.2. *Let E and F be Banach spaces. An arbitrary mapping $f : E \to F$ is absolutely p-summing at $a \in E$ if and only if there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{E'}$ such that*

$$\|f(a+x) - f(a)\| \leqslant C\left(\int_{B_{E'}} \left|\varphi(x)\right|^p \, d\mu(\varphi)\right)^{\frac{1}{p}}$$

for all $x \in E$.

Proof. Using a clever argument from [5, p. 2] (also credited to M. Mendel and G. Schechtman), applied by Farmer and Johnson in the context of Lipschitz summing mappings, one can see that f is absolutely p-summing at a if and only if there is a $C \geqslant 0$ so that

$$\sum_{j=1}^{m} |b_j| \left\| f(a+x_j) - f(a)\right\|^p \leqslant C \sup_{\varphi \in B_{E'}} \sum_{j=1}^{m} |b_j| \left|\varphi(x_j)\right|^p \tag{4.1}$$

for every $m \in \mathbb{N}$, $x_1, \dots, x_m \in E$ and scalars $b_1, \dots, b_m$. The idea of the argument is the following: by approximation, it is enough to deal with rationals b_i and, by cleaning denominators, we can resume to integers b_i. Then, for b_i (integer) and $x_1, \dots, x_m$, we consider the new collection of vectors in which each x_i is repeated b_i times.

Putting $X = E$, $Y = F$,

$$\mathcal{H} = \{f : E \to F\},$$

$x_0 = 0$, $K = B_{E'}$, $G = \mathbb{K}$ and R and S defined by:

$R : B_{E'} \times E \times \mathbb{K} \to [0, \infty)$,

$$R(\varphi, x, \lambda) = |\lambda| \left|\varphi(x)\right|,$$

$S : \mathcal{H} \times E \times \mathbb{K} \to [0, \infty)$,

$$S(f, x, \lambda) = |\lambda| \left\| f(a+x) - f(a)\right\|,$$

in view of characterization (4.1) it follows that f is absolutely p-summing if and only if f is R–S abstract p-summing. So, Theorem 2.2 completes the proof. □

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References

[1] G. Botelho, D. Pellegrino, P. Rueda, A nonlinear Pietsch domination theorem, Monatsh. Math. 158 (2009) 247–257.
[2] E. Çaliskan, D.M. Pellegrino, On the multilinear generalizations of the concept of absolutely summing operators, Rocky Mountain J. Math. 37 (2007) 1137–1154.
[3] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge University Press, 1995.
[4] V. Dimant, Strongly p-summing multilinear mappings, J. Math. Anal. Appl. 278 (2003) 182–193.
[5] J. Farmer, W.B. Johnson, Lipschitz p-summing operators, Proc. Amer. Math. Soc. 137 (2009) 2989–2995.
[6] S. Geiss, Ideale multilinearer Abbildungen, Diplomarbeit, 1985.
[7] F. Martínez-Giménez, E.A. Sánchez-Pérez, Vector measure range duality and factorizations of (D, p)-summing operators from Banach function spaces, Bull. Braz. Math. Soc. (N.S.) 35 (2004) 51–69.
[8] M.C. Matos, Absolutely summing holomorphic mappings, An. Acad. Brasil. Cienc. 68 (1996) 1–13.
[9] M.C. Matos, Nonlinear absolutely summing mappings, Math. Nachr. 258 (2003) 71–89.
[10] X. Mujica, $\tau(p; q)$-Summing mappings and the domination theorem, Port. Math. (N.S.) 65 (2008) 221–226.
[11] D. Pellegrino, Aplicações entre espaços de Banach relacionadas à convergência de séries, thesis, Universidade Estadual de Campinas (UNICAMP), 2002.
[12] A. Pietsch, Ideals of multilinear functionals (designs of a theory), in: Proceedings of the Second International Conference on Operator Algebras, Ideals, and Their Applications in Theoretical Physics, Leipzig, 1983, in: Teubner-Texte Math., vol. 67, Teubner, Leipzig, 1984, pp. 185–199.