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Asymptotic expansions of Bessel, Anger and Weber transformations ☆

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ABSTRACT

In this paper we explore higher order numerical quadrature for the integrations of systems containing Bessel, Anger and Weber functions. The method is constructed by finding the approximate solution of the differential equation and truncating the asymptotic series. Numerical examples based on theoretical results are presented to demonstrate the efficiency of the method.

1. Introduction

The computation of the highly oscillatory integrals is one of the oldest and most important issues in numerical analysis. In many areas of applied mathematics, such as physics, chemistry and engineering, one always encounters the problems of computing the integrals involving oscillatory functions

$$\int_a^b f(x)J_v(rx)\,dx, \qquad \int_a^b f(x)\cos(r_1x)J_v(rx)\,dx, \qquad \int_a^b f(x)A_v(rx)\,dx, \qquad \int_a^b f(x)W_v(rx)\,dx, \tag{1.1}$$

where $J_v(rx)$ is Bessel function of the first kind and of order v, $A_v(rx)$ and $W_v(rx)$ are Anger function and Weber function of order v, respectively, v is arbitrary positive real number and r is large. For large r, the integrands become highly oscillatory and present serious difficulties in obtaining numerical convergence of the integrations.

Levin [1,2] presented a collocation method for $\int_a^b f(x)J_v(rx)\,dx$ with an error bound $O(r^{-2.5})$ [3] under the assumption that $0 \notin [a,b]$. The collocation method is applicable to a wide class of oscillatory integrals with weight functions satisfying certain differential conditions. But the method is not efficient for the case $\int_a^b f(x)A_v(rx)\,dx$ and $\int_a^b f(x)W_v(rx)\,dx$ where v is not the integer.

The homotopy perturbation method (HPM) is a powerful tool in nonlinear problems [4,5]. This method is to continuously deform a simple problem which is easy to solve into the under study problem. It is worth mentioning that the origin of the homotopy method can be traced back to Layne T. Watson [6–8]. To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation

$$A(u) - f(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega \tag{1.2}$$

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with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad \mathbf{r} \in \Gamma \tag{1.3}$$

where A is a general differential operator, B is a boundary operator, $f(\mathbf{r})$ is a known analytic function, Γ is the boundary of the domain Ω. The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (1.2), therefore, can be rewritten as follows

$$L(u) + N(u) - f(\mathbf{r}) = 0. \tag{1.4}$$

By the homotopy technique, we construct a homotopy $v(r, p): \Omega \times [0, 1] \to R^n$ which satisfies

$$L(v) - L(u_0) + pL(u_0) + p\big[N(v) - f(\mathbf{r})\big] = 0, \tag{1.5}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1.2), which satisfies generally the boundary conditions.

The basic assumption is that the solution of Eq. (1.5) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots, \tag{1.6}$$

then

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{1.7}$$

Recently, Molabahrami [9] applied the HPM to calculate Fourier transformations $\int_a^b f(x)e^{i\omega g(x)}\,dx$ under the condition that the oscillatory function $g(x)$ has not critical point at the endpoints of integration region. Molabahrami [9] obtained numerical solution of highly oscillatory integrals

$$I[f] = \int_a^b f(x)e^{i\omega g(x)}\,dx$$

based on the first-order differential equation

$$u'(x) + i\omega g'(x)u(x) = f(x). \tag{1.8}$$

The one of particular solution of Eq. (1.8) can be expressed by the following integration

$$u(x)e^{i\omega g(x)} = \int f(x)e^{i\omega g(x)}\,dx. \tag{1.9}$$

Assume that $L(u) = i\omega g'(x)u(x)$ and $N(u) = u'(x)$, then the HPM can be illustrated as follows

$$L(v) - L(y_0) + pL(y_0) + p\big[N(v) - f(x)\big] = 0, \tag{1.10}$$

where y_0 is an initial approximation of (1.10) and $p \in [0, 1]$ is an imbedding parameter. Substituting (1.6) into (1.10) and equating the terms with the identical powers of p, we can obtain all v_j. Therefore, for $u'(x) + i\omega g'(x)u(x) = f(x)$, it is true that

$$I[f] = \int_a^b f(x)e^{i\omega g(x)}\,dx = u(x)e^{i\omega g(x)}\Big|_{x=a}^{b} \sim \left\{\sum_{j=0}^{\infty} v_j(x)e^{i\omega g(x)}\right\}\Bigg|_{x=a}^{b}. \tag{1.11}$$

In [10], Chen and Xiang extended the HPM to calculate the multivariate vector-value highly oscillatory integrals.

In this paper, we apply the HPM for computing the integrals (1.1) by constructing an entirely different homotopy from the paper [10]. The paper is organized as follows. In Section 2, we present the details of the HPM for $\int_a^b f(x)J_v(rx)\,dx$ and $\int_a^b f(x)\cos(r_1 x)J_v(rx)\,dx$, including the asymptotic formulas and these asymptotic orders. In Section 3, the HPM is applied for $\int_a^b f(x)A_v(rx)\,dx$ and $\int_a^b f(x)W_v(rx)\,dx$. Finally, we conclude this paper in Section 4.

2. The HPM for Bessel transformations

2.1. The case $\int_a^b f(x)J_v(rx)\,dx$ $(0 \notin [a,b])$

Assume that $\{U_k(x)\}_{k=1}^{\infty}$ are the solution of the first-order differential equation

$$\left[\sum_{k=1}^{\infty} V_k(x)J_{v+k}(rx)(rx)^{v+k}\right]' = f(x)J_v(rx),$$

that is,

$$\left[\sum_{k=1}^{\infty} U_k(x)J_{v+k}(rx)(rx)^{v+k}\right]' = f(x)J_v(rx). \tag{2.1}$$

Apparently, (2.1) satisfies

$$\sum_{k=1}^{\infty} U_k(x)J_{v+k}(rx)(rx)^{v+k} = \int f(x)J_v(rx)\,dx. \tag{2.2}$$

Then

$$\int_a^b f(x)J_v(rx)\,dx = \sum_{k=1}^{\infty} U_k(x)J_{v+k}(rx)(rx)^{v+k}\Big|_{x=a}^{b}. \tag{2.3}$$

Expanding (2.1) and rearranging, we can construct a homotopy

$$\sum_{k=1}^{\infty} rU_k(x)J_{v+k-1}(rx)(rx)^{v+k} + p\left[\sum_{k=1}^{\infty} U_k'(x)J_{v+k}(rx)(rx)^{v+k} - f(x)J_v(rx)\right] = 0. \tag{2.4}$$

Suppose that the solution $U_k(x)$ of (2.4) can be expressed as a series in p,

$$U_k(x) = U_{k,0}(x) + pU_{k,1}(x) + p^2U_{k,2}(x) + p^3U_{k,3}(x) + \cdots \tag{2.5}$$

for $p \to 1$. Substituting (2.5) into (2.4) and equating the terms with the identical powers of p, we have

$$\begin{cases} p^0\colon\quad \displaystyle\sum_{k=1}^{\infty} rU_{k,0}(x)J_{k+v-1}(rx)(rx)^{v+k} = 0 \quad\Rightarrow\quad U_{k,0}(x) = 0,\quad k \geqslant 1, \\ p^1\colon\quad \displaystyle\sum_{k=1}^{\infty}\bigl(rU_{k,1}(x)J_{k+v-1}(rx) + U_{k,0}'(x)J_{k+v}(rx)\bigr)(rx)^{v+k} - f(x)J_v(rx) = 0 \\ \qquad\quad \Rightarrow\quad U_{1,1}(x) = \dfrac{1}{r^{v+2}}\dfrac{f(x)}{x^{v+1}},\quad U_{k,1}(x) = 0,\quad k \geqslant 2, \\ p^2\colon\quad \displaystyle\sum_{k=1}^{\infty}\bigl(rU_{k,2}(x)J_{k+v-1}(rx) + U_{k,1}'(x)J_{k+v}(rx)\bigr)(rx)^{v+k} = 0 \\ \qquad\quad \Rightarrow\quad U_{1,2}(x) = 0,\quad U_{2,2}(x) = -\dfrac{1}{r^2}\dfrac{U_{1,1}'(x)}{x},\quad U_{k,2}(x) = 0,\quad k \geqslant 3, \\ \quad\vdots \\ p^n\colon\quad \displaystyle\sum_{k=1}^{\infty}\bigl(rU_{k,n}(x)J_{k+v-1}(rx) + U_{k,n-1}'(x)J_{k+v}(rx)\bigr)(rx)^{v+k} = 0 \\ \qquad\quad \Rightarrow\quad U_{n,n}(x) = -\dfrac{1}{r^2}\dfrac{U_{n-1,n-1}'(x)}{x},\quad U_{k,n}(x) = 0,\quad 1 \leqslant k \leqslant n-1 \text{ and } k \geqslant n+1. \end{cases} \tag{2.6}$$

From (2.3), we get

$$I[f] = \int_a^b f(x)J_v(rx)\,dx \sim \left\{\sum_{k=1}^{\infty} U_{k,k}J_{v+k}(rx)(rx)^{v+k}\right\}\Big|_{x=a}^{b}. \tag{2.7}$$

If the n-terms approximations are sufficient, the HPM for $\int_a^b f(x)J_v(rx)\,dx$ can be defined as follows

$$I_n[f]=\left\{\sum_{k=1}^{n}U_{k,k}(x)J_{v+k}(rx)(rx)^{v+k}\right\}\Bigg|_{x=a}^{b}. \tag{2.8}$$

We can evaluate all of components $U_{k,k}(x)$ by Maple 11.

Theorem 2.1. *For the integrals $\int_a^b f(x)J_v(rx)\,dx$ $(0\notin[a,b])$, let $f(x)$ be a sufficiently smooth function in $[a,b]$, then the error of the HPM satisfies*

$$\big|I[f]-I_n[f]\big|=O\big(r^{-n-3/2}\big),\quad r\gg 1, \tag{2.9}$$

where $I_n[f]$ is defined by (2.8)*.*

Proof. By $\frac{d}{dx}x^{k+1}J_{k+1}(x)=x^{k+1}J_k(x)$ [13], we have $\frac{d}{r\,dx}(rx)^{k+1}J_{k+1}(rx)=(rx)^{k+1}J_k(rx)$. Notice that

$$\begin{aligned}
I[f]&=\int_a^b f(x)J_v(rx)\,dx=\frac{1}{r^{v+1}}\int_a^b f(x)x^{-v-1}(rx)^{v+1}J_v(rx)\,dx\\
&=\frac{1}{r^{v+2}}\int_a^b f(x)x^{-v-1}\,d(rx)^{v+1}J_{v+1}(rx)\\
&=\frac{1}{r^{v+2}}f(x)x^{-v-1}(rx)^{v+1}J_{v+1}(rx)\Big|_{x=a}^{b}-\frac{1}{r^{v+2}}\int_a^b\big[f(x)x^{-v-1}\big]'(rx)^{v+1}J_{v+1}(rx)\,dx\\
&=U_{1,1}(x)(rx)^{v+1}J_{v+1}(rx)\big|_{x=a}^{b}-\int_a^b U'_{1,1}(x)(rx)^{v+1}J_{v+1}(rx)\,dx,
\end{aligned}$$

then by integrating repeatedly by parts and using (2.6),

$$I[f]=\sum_{k=1}^{n}U_{k,k}(x)J_{v+k}(rx)(rx)^{v+k}\Big|_{x=a}^{b}-\int_a^b U'_{n,n}(x)(rx)^{v+n}J_{v+n}(rx)\,dx.$$

According to the definition of $I_n[f]$ and $U_{k,k}(x)$, we get

$$\begin{aligned}
\big|I[f]-I_n[f]\big|&=\left|\int_a^b U'_{n,n}(x)(rx)^{v+n}J_{v+n}(rx)\,dx\right|=\left|\int_a^b\frac{U'_{n,n}(x)}{r^2x}\big[(rx)^{v+n+1}J_{v+n+1}(rx)\big]'\,dx\right|\\
&\leqslant\left|U_{n+1,n+1}(x)(rx)^{v+n+1}J_{v+n+1}(rx)\big|_{x=a}^{b}\right|+\left|\int_a^b U'_{n+1,n+1}(x)(rx)^{v+n+1}J_{v+n+1}(rx)\,dx\right|.
\end{aligned}$$

Due to the facts that $\|J_v(rx)\|_\infty=O(r^{-1/2})$ $(r\to\infty,\ 0<a\leqslant x\leqslant b)$, $\|U_{k,k}(x)\|_\infty=O(r^{-v-2k})$ and $\|U'_{k,k}(x)\|_\infty=O(r^{-v-2k})$, the error of the HPM $I_n[f]$ satisfies

$$\big|I[f]-I_n[f]\big|=O\big(r^{-v-2(n+1)}r^{v+n+1}r^{-1/2}\big)=O\big(r^{-n-3/2}\big).\qquad\square$$

Example 1. Consider the HPM for $\int_1^2\frac{1}{1+x^2}J_2(rx)\,dx$ (see Fig. 1).

2.2. *The case $\int_a^b f(x)\cos(r_1x)J_v(rx)\,dx$ $(0\notin[a,b])$*

The integrals $\int_a^b f(x)\cos(r_1x)J_v(rx)\,dx$ can be rewritten as

$$\int_a^b f(x)\cos(r_1x)J_v(rx)\,dx=Re\int_a^b f(x)e^{ir_1x}J_v(rx)\,dx. \tag{2.10}$$

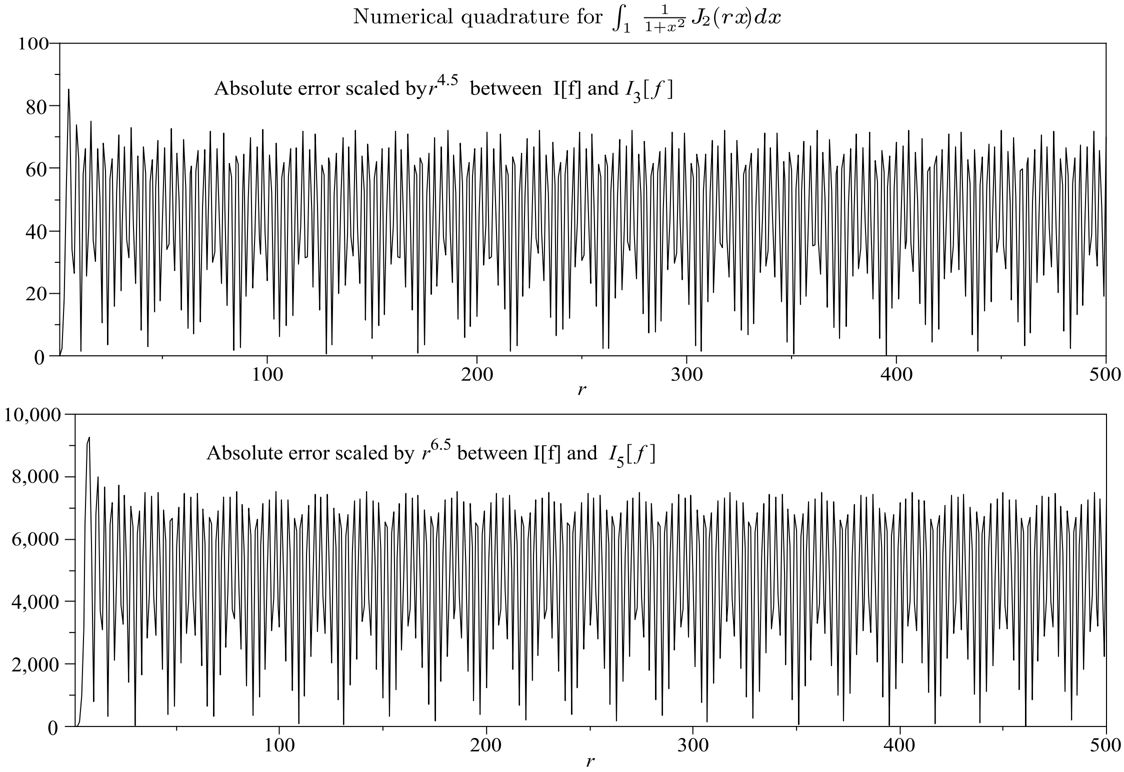

Fig. 1. The HPM $I_n[f]$ for $\int_1^2 \frac{1}{1+x^2} J_2(rx)\,dx$ with $n=3$ or 5. The absolute errors are scaled by $r^{4.5}$ or $r^{6.5}$, respectively.

Assume that

$$\left[\sum_{k=1}^{\infty} U_k(x) J_{v+k}(rx)(rx)^{v+k}\right]' = f(x)e^{ir_1x} J_v(rx), \quad r_1 \leqslant r \tag{2.11}$$

or

$$\left[U(x)e^{ir_1x}\right]' = f(x)e^{ir_1x} J_v(rx), \quad r_1 > r. \tag{2.12}$$

Expanding (2.11) and (2.12), and rearranging, we have

$$\sum_{k=1}^{\infty} U_k'(x) J_{v+k}(rx)(rx)^{v+k} + r\sum_{k=1}^{\infty} U_k(x) J_{v+k-1}(rx)(rx)^{v+k} - f(x)e^{ir_1x} J_v(rx) = 0 \tag{2.13}$$

or

$$U'(x) + ir_1 U(x) - f(x) J_v(rx) = 0. \tag{2.14}$$

We construct the following homotopies:

$$(1)\quad r\sum_{k=1}^{\infty} U_k(x) J_{v+k-1}(rx)(rx)^{v+k} + p\left[\sum_{k=1}^{\infty} U_k'(x) J_{v+k}(rx)(rx)^{v+k} - f(x)e^{ir_1x} J_v(rx)\right] = 0, \quad \text{when } r \geqslant r_1, \tag{2.15}$$

$$(2)\quad ir_1 U(x) + p\left[U'(x) - f(x) J_v(rx)\right] = 0, \quad \text{when } r < r_1. \tag{2.16}$$

We have the following results:

(1) For $r \geqslant r_1$, substituting (2.5) into (2.15), there exists

$$\begin{cases} p^0\colon & \sum_{k=1}^{\infty} rU_{k,0}(x)J_{k+v-1}(rx)(rx)^{v+k} = 0 \quad \Rightarrow \quad U_{k,0}(x) = 0, \quad k \geqslant 1, \\ p^1\colon & \sum_{k=1}^{\infty} rU_{k,1}(x)J_{k+v-1}(rx)(rx)^{v+k} + \sum_{k=1}^{\infty} U'_{k,0}(x)J_{k+v}(rx)(rx)^{v+k} - f(x)e^{ir_1x}J_v(rx) = 0 \\ & \quad \Rightarrow \quad U_{1,1}(x) = \frac{1}{r^{v+2}}\frac{f(x)e^{ir_1x}}{x^{v+1}}, \quad U_{k,1}(x) = 0, \quad k \geqslant 2, \\ p^2\colon & \sum_{k=1}^{\infty} rU_{k,2}(x)J_{k+v-1}(rx)(rx)^{v+k} + \sum_{k=1}^{\infty} U'_{k,1}(x)J_{k+v}(rx)(rx)^{v+k} = 0 \\ & \quad \Rightarrow \quad U_{1,2}(x) = 0, U_{2,2}(x) = -\frac{1}{r^2}\frac{U'_{1,1}(x)}{x}, \quad U_{k,2}(x) = 0, \quad k \geqslant 3, \\ \vdots & \\ p^n\colon & \sum_{k=1}^{\infty} rU_{k,n}(x)J_{k+v-1}(rx)(rx)^{v+k} + \sum_{k=1}^{\infty} U'_{k,n-1}(x)J_{k+v}(rx)(rx)^{v+k} = 0 \\ & \quad \Rightarrow \quad U_{n,n}(x) = -\frac{1}{r^2}\frac{U'_{n-1,n-1}(x)}{x}, \quad U_{k,n}(x) = 0, \quad 1 \leqslant k \leqslant n-1 \text{ and } k \geqslant n+1. \end{cases} \tag{2.17}$$

(2) For $r < r_1$, let $U(x) = U_0(x) + pU_1(x) + p^2U_2(x) + p^3U_3(x) + \cdots$, then

$$\begin{cases} p^0\colon & ir_1U_0(x) = 0 \quad \Rightarrow \quad U_0(x) = 0, \\ p^1\colon & ir_1U_1(x) + U'_0(x) - f(x)J_v(rx) = 0 \quad \Rightarrow \quad U_1(x) = \frac{f(x)J_v(rx)}{ir_1}, \\ p^2\colon & ir_1U_2(x) + U'_1(x) = 0 \quad \Rightarrow \quad U_2(x) = -\frac{U'_1(x)}{ir_1}, \\ \vdots & \\ p^n\colon & ir_1U_n(x) + U'_{n-1}(x) = 0 \quad \Rightarrow \quad U_n(x) = -\frac{U'_{n-1}(x)}{ir_1}. \end{cases} \tag{2.18}$$

We get

$$I[f] = \int_a^b f(x)\cos(r_1x)J_v(rx)\,dx \sim \left\{Re\sum_{k=1}^{\infty} U_{k,k}(x)J_{v+k}(rx)(rx)^{v+k}\right\}\Bigg|_{x=a}^{b}, \tag{2.19}$$

where $r \geqslant r_1$ and $U_{k,k}(x)$ is defined by (2.17), and

$$I[f] = \int_a^b f(x)\cos(r_1x)J_v(rx)\,dx \sim \left\{Re\sum_{k=1}^{\infty} U_k(x)e^{ir_1x}\right\}\Bigg|_{x=a}^{b}, \tag{2.20}$$

where $r < r_1$ and $U_k(x)$ is defined by (2.18).

Define the HPM for $\int_a^b f(x)\cos(r_1x)J_v(rx)\,dx$ as

$$I_n[f] = \left\{Re\sum_{k=1}^{n} U_{k,k}(x)J_{v+k}(rx)(rx)^{v+k}\right\}\Bigg|_{x=a}^{b}, \quad r \geqslant r_1 \text{ and } r \gg 1 \tag{2.21}$$

or

$$I_n[f] = \left\{Re\sum_{k=1}^{n} U_k(x)e^{ir_1x}\right\}\Bigg|_{x=a}^{b}, \quad r < r_1 \text{ and } r_1 \gg 1. \tag{2.22}$$

Theorem 2.2. *For $\int_a^b f(x)\cos(r_1x)J_v(rx)\,dx$ $(0 \notin [a,b])$, if $f(x)$ is a sufficiently smooth function in $[a,b]$ and $\max(r,r_1) \gg 1$, then the error of the HPM satisfies*

$$\left|I[f]-I_n[f]\right| = O\left(r^{-1/2}\left(\frac{r}{r_1}\right)^{-n-1}\right), \quad r \geqslant r_1 \quad or \quad O\left(r^{-1/2}\left(\frac{r_1}{r}\right)^{-n-1}\right), \quad r < r_1. \tag{2.23}$$

Proof. If $r \geqslant r_1$ and $r \gg 1$, then

$$\begin{aligned}
I[f] &= Re\frac{1}{r^{v+1}}\int_a^b \frac{f(x)e^{ir_1x}}{x^{v+1}}(rx)^{v+1}J_v(rx)\,dx \\
&= Re\frac{1}{r^{v+2}}\int_a^b \frac{f(x)e^{ir_1x}}{x^{v+1}}\,d(rx)^{v+1}\,J_{v+1}(rx) \\
&= Re\left[U_{1,1}(x)(rx)^{v+1}J_{v+1}(rx)\Big|_{x=a}^b - \int_a^b U'_{1,1}(x)(rx)^{v+1}J_{v+1}(rx)\,dx\right] \\
&= Re\left[\sum_{k=1}^{2}U_{k,k}(x)(rx)^{v+k}J_{v+k}(rx)\Big|_{x=a}^b - \int_a^b U'_{2,2}(x)(rx)^{v+2}J_{v+2}(rx)\,dx\right].
\end{aligned} \tag{2.24}$$

Integrating repeatedly by parts, we have

$$I[f] = Re\left[\sum_{k=1}^{n}U_{k,k}(x)(rx)^{v+k}J_{v+k}(rx)\Big|_{x=a}^b - \int_a^b U'_{n,n}(x)(rx)^{v+n}J_{v+n}(rx)\,dx\right]. \tag{2.25}$$

From (2.21), we get

$$\left|I[f]-I_n[f]\right| = \left|Re\left[\int_a^b U'_{n,n}(x)(rx)^{v+n}J_{v+n}(rx)\,dx\right]\right|. \tag{2.26}$$

Thus,

$$\left|I[f]-I_n[f]\right| \leqslant Re\left[\left|U_{n+1,n+1}J_{v+n+1}(rx)(rx)^{v+n+1}\Big|_{x=a}^b\right| + \left|\int_a^b U'_{n+1,n+1}J_{v+n+1}(rx)(rx)^{v+n+1}\,dx\right|\right]. \tag{2.27}$$

Considering that $\|J_v(rx)\|_\infty = O(r^{-1/2})$ for $0 \notin [a,b]$, $\|U_{n+1,n+1}(x)\|_\infty = O(\frac{r_1^n}{r^{v+2(n+1)}})$ and $\|U'_{n+1,n+1}(x)\|_\infty = O(\frac{r_1^{n+1}}{r^{v+2(n+1)}})$, we have

$$\left|I[f]-I_n[f]\right| = O\left(\max\left(\frac{r_1^n}{r^{n+1}}r^{-1/2}, \frac{r_1^{n+1}}{r^{n+1}}r^{-1/2}\right)\right) = O\left(r^{-1/2}\left(\frac{r_1}{r}\right)^{n+1}\right). \tag{2.28}$$

If $r < r_1$ and $r_1 \gg 1$, then

$$\begin{aligned}
I[f] &= Re\int_a^b \frac{f(x)J_v(rx)}{ir_1}\,de^{ir_1x} \\
&= Re\left[U_1(x)e^{ir_1x}\Big|_{x=a}^b - \int_a^b U'_1(x)e^{ir_1x}\,dx\right] \\
&= Re\left[U_1(x)e^{ir_1x}\Big|_{x=a}^b + \int_a^b U_2(x)\,de^{ir_1x}\right] \\
&= Re\left[\sum_{k=1}^{2}U_k(x)e^{ir_1x}\Big|_{x=a}^b - \int_a^b U'_2(x)e^{ir_1x}\,dx\right].
\end{aligned} \tag{2.29}$$

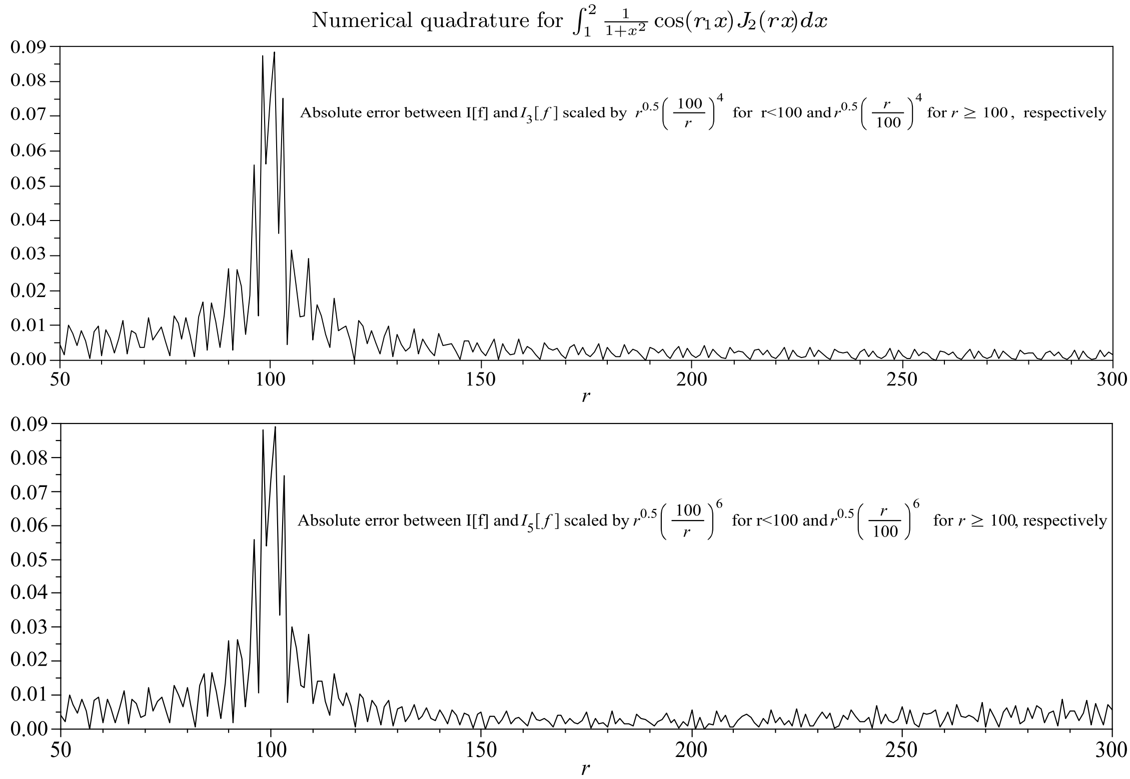

Fig. 2. The HPM $I_n[f]$ for approximating $\int_1^2 \frac{1}{1+x^2}\cos(100x) J_2(rx)\,dx$ with $n = 3$ or 5. The absolute error of $I_3[f]$ is scaled by $\frac{r^{1/2}}{(\frac{100}{r})^4}$ for $r < 100$ and by $\frac{r^{1/2}}{(\frac{r}{100})^4}$ for $r \geqslant 100$, respectively (on the top); The absolute error of $I_5[f]$ is scaled by $\frac{r^{1/2}}{(\frac{100}{r})^6}$ for $r < 100$ and by $\frac{r^{1/2}}{(\frac{r}{100})^6}$ for $r \geqslant 100$, respectively (on the bottom).

Using repeatedly this technique, we get

$$I[f] = I_n[f] - Re\int_a^b U_n'(x)e^{ir_1x}\,dx. \tag{2.30}$$

Therefore, the error satisfies

$$\begin{aligned}\bigl|I[f] - I_n[f]\bigr| &= Re\left|\int_a^b U_n'(x)e^{ir_1x}\,dx\right| \\ &\leqslant Re\left[\left|U_{n+1}(x)e^{ir_1x}\Big|_{x=a}^b\right| + \left|\int_a^b U_{n+1}'(x)e^{ir_1x}\,dx\right|\right]. \end{aligned}\tag{2.31}$$

By means of $\|U_{n+1}(x)\|_\infty = O(\frac{r^{n-1/2}}{r_1^{n+1}})$ and $\|U_{n+1}'(x)\|_\infty = O(\frac{r^{n+1/2}}{r_1^{n+1}})$, we obtain

$$\bigl|I[f] - I_n[f]\bigr| = O\left(\max\left(\frac{r^{n-1/2}}{r_1^{n+1}}, \frac{r^{n+1/2}}{r_1^{n+1}}\right)\right) = O\left(r^{-1/2}\left(\frac{r}{r_1}\right)^{n+1}\right). \tag{2.32}$$

The results are true. □

Example 2. Let us consider $\int_1^2 \frac{1}{1+x^2}\cos(100x) J_2(rx)\,dx$ (see Fig. 2).

3. Applications to Anger and Weber transformations

A solution of the inhomogeneous Bessel equation [11,12]

$$x^2y'' + xy' + \bigl(x^2 - v^2\bigr)y = \frac{(x-v)\sin(v\pi)}{\pi} \tag{3.1}$$

is denoted by the Anger function $A_v(x)$

$$A_v(x) = \frac{1}{\pi}\int_0^{\pi} \cos\bigl(v\theta - x\sin(\theta)\bigr)\,d\theta. \tag{3.2}$$

If v is an integer, then $A_v(x) = J_v(x)$, where $J_v(x)$ is the Bessel function of the first kind. A solution of the inhomogeneous Bessel equation [11,12]

$$x^2y'' + xy' + \bigl(x^2 - v^2\bigr)y = \frac{(v-x)\cos(v\pi) - (v+x)}{\pi} \tag{3.3}$$

is denoted by the Weber function $W_v(x)$

$$W_v(x) = \frac{1}{\pi}\int_0^{\pi} \sin\bigl(v\theta - x\sin(\theta)\bigr)\,d\theta. \tag{3.4}$$

For large values r, $A_v(rx)$ and $W_v(rx)$ become highly oscillatory. Next we consider the numerical quadrature of the Anger transformations $\int_a^b f(x)A_v(rx)\,dx$ and Weber transformations $\int_a^b f(x)W_v(rx)\,dx$.

Suppose that

$$\left[\sum_{k=1}^{\infty} U_k(x)A_{v+k}(rx)(rx)^{v+k} + P(x)\right]' = f(x)A_v(rx) \tag{3.5}$$

and

$$\left[\sum_{k=1}^{\infty} U_k(x)W_{v+k}(rx)(rx)^{v+k} + Q(x)\right]' = f(x)W_v(rx). \tag{3.6}$$

Substituting [12]

$$\bigl[(rx)^v A_v(rx)\bigr]' = r(rx)^v A_{v-1}(rx) + r(rx)^{v-1}\frac{\sin(v\pi)}{\pi}, \tag{3.7}$$

$$\bigl[(rx)^v W_v(rx)\bigr]' = r(rx)^v W_{v-1}(rx) + r(rx)^{v-1}\frac{1-\cos(v\pi)}{\pi} \tag{3.8}$$

into (3.5) and (3.6), respectively, we get the following homotopies:

$$\begin{aligned}
&\sum_{k=1}^{\infty} rU_k(x)A_{v+k-1}(rx)(rx)^{v+k} + p\left[\sum_{k=1}^{\infty} U_k'(x)A_{v+k}(rx)(rx)^{v+k}\right.\\
&\quad\left. + \sum_{k=1}^{\infty} rU_k(x)(rx)^{v+k}\frac{\sin((v+k)\pi)}{\pi rx} + P'(x) - f(x)A_v(rx)\right] = 0, \qquad (3.9)\\
&\sum_{k=1}^{\infty} rU_k(x)W_{v+k-1}(rx)(rx)^{v+k} + p\left[\sum_{k=1}^{\infty} U_k'(x)W_{v+k}(rx)(rx)^{v+k}\right.\\
&\quad\left. + \sum_{k=1}^{\infty} rU_k(x)(rx)^{v+k}\frac{1-\cos((v+k)\pi)}{\pi rx} + Q'(x) - f(x)W_v(rx)\right] = 0. \qquad (3.10)
\end{aligned}$$

Let

$$P = P_0 + pP_1 + p^2P_2 + p^3P_3 + \cdots, \tag{3.11}$$
$$Q = Q_0 + pQ_1 + p^2Q_2 + p^3Q_3 + \cdots \tag{3.12}$$

and

$$U_k(x) = U_{k,0}(x) + pU_{k,1}(x) + p^2U_{k,2}(x) + p^3U_{k,3}(x) + \cdots, \tag{3.13}$$

then by equating the terms with the identical powers of p, we have

$$\begin{aligned}
&U_{k,0}(x)=0,\quad k\geqslant 1,\qquad P_0=0,\\
&U_{1,1}(x)=\frac{f(x)}{r^{v+2}x^{v+1}},\qquad P_1=-\frac{r^{v+1}\sin((v+1)\pi)}{\pi}\int_a^x t^v U_{1,1}(t)\,dt,\qquad U_{k,1}(x)=0,\quad k\neq 1,\\
&U_{2,2}(x)=-\frac{U'_{1,1}(x)}{r^2x},\qquad P_2=-\frac{r^{v+2}\sin((v+2)\pi)}{\pi}\int_a^x t^{v+1} U_{2,2}(t)\,dt,\qquad U_{k,2}(x)=0,\quad k\neq 2,\\
&\quad\vdots\\
&U_{n,n}(x)=-\frac{U'_{n-1,n-1}(x)}{r^2x},\qquad P_n=-\frac{r^{v+n}\sin((v+n)\pi)}{\pi}\int_a^x t^{v+n-1} U_{n,n}(t)\,dt,\qquad U_{k,n}(x)=0,\quad k\neq n
\end{aligned}\tag{3.14}$$

for (3.9), and

$$\begin{aligned}
&U_{k,0}(x)=0,\quad k\geqslant 1,\qquad Q_0=0,\\
&U_{1,1}(x)=\frac{f(x)}{r^{v+2}x^{v+1}},\qquad Q_1=-\frac{r^{v+1}(1-\cos((v+1)\pi))}{\pi}\int_a^x t^v U_{1,1}(t)\,dt,\qquad U_{k,1}(x)=0,\quad k\neq 1,\\
&U_{2,2}(x)=-\frac{U'_{1,1}(x)}{r^2x},\qquad Q_2=-\frac{r^{v+2}(1-\cos((v+2)\pi))}{\pi}\int_a^x t^{v+1} U_{2,2}(t)\,dt,\qquad U_{k,2}(x)=0,\quad k\neq 2,\\
&\quad\vdots\\
&U_{n,n}(x)=-\frac{U'_{n-1,n-1}(x)}{r^2x},\qquad Q_n=-\frac{r^{v+n}(1-\cos((v+n)\pi))}{\pi}\int_a^x t^{v+n-1} U_{n,n}(t)\,dt,\qquad U_{k,n}(x)=0,\quad k\neq n
\end{aligned}\tag{3.15}$$

for (3.10).

Define the HPM for $\int_a^b f(x)A_v(rx)\,dx$ as

$$I_n[f]=\left\{\sum_{k=1}^n U_{k,k}(x)A_{v+k}(rx)(rx)^{v+k}+\sum_{k=1}^n P_k\right\}\bigg|_{x=a}^b,\tag{3.16}$$

and for $\int_a^b f(x)W_v(rx)\,dx$ as

$$I_n[f]=\left\{\sum_{k=1}^n U_{k,k}(x)W_{v+k}(rx)(rx)^{v+k}+\sum_{k=1}^n Q_k\right\}\bigg|_{x=a}^b,\tag{3.17}$$

where $U_{k,k}(x)$, P_k and Q_k are denoted by (3.14) and (3.15).

Theorem 3.1. *Let $f(x)$ be a sufficiently smooth function in $[a,b]$ and assume that $U_{k,k}(x)$, P_k and Q_k satisfy the formulas* (3.14) *and* (3.15)*, then the errors for $I[f]=\int_a^b f(x)A_v(rx)\,dx$ or $\int_a^b f(x)W_v(rx)\,dx$ satisfy*

$$\big|I[f]-I_n[f]\big|=O\big(r^{-n-1}\big),\tag{3.18}$$

where $I_n[f]$ is denoted by (3.16) *or* (3.17)*.*

Proof. The transformations with the assumption that $0\notin[a,b]$

$$I[f]=\int_a^b f(x)A_v(rx)\,dx$$

can be rewritten as

$$I[f]=\int_a^b \frac{f(x)}{(rx)^{v+1}}(rx)^{v+1}A_v(rx)\,dx.$$

From (3.7), we have

$$\begin{aligned}
I[f] &= \int_a^b \frac{f(x)}{r^{v+2}x^{v+1}}\big[(rx)^{v+1}A_{v+1}(rx)\big]'\,dx - \frac{r^{v+1}\sin v\pi}{\pi}\int_a^b \frac{f(x)}{r^{v+2}x^{v+1}}x^v\,dx \\
&= \int_a^b U_{1,1}(x)\big[(rx)^{v+1}A_{v+1}(rx)\big]'\,dx + P_1 \\
&= U_{1,1}(x)(rx)^{v+1}A_{v+1}(rx)\big|_{x=a}^b - \int_a^b U'_{1,1}(x)(rx)^{v+1}A_{v+1}(rx)\,dx + P_1 \\
&= U_{1,1}(x)(rx)^{v+1}A_{v+1}(rx)\big|_{x=a}^b - \int_a^b \frac{U'_{1,1}(x)}{rx}(rx)^{v+2}A_{v+1}(rx)\,dx + P_1 \\
&= U_{1,1}(x)(rx)^{v+1}A_{v+1}(rx)\big|_{x=a}^b + \int_a^b U_{2,2}(x)\big[(rx)^{v+2}A_{v+2}(rx)\big]'\,dx + P_1 + P_2 \\
&= \sum_{k=1}^2 U_{k,k}(x)(rx)^{v+k}A_{v+k}(rx)\bigg|_{x=a}^b + \sum_{k=1}^2 P_k - \int_a^b U'_{2,2}(x)(rx)^{v+2}A_{v+2}(rx)\,dx.
\end{aligned}$$

Using repeatedly this technique, we get

$$I[f] = I_n[f] - \int_a^b U'_{n,n}(x)(rx)^{v+n}A_{v+n}(rx)\,dx.$$

That is, the error satisfies

$$\begin{aligned}
\big|I[f]-I_n[f]\big| &= \left|\int_a^b U'_{n,n}(x)(rx)^{v+n}A_{v+n}(rx)\,dx\right| \\
&\leqslant \left|\int_a^b \frac{U'_{n,n}(x)}{r^2x}\big[(rx)^{v+n+1}A_{v+n+1}(rx)\big]'\,dx\right| + |P_{n+1}| \\
&\leqslant \big|U_{n+1,n+1}(x)(rx)^{v+n+1}A_{v+n+1}(rx)\big|_{x=a}^b\big| \\
&\quad + \left|\int_a^b U'_{n+1,n+1}(x)(rx)^{v+n+1}A_{v+n+1}(rx)\,dx\right| + |P_{n+1}|.
\end{aligned}$$

Notice that $\|U_{n+1,n+1}(x)\|_\infty = O(r^{-v-2(n+1)})$, $\|U'_{n+1,n+1}(x)\|_\infty = O(r^{-v-2(n+1)})$, $\|P_{n+1}\|_\infty = O(r^{-(n+1)})$, and

$$\big\|A_v(rx)\big\|_\infty = O\big(r^{-1/2}\big), \qquad \big\|W_v(rx)\big\|_\infty = O\big(r^{-1/2}\big) \quad [13].$$

Thus

$$\big|I[f]-I_n[f]\big| = O\big(\max\big(r^{-n-3/2}, r^{-n-1}\big)\big) = O\big(r^{-n-1}\big).$$

Similarly, we can prove the result for the transformations $\int_a^b f(x)W_v(rx)\,dx$. □

Example 3. Let us consider $\int_1^2 \frac{1}{1+x^2}A_\pi(rx)\,dx$ (see Fig. 3).

Example 4. Let us consider $\int_1^2 \frac{1}{1+x^2}W_{4\pi}(rx)\,dx$ (see Fig. 4).

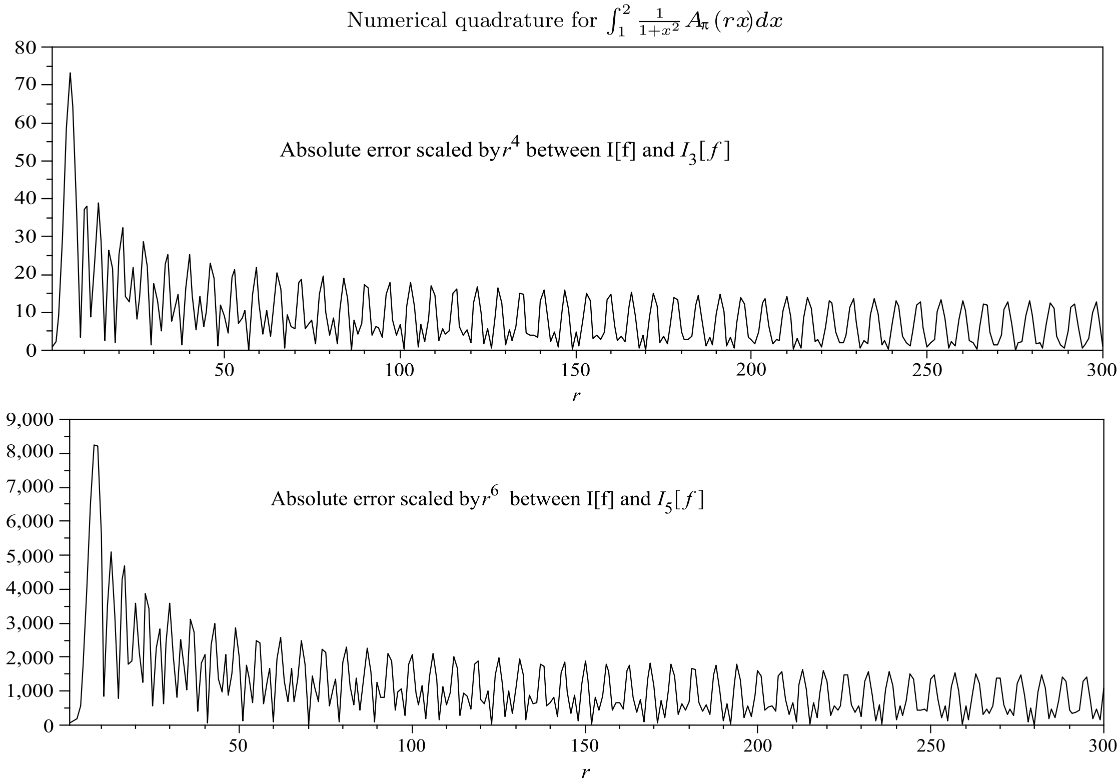

Fig. 3. The HPM $I_n[f]$ for $\int_1^2 \frac{1}{1+x^2} A_\pi(rx)\,dx$ with $n=3$ or 5. The absolute errors are scaled by r^4 or r^6, respectively.

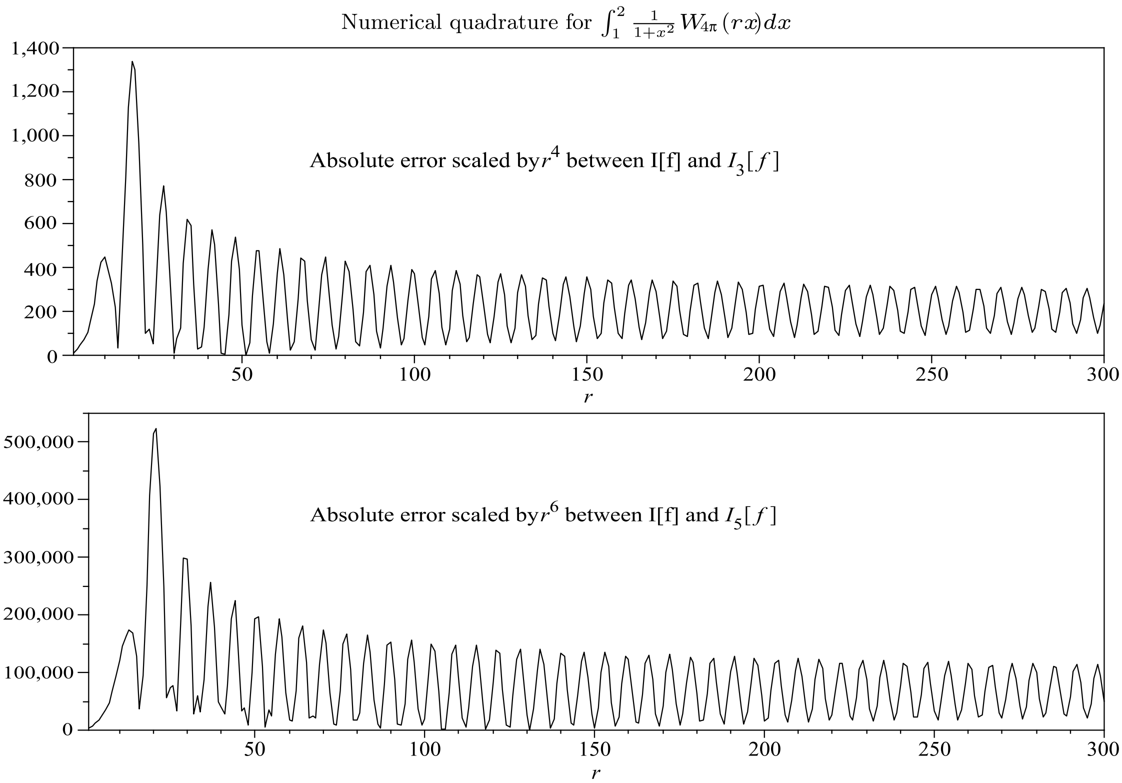

Fig. 4. The HPM $I_n[f]$ for $\int_1^2 \frac{1}{1+x^2} W_{4\pi}(rx)\,dx$ with $n=3$ or 5. The absolute errors are scaled by r^4 or r^6, respectively.

4. Conclusion

In this paper we have applied the HPM for obtaining the approximations of the integrations containing Bessel, Anger and Weber functions, and the asymptotic order of the method. The method is efficient for $\int_a^b f(x) J_v(rx)\,dx$, $\int_a^b f(x) A_v(rx)\,dx$ and $\int_a^b f(x) W_v(rx)\,dx$. For the case $|r_1 - r| \gg 1$, we can obtain accurate approximation of $\int_a^b f(x)\cos(r_1 x) J_v(rx)\,dx$ too. The disadvantage is that the method for $\int_a^b f(x)\cos(r_1 x) J_v(rx)\,dx$ is failing for small value $|r_1 - r|$. All numerical examples show that our results are true for computing these class of the integrals.

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