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Energy decay rates of elastic waves in unbounded domain with potential type of damping

Ruy C. Charão [a,*], Ryo Ikehata [b]

[a] Federal University of Santa Catarina, Department of Mathematics, Brazil
[b] Hiroshima University, Department of Mathematics, Graduate School of Education, Higashi-Hiroshima 739-8524, Japan

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ABSTRACT

In this paper we first show that the total energy of solutions for a semilinear system of elastic waves in $\mathbb{R}^n$ with a potential type of damping decays in an algebraic rate to zero. We study the critical potential case and we assume that the initial data have a compact support. An application for the Euler–Poisson–Darboux type dissipation $V(t,x)$ is obtained and in this case the compactness of the support on the initial data is not necessary. Finally, we shall discuss the energy concentration region for the linear system of elastic waves in an exterior domain.

1. Introduction

In this work we first study asymptotic properties for the semilinear system of elastic waves in $\mathbb{R}^n$. The associated vector displacement $u = u(t,x)$ is given by the following initial value problem

$$\begin{aligned} & u_{tt} - a^2\Delta u - \left(b^2 - a^2\right)\nabla \operatorname{div} u + V(t,x)u_t + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ & u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{1.1}$$

with a power type of nonlinearity $|u|^{p-1}u$ together with

$$1 < p < +\infty \quad (n = 2) \quad \text{and} \quad 1 < p < \frac{n+2}{n-2} \quad (n \geqslant 3), \tag{1.2}$$

where the Lamé coefficients $a > 0$ and $b > 0$ satisfy $0 < a^2 < b^2$. The notation $|u|$ indicates the usual Euclidean norm of the vector $u = u(t,x) \in \mathbb{R}^n$.

In this work, we set the vector displacement $u = (u^1, \ldots, u^n)$ with $u^i = u^i(t,x)$, $\Delta u = (\Delta u^1, \ldots, \Delta u^n)$, $u_t = (u_t^1, \ldots, u_t^n)$, $u_{tt} = (u_{tt}^1, \ldots, u_{tt}^n)$, ∇ is the gradient operator and $\operatorname{div} u$ is the divergence of u.

For a technical reason we have to impose a rather strong hypothesis on the initial data

$$u_0 \in \left(H^1(\mathbb{R}^n)\right)^n, \quad u_1 \in \left(L^2(\mathbb{R}^n)\right)^n, \quad \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \left\{|x| \leqslant R\right\}, \tag{1.3}$$

where $R > 0$ is an arbitrarily fixed real number, and $|x|$ is the usual Euclidean norm of $x \in \mathbb{R}^n$.

* Corresponding author.
E-mail addresses: charao@mtm.ufsc.br (R.C. Charão), ikehatar@hiroshima-u.ac.jp (R. Ikehata).

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It is easy to verify that under these conditions the problem (1.1) has a unique weak solution $u = u(t,x)$ in the class

$$u \in C\big([0,+\infty); \big(H^1\big(\mathbb{R}^n\big)\big)^n\big) \cap C^1\big([0,+\infty); \big(L^2\big(\mathbb{R}^n\big)\big)^n\big),$$

and satisfies the finite propagation speed property:

$$u(t,x) = 0 \quad \text{for } |x| \geqslant bt + R,$$

provided that $V \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ and $V(x) \geqslant 0$. For these basic results we can refer the reader to Ikawa [3] and Strauss [17].

In the scalar valued wave equation case with a potential type of damping:

$$u_{tt} - \Delta u + V(t,x)u_t + \eta |u|^{p-1}u = 0,$$

there are several references on the decay and non-decay of the total energy, and we can refer the reader to [11,14,15, 21] and the references therein (all $\eta = 0$ cases). From their results we can find that for a (time-independent) potential $V(x) \approx (1+|x|)^{-\gamma}$, $\gamma = 1$ is critical. On the other hand, recently Todorova and Yordanov [20] have investigated its precise decay rate of the total energy in the case when γ above satisfies $0 \leqslant \gamma < 1$ and $\eta = 0$. They also studied the nonlinear equation case ($\eta = 1$) with absorption (see [19]). Quite recently Ikehata and Inoue [7] derived the total energy decay results to the nonlinear wave equation in the case when $\gamma = 1$ and $\eta = 1$. They proposed a quite simple way to compute the decay rate of the total energy by modifying the method due to Todorova and Yordanov [20].

In the present paper we apply a simple method due to Ikehata and Inoue [7] to a semilinear system of elastic waves having a non-constant damping coefficient with a critical parameter $\gamma = 1$ in order to obtain polynomial decay of the total energy. Although we have many decay and non-decay results concerning the scalar valued wave equation case, there are few related results on the system of elastic waves with a non-compactly supported critical potential of damping. For instance, Charão and Ikehata [2] studied the system of elastic waves in an exterior domain, with a dissipation localized near infinity but, to show polynomial decay of solutions, they imposed an extra assumption on the Lamé coefficients of the system: $b^2 < 4a^2$. In this paper we do not impose such a condition. Furthermore, Kapitonov [9] and Charão [1] studied decay rates of local energy for non-damped linear system of elastic waves in three dimension. In the present paper we also study the problem of energy concentration areas for the linear system of elastic waves in an exterior domain of $\mathbb{R}^n$ by using the (modified) Todorova and Yordanov [18] method. We obtain unbounded regions where the energy decays exponentially under effect of a non-constant potential type of damping depending on a real parameter σ. This is an important result because as we know, the total energy for hyperbolic dissipative equations, even in the linear case and under effect of a constant potential type of damping, in general, decays with polynomial rates (see Nakao [16], Charão and Ikehata [2]).

Menzala and Ferreira [12] and Menzala and da Luz [13] studied asymptotic properties for the system of elastic waves coupled with electromagnetic waves in exterior domains with a constant potential type of damping. For the other scalar valued wave equation cases we also can refer the reader to Ikehata [4,6], Zuazua [24] and the references therein.

2. Identities of energy

In this section we consider that the potential type of damping is time-independent, that is, the coefficient of damping $V = V(x)$.

We consider smooth functions $f = f(t)$ and $g = g(t)$ to be specified later. Using the simple multiplier technique we determine two functions $e(t,x)$ and $G(t,x)$ such that

$$[gu + fu_t] \cdot \big[u_{tt} - a^2\Delta u - \big(b^2 - a^2\big)\nabla \operatorname{div} u + V(x)u_t + |u|^{p-1}u\big] = \frac{d}{dt}e(t,x) + G(t,x), \tag{2.1}$$

where $x \cdot y$ denotes the usual inner product between vectors $x, y \in \mathbb{R}^n$.

In fact, we note that

$$e(t,x) = \frac{f(t)}{2}\left\{|u_t|^2 + a^2|\nabla u|^2 + \big(b^2 - a^2\big)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right\} + g(t)u \cdot u_t + \frac{1}{2}\big\{V(x)g(t) - g_t(t)\big\}|u|^2$$

and

$$\begin{aligned} G(t,x) = {} & |u_t|^2\left\{f(t)V(x) - g(t) - \frac{1}{2}f_t(t)\right\} + |u|^2\left\{\frac{1}{2}g_{tt}(t) - \frac{1}{2}V(x)g_t(t)\right\} \\ & + a^2|\nabla u|^2\left\{g(t) - \frac{1}{2}f_t(t)\right\} + \big(b^2 - a^2\big)(\operatorname{div} u)^2\left\{g(t) - \frac{1}{2}f_t(t)\right\} \\ & - a^2 g(t)\operatorname{div}(u : \nabla u) - a^2 f(t)\operatorname{div}(u_t : \nabla u) - \big(b^2 - a^2\big)g(t)\operatorname{div}(u \operatorname{div} u) \\ & - \big(b^2 - a^2\big)f(t)\operatorname{div}(u_t \operatorname{div} u) + |u|^{p+1}\left\{g(t) - \frac{f_t(t)}{p+1}\right\}. \end{aligned}$$

Here, for $u = (u^1, u^2, \dots, u^n)$ and $v = (v^1, v^2, \dots, v^n)$, $v : \nabla u$ denotes the vector

$$v^1 \nabla u^1 + v^2 \nabla u^2 + \cdots + v^n \nabla u^n = \sum_{i=1}^{n} v^i \nabla u^i \in \mathbb{R}^n$$

and we have also denoted

$$|\nabla u|^2 = \sum_{i=1}^{n} \left|\nabla u^i\right|^2.$$

To obtain the identity (2.1) we have used the following elementary identities

$$\operatorname{div}(u : \nabla u) = \operatorname{div}\left(\sum_{i=1}^{n} u^i \nabla u^i\right) = \sum_{i=1}^{n} \left[u^i \Delta u^i + \nabla u^i \cdot \nabla u^i\right] = u \cdot \Delta u + |\nabla u|^2,$$

$$\operatorname{div}(u_t : \nabla u) = \operatorname{div}\left(\sum_{i=1}^{n} u_t^i \nabla u^i\right) = \sum_{i=1}^{n} \left[u_t^i \Delta u^i + \nabla u_t^i \cdot \nabla u^i\right] = u_t \cdot \Delta u + \frac{d}{2\,dt} |\nabla u|^2,$$

$$\operatorname{div}(u \operatorname{div} u) = (\operatorname{div} u)^2 + u \cdot \nabla(\operatorname{div} u)$$

and

$$\operatorname{div}(u_t \operatorname{div} u) = \frac{d}{2\,dt} (\operatorname{div} u)^2 + u_t \cdot \nabla(\operatorname{div} u).$$

Thus, for $u = u(x,t)$ the solution of the problem (1.1), from identity (2.1) we have the following energy identity

$$\frac{d}{dt} e(t,x) + G(t,x) = 0. \tag{2.2}$$

Using the finite speed of propagation property and the divergence formula we can integrate (2.2) to obtain

$$\frac{d}{dt} E(t) + F(t) = 0, \quad t > 0, \tag{2.3}$$

with

$$E(t) = \int_{\mathbb{R}^n} e(t,x)\,dx = \int_{\Omega(t)} e(t,x)\,dx$$

and

$$\begin{aligned} F(t) = {} & \int_{\Omega(t)} \left[V(x) f - g - \frac{1}{2} f_t\right] |u_t|^2\,dx + \int_{\Omega(t)} \left[\frac{1}{2} g_{tt} - \frac{1}{2} V(x) g_t\right] |u|^2\,dx \\ & + \int_{\Omega(t)} \left\{a^2 \left[g - \frac{1}{2} f_t\right] |\nabla u|^2 + \left(b^2 - a^2\right) \left[g - \frac{1}{2} f_t\right] |\operatorname{div} u|^2 + \left[g - \frac{f_t}{p+1}\right] |u|^{p+1}\right\} dx, \end{aligned} \tag{2.4}$$

where

$$\Omega(t) = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \colon |x| \leqslant bt + R\right\}$$

and R is the radius of support of the initial data.

Remark 2.1. We observe that by taking $f(t) \equiv 1$ and $g(t) \equiv 0$ the functions in identity (2.3) become

$$F(t) = \int_{\Omega(t)} V(x) |u_t|^2\,dx$$

and

$$E(t) = \int_{\Omega(t)} \left\{|u_t|^2 + a^2 |\nabla u|^2 + \left(b^2 - a^2\right) (\operatorname{div} u)^2 + \frac{2}{p+1} |u|^{p+1}\right\} dx.$$

So, the energy of the system (1.1) given by the above expression is a nonincreasing function of t and the term $V(x) u_t$, with $V(x) \geqslant 0$, is dissipative.

3. Energy estimates

We suppose that the potential $V(x) \in L^\infty(\mathbb{R}^n)$ of Eq. (1.1) has the form: there exists a constant $C_0 > 0$ such that

$$V(x) \geqslant \frac{C_0}{1+|x|}. \tag{3.1}$$

Now, choose the following positive functions

$$h(t) = 1+t, \qquad f(t) = (1+t)^{1-\delta} \quad \text{and} \quad g(t) = \frac{1-\delta}{2}(1+t)^{-\delta} \tag{3.2}$$

defined for $t \geqslant 0$, where $\delta > 0$ is a number such that

$$1 - \frac{C_0}{b} < \delta < 1$$

if $0 < C_0 \leqslant b$, and

$$0 \leqslant \delta < 1$$

if $C_0 > b > 0$.

In the next lemma, b stands for the coefficient, which appears in the system of elastic waves and it is the propagation speed of this system.

Lemma 3.1. *Suppose* (3.1)*. Then there exists $t_0 > 0$ such that the functions h, f and g satisfy*

(i) $2V(x)f - f_t - 2g \geqslant 0$, $x \in \Omega(t)$,
(ii) $2g - f_t \geqslant 0$,
(iii) $g_{tt} - V(x)g_t \geqslant 0$, $x \in \Omega(t)$,
(iv) $g - \frac{f_t}{p+1} \geqslant 0$,
(v) $V(x)g - g_t - h(t)^{-1}g \geqslant 0$,

for all $t \geqslant t_0$, where $\Omega(t) = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n\colon |x| \leqslant bt + R\}$ with R being the radius of support of the initial data.

A similar proof of this lemma appears in Ikehata and Inoue [7, Lemma 2.3] in the scalar-valued wave equation case. Since the proof of the other part is easy or similar, we shall give the proof of only (v) in order to make this paper self-contained.

Proof. Let us prove only (v).

$$V(x)g(t) - g_t(t) - h(t)^{-1}g(t) = \frac{1-\delta}{2}(1+t)^{-1-\delta}\big[(1+t)V(x) + \delta - 1\big] \geqslant \frac{1-\delta}{2}(1+t)^{-1-\delta}\left(C_0\frac{1+t}{1+|x|} + \delta - 1\right)$$
$$\geqslant \frac{1-\delta}{2}(1+t)^{-1-\delta}\left(C_0\frac{1+t}{1+bt+R} - 1 + \delta\right) > 0,$$

for all $x \in \Omega(t)$ with sufficiently large $t \geqslant t_0 \gg 1$, because we have

$$\lim_{t\to+\infty}\left(C_0\frac{1+t}{1+bt+R} - 1 + \delta\right) = \frac{C_0}{b} - 1 + \delta > 0,$$

under the assumptions that

$$1 > \delta > 1 - \frac{C_0}{b} > 0 \quad \text{if } 0 < C_0 \leqslant b,$$

or

$$0 \leqslant \delta < 1 \quad \text{if } 0 < b < C_0.$$

The proof of (i) is similar. The other part is quite easy to be checked. □

By considering the functions given in (3.2) and the definition of $F(t)$ in (2.4) we conclude from Lemma 3.1 that

$$F(t) \geqslant 0, \quad t \geqslant t_0.$$

The above estimate for $F(t)$, combined with (2.3), implies that

$$E(t) \leqslant E(t_0), \quad t \geqslant t_0 > 0, \tag{3.3}$$

so one has:

Lemma 3.2. *Let h, f and g be given by* (3.2) *and t_0 given in Lemma* 3.1*. Then, for $t \geqslant t_0$ and $x \in \Omega(t)$ it holds that*

$$\big[V(x)g - g_t\big]|u|^2 + hg|u_t|^2 + 2gu\cdot u_t \geqslant 0. \tag{3.4}$$

Proof. We have

$$|2gu\cdot u_t| \leqslant h(t)g|u_t|^2 + h(t)^{-1}g|u|^2.$$

Then, from property (v) in Lemma 3.1, we get

$$\begin{aligned}\big[V(x)g - g_t\big]|u|^2 + 2gu\cdot u_t &\geqslant \big[V(x)g - g_t\big]|u|^2 - hg|u_t|^2 - h(t)^{-1}g|u|^2\\ &= \big[V(x)g - g_t - h(t)^{-1}g\big]|u|^2 - hg|u_t|^2 \geqslant -hg|u_t|^2. \qquad \square\end{aligned}$$

Now, using the definition of $E(t)$ and the estimate (3.4) we conclude that

$$E(t) \geqslant \frac{1}{2}\int\limits_{\Omega(t)} \left[(f-hg)|u_t|^2 + f\left\{a^2|\nabla u|^2 + \big(b^2-a^2\big)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right\}\right]dx.$$

The above estimate combined with (3.3) implies the following result.

Lemma 3.3. *Let u be the weak solution of* (1.1) *with* (3.1)*, and let f, g and h be the functions in* (3.2)*. Then, for $t \geqslant t_0$ it is true that*

$$\frac{1}{2}\int\limits_{\Omega(t)} \left[(f-hg)|u_t|^2 + f\left\{a^2|\nabla u|^2 + \big(b^2-a^2\big)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right\}\right]dx \leqslant E(t_0). \tag{3.5}$$

4. Main results

The following main result of this paper is obtained by substituting the expressions of the functions f, g and h (see (3.2)) in the estimate (3.5).

Theorem 4.1. *Let u be the weak solution of* (1.1) *with* (3.1)*. Let t_0 be given by Lemma* 3.1*. Then for $t \geqslant t_0$ it holds that*

$$\frac{1}{2}\int\limits_{\mathbb{R}^n}\left[a^2|\nabla u|^2 + \big(b^2-a^2\big)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right]dx \leqslant E(t_0)(1+t)^{-(1-\delta)}$$

and

$$\frac{1}{2}\int\limits_{\mathbb{R}^n}|u_t|^2\,dx \leqslant \frac{2}{1+\delta}E(t_0)(1+t)^{-(1-\delta)},$$

where δ is any number satisfying

$$1 - \frac{C_0}{b} < \delta < 1$$

for the case $0 < C_0 \leqslant b$, and

$$0 \leqslant \delta < 1$$

in the case $C_0 > b > 0$.

For time-dependent potentials we also have a similar result.

Theorem 4.2. *Let u be the solution of* (1.1) *with the smooth potential $V = V(t,x) \in L^\infty(\mathbb{R}^n \times [0,+\infty))$ satisfying*

$$V(t,x) \geqslant \frac{C_0}{1+|x|+t}, \qquad V_t(t,x) \leqslant 0$$

with a constant $C_0 > 0$. Then, there exists $t_0 > 0$ such that the solution u has the following asymptotic behavior for $t \geqslant t_0$,

$$\frac{1}{2}\int\limits_{\mathbb{R}^n}\left[a^2|\nabla u|^2 + \big(b^2-a^2\big)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right]dx \leqslant E(t_0)(1+t)^{-(1-\delta)},$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2\,dx \leqslant C_\delta E(t_0)(1+t)^{-(1-\delta)},$$

where δ is any number satisfying

$$1-\frac{C_0}{1+b}<\delta<1$$

for the case $0<C_0\leqslant 1+b$, and

$$0\leqslant\delta<1$$

in the case $C_0>1+b$.

Proof. The uniqueness and existence part of weak solutions is quite similar to the time independent potential case, so we shall restrict ourselves to the proof of decay property of the total energy.

The proof is, however, similar to the proof of Theorem 4.1 since we can use the monotonicity of the function $t\mapsto V(x,t)$ and the identity

$$\frac{d}{dt}E(t)+F_1(t)=0,$$

where

$$\begin{aligned}E(t)&=\frac{1}{2}\int_{\mathbb{R}^n} f(t)\big[|u_t|^2+a^2|\nabla u|^2+\big(b^2-a^2\big)(\operatorname{div}u)^2\big]\,dx\\&\quad+\int_{\mathbb{R}^n}\left[\frac{f}{p+1}|u|^{p+1}+2gu\cdot u_t+\big\{V(t,x)g(t)-g_t(t)\big\}|u|^2\right]dx\end{aligned}$$

and

$$\begin{aligned}F_1(t)&=\int_{\mathbb{R}^n}\left\{\left[V(t,x)f-g-\frac{1}{2}f_t\right]|u_t|^2+\left[\frac{1}{2}g_{tt}-\frac{1}{2}V(t,x)g_t\right]|u|^2\right\}dx-\int_{\mathbb{R}^n}\frac{gV_t(t,x)}{2}|u|^2\,dx\\&\quad+\int_{\mathbb{R}^n}\left\{a^2\left[g-\frac{1}{2}f_t\right]|\nabla u|^2+\big(b^2-a^2\big)\left[g-\frac{1}{2}f_t\right]|\operatorname{div}u|^2+\left[g-\frac{f_t}{p+1}\right]|u|^{p+1}\right\}dx.\end{aligned}$$

Because of the assumption on $V(t,x)$ and Lemma 3.2 we can drop the second term of $F_1(t)$ in order to estimate similarly to the time independent case. □

5. Another type of dissipation

As an application of the former part we shall consider the so called Euler–Poisson–Darboux type dissipation $V(t,x)$ satisfying

$$(1+t)V(t,x)\geqslant A(x),\qquad V_t\leqslant 0, \tag{5.1}$$

where the smooth bounded function $A(x)$ on $[0,+\infty)\times\mathbb{R}^n$ satisfies

$$\alpha=\inf_{x\in\mathbb{R}^n}A(x)>0. \tag{5.2}$$

In [22] Uesaka considered the semilinear wave equation with such a type of dissipation, and obtained total energy decay results under appropriate assumptions on $A(x)$. Note that a monotone nonlinear perturbation $f(u)$ with $uf(u)\geqslant 0$ were dealt with in [22] as just considered in this paper. This type of dissipation was also considered in Wirth [23, Corollary 4.2], and a precise L^p-L^q type of decay estimate was obtained to a scalar-valued wave equation with a weak dissipation like $V(t,x)=\mu(1+t)^{-1}$ $(\mu>0)$.

Now we shall study the Cauchy problem (1.1)–(1.2) with the Euler–Poisson–Darboux type dissipation $V(t,x)$ satisfying (5.1) and (5.2). Our result in this section is as follows. In this case we do not necessarily assume compactness of the support on the initial data.

Theorem 5.1. *Let $[u_0, u_1] \in (H^1(\mathbb{R}^n))^n \times (L^2(\mathbb{R}^n))^n$, and suppose* (1.2)*. Furthermore, let u be the global weak solution of* (1.1) *with $V(x)$ replaced by $V(t,x)$ satisfying* (5.1) *and* (5.2)*. Then for $t \geqslant 0$ it is true that*

$$\frac{1}{2}\int_{\mathbb{R}^n}\left[a^2|\nabla u|^2 + \left(b^2 - a^2\right)(\operatorname{div} u)^2 + \frac{2}{p+1}|u|^{p+1}\right]dx \leqslant E(0)(1+t)^{-(1-\delta)}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n}|u_t|^2\,dx \leqslant \frac{2}{1+\delta}E(0)(1+t)^{-(1-\delta)},$$

where δ is any number satisfying

$$1-\alpha \leqslant \delta < 1$$

for the case $0 < \alpha < 1$, and

$$0 \leqslant \delta < 1$$

in the case $\alpha \geqslant 1$.

Remark 5.2. Since we do not rely on the finite propagation speed property, a relation between the parameter b and α is no need in this case.

Proof. It is sufficient to check the five conditions in Lemma 3.1 for $(t,x) \in [0,+\infty) \times \mathbb{R}^n$. Since the other part is almost similar to the proof of Lemma 3.1, we shall prove only (i) of Lemma 3.1. Indeed, from (3.2), (5.1) and (5.2) one has

$$\begin{aligned}2V(t,x)f(t) - f_t(t) - 2g(t) &= 2(1+t)^{-\delta}\left[(1+t)V(t,x) - 1 + \delta\right]\\ &\geqslant 2(1+t)^{-\delta}\left(A(x) - 1 + \delta\right) \geqslant 2(1+t)^{-\delta}(\alpha - 1 + \delta) \geqslant 0\end{aligned}$$

for all $(t,x) \in [0,+\infty) \times \mathbb{R}^n$ under the assumptions on the δ and α. □

6. The linear system in an exterior domain

In this section we shall study some asymptotic properties of the *linear* system of elastic waves in an exterior domain, and in particular, the energy concentration region. So, we are concerned with the following initial boundary value problem

$$\begin{aligned}&u_{tt} - a^2\Delta u - \left(b^2 - a^2\right)\nabla \operatorname{div} u + V(t,x)u_t = 0, \quad (t,x) \in (0,\infty) \times \Omega,\\ &u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x \in \Omega,\\ &u|_{\partial\Omega} = 0, \quad t > 0,\end{aligned} \tag{6.1}$$

where

$$V = V(t,x) \in L^\infty\left((0,\infty) \times \Omega\right) \tag{6.2}$$

and $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ is an exterior domain with a compact smooth boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$.

The total energy $E(t)$ associated with the problem (6.1) is defined by

$$E(t) = \frac{1}{2}\int_{\Omega}\left[\left|u_t(t,x)\right|^2 + a^2\left|\nabla u(t,x)\right|^2 + \left(b^2 - a^2\right)\left(\operatorname{div} u(t,x)\right)^2\right]dx.$$

Concerning the well-posedness to problem (6.1) with (6.2) (cf. Lions and Strauss [10]) we have the following result.

Proposition 6.1. *Let $n \geqslant 2$. Then for each $[u_0, u_1] \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$, there exists a unique solution $u \in C([0,\infty); (H_0^1(\Omega))^n) \cap C^1([0,\infty); (L^2(\Omega))^n)$ to problem* (6.1) *satisfying*

$$E(t) + \int_0^t\int_{\Omega} V(t,x)\left|u_t(t,x)\right|^2\,dx\,dt = E(0), \tag{6.3}$$

for all $t \geqslant 0$.

In this section, we first deal with the time-independent potential case, so we assume that the potential $V(t,x)$ of the damping term is time-independent, and is given by

$$V(t,x) \equiv V(x) = \frac{\delta}{|x|^{\sigma}}, \quad \delta > 0,\ \sigma \in [0,+\infty). \tag{6.4}$$

Note that for all $\sigma \geqslant 0$ one has $V \in L^{\infty}((0,+\infty) \times \Omega)$ in (6.4). The main result in this section is:

Theorem 6.2. *Let $n \geqslant 2$, $\sigma \geqslant 0$ and $\sigma \neq 2$. We suppose that the initial data to problem* (6.1) *satisfy $[u_0, u_1] \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$ and*

$$K_0 = \int_{\Omega} \exp\left(\frac{2\delta|x|^{2-\sigma}}{b^2(2-\sigma)^2}\right)\left[\left|u_1(x)\right|^2 + a^2\left|\nabla u_0(x)\right|^2 + \left(b^2 - a^2\right)\left(\operatorname{div} u_0(x)\right)^2\right] dx < +\infty.$$

Then the unique solution $u(t,x)$ to problem (6.1) *given by Proposition* 6.1 *satisfies*

$$\int_{\Omega} \exp\left(\frac{2\delta|x|^{2-\sigma}}{b^2(2-\sigma)^2(1+t)}\right)\left[\left|u_t(t,x)\right|^2 + a^2\left|\nabla u(t,x)\right|^2 + \left(b^2 - a^2\right)\left(\operatorname{div} u(t,x)\right)^2\right] dx \leqslant K_0. \tag{6.5}$$

Remark 6.3. By Theorem 6.2 we see that only the case $\sigma = 2$ is excluded. In this sense the $\sigma = 2$ case seems critical. On the other hand, the condition $\sigma \geqslant 0$ is not essential, however, for simplicity of discussion we deal with this case without loss of generality.

As a direct consequence of Theorem 6.2 one has the following interesting result, which is a generalization of that obtained to the scalar wave equation with $\sigma = 0$ in Todorova and Yordanov [18], that is, the following corollary says that the solution decays exponentially outside the ball $\{|x| \leqslant (1+t)^{\frac{1+\eta}{2-\sigma}}\}$.

Corollary 6.4. *Let $n \geqslant 2$, $0 \leqslant \sigma < 2$ and $\eta > 0$. Then, under the assumption as in Theorem* 6.2 *it is true that*

$$\int_{|x| \geqslant (1+t)^{\frac{1+\eta}{2-\sigma}}} \left[\left|u_t(t,x)\right|^2 + a^2\left|\nabla u(t,x)\right|^2 + \left(b^2 - a^2\right)\left(\operatorname{div} u(t,x)\right)^2\right] dx \leqslant K_0 \exp\left(-\frac{2\delta}{b^2(2-\sigma)^2}(1+t)^{\eta}\right).$$

Remark 6.5. In the case when $0 \leqslant \sigma < 1$ and $\eta \in (0, 1-\sigma)$, because of $(1+\eta)/(2-\sigma) < 1$ the total energy concentrates in a ball $\{|x| \leqslant (1+t)^{\frac{1+\eta}{2-\sigma}}\}$ much smaller than the forward light cone $\{|x| \leqslant bt\}$ as t is large enough even if the initial data do not have a compact support. This is due to the strong effect of the damping term $V(x)$ with $\sigma \in [0,1)$. This observation seems rather new even in the elastic wave equation case. On the other hand, in the case when $\sigma \geqslant 1$ we do not find any remarkable properties more than the usual finite speed propagation property of the elastic waves concerning the energy concentration region.

The proof of Theorem 6.2 can be done by use of the (modified) Todorova–Yordanov method [18]. Their method was originally applied to the scalar-valued damped wave equations with $\sigma = 0$. The next lemma tells us that the Todorova–Yordanov method is also applicable to the potential type of damped systems of elastic waves if we choose a weight function $\psi(t,x)$ appropriately. In this paper it is essential how we choose $\psi(t,x)$. In fact, for $\sigma \neq 2$ we take as a weight function:

$$\psi(t,x) = \frac{\delta|x|^{2-\sigma}}{b^2(2-\sigma)^2(1+t)}. \tag{6.6}$$

It is very easy to verify that the function (6.6) satisfies

$$V(x)\psi_t(t,x) + b^2\left|\nabla\psi(t,x)\right|^2 = 0, \quad \psi_t(t,x) < 0. \tag{6.7}$$

Remark 6.6. In the case when $\sigma = 0$, the weight function for the scalar wave equation has already been found in Ikehata and Tanizawa [8] and for the system of elastic waves in Charão and Ikehata [2]. In the case when $\delta = 0$, the weight function is chosen in Ikehata [5] like $\psi(t,x) = |x| - t$, and this function played an essential role in deriving the local energy decay of free wave equations.

Now, we can prove Theorem 6.2.

Proof of Theorem 6.2. The proof follows from the new weighted energy method due to Todorova and Yordanov [18]. Since we are dealing with a weak solution, by density argument we may assume that the initial data and the corresponding solution are sufficiently smooth and vanish as $|x| \to +\infty$.

We set

$$E(t,x) = \frac{1}{2}\Big[\big|u_t(t,x)\big|^2 + a^2\big|\nabla u(t,x)\big|^2 + \big(b^2-a^2\big)\big(\operatorname{div} u(t,x)\big)^2\Big].$$

We take inner product between the vector equation in (6.1) and the Todorova–Yordanov multiplier $e^{2\psi(t,x)}u_t(t,x)$. Then one has the identity:

$$\begin{aligned} &e^{2\psi}u_t\cdot\big[u_{tt} - a^2\Delta u - \big(b^2-a^2\big)\nabla(\operatorname{div} u) + V(x)u_t\big] \\ &\quad= \frac{\partial}{\partial t}\big[e^{2\psi}E(t,x)\big] - a^2\sum_{j=1}^{N}\operatorname{div}\big(e^{2\psi}u_t^j\nabla u^j\big) - \big(b^2-a^2\big)\operatorname{div}\big[(\operatorname{div} u)e^{2\psi}u_t\big] \\ &\qquad - \frac{e^{2\psi}}{\psi_t}\left[a^2\sum_{j=1}^{N}\big|\psi_t\nabla u^j - u_t^j\nabla\psi\big|^2 + \big(b^2-a^2\big)\big[(\operatorname{div} u)\psi_t - (\nabla\psi\cdot u_t)\big]^2\right] \\ &\qquad + \frac{e^{2\psi}}{\psi_t}\big[V(x)\psi_t|u_t|^2 + a^2|\nabla\psi|^2|u_t|^2 + \big(b^2-a^2\big)(\nabla\psi\cdot u_t)^2\big] - \psi_t|u_t|^2e^{2\psi}. \end{aligned} \tag{6.8}$$

Now, due to the hypothesis on ψ_t in (6.7), we obtain from identity (6.8) and the Schwarz inequality that the solution $u(t,x)$ to problem (6.1) satisfies

$$\begin{aligned} 0 &\geqslant \frac{\partial}{\partial t}\big[e^{2\psi}E(t,x)\big] - a^2\sum_{j=1}^{N}\operatorname{div}\big[e^{2\psi}u_t^j\nabla u^j\big] - \big(b^2-a^2\big)\operatorname{div}\big[(\operatorname{div} u)e^{2\psi}u_t\big] \\ &\quad + \frac{e^{2\psi}}{\psi_t}|u_t|^2\big[V(x)\psi_t + a^2|\nabla\psi|^2 + \big(b^2-a^2\big)|\nabla\psi|^2\big], \end{aligned}$$

that is,

$$0 \geqslant \frac{\partial}{\partial t}\big[e^{2\psi}E(t,x)\big] - a^2\sum_{j=1}^{N}\operatorname{div}\big[e^{2\psi}u_t^j\nabla u^j\big] - \big(b^2-a^2\big)\operatorname{div}\big[(\operatorname{div} u)e^{2\psi}u_t\big] + \frac{e^{2\psi}}{\psi_t}|u_t|^2\big[V(x)\psi_t + b^2|\nabla\psi|^2\big].$$

Finally, the fact that ψ is the solution of the differential equation in (6.7), from previous inequality, we have arrived at the following inequality

$$\frac{\partial}{\partial t}\big[e^{2\psi}E(t,x)\big] \leqslant a^2\sum_{j=1}^{N}\operatorname{div}\big(e^{2\psi}u_t^j\nabla u^j\big) + \big(b^2-a^2\big)\operatorname{div}\big[(\operatorname{div} u)e^{2\psi}u_t\big]. \tag{6.9}$$

At this point, integrating the inequality (6.9) over $[0,t]\times\Omega$, using the divergence theorem and the Dirichlet boundary condition on the solution $u(x,t)$, we see that

$$\int_\Omega e^{2\psi}E(t,x)\,dx \leqslant \int_\Omega e^{2\psi(0,x)}E(0,x)\,dx.$$

This implies the estimate of Theorem 6.2. □

Next we shall deal with the time-dependent potential case with a special form. In fact, we shall consider the case when

$$V(t,x) = \frac{C}{|x|^\sigma(1+t)^{\frac{1}{2}}}, \quad C>0,\ \sigma\in[0,+\infty),\ \sigma\neq 2. \tag{6.10}$$

Then, we have a similar result to the case when the coefficient of the damping term is time-independent. The result is:

Theorem 6.7. *Let $n \geqslant 2$, $\sigma \geqslant 0$ and $\sigma \neq 2$. We suppose that the initial data to problem* (6.1) *satisfy* $[u_0,u_1]\in(H_0^1(\Omega))^n\times(L^2(\Omega))^n$ *and*

$$K_0 = \int_\Omega \exp\left(\frac{6C|x|^{2-\sigma}}{2b^2(2-\sigma)^2}\right)\Big[\big|u_1(x)\big|^2 + a^2\big|\nabla u_0(x)\big|^2 + \big(b^2-a^2\big)\big(\operatorname{div} u_0(x)\big)^2\Big]dx < +\infty.$$

Then, for $V(t,x)$ given by (6.10) *the unique solution $u(t,x)$ to problem* (6.1) *given by Proposition* 6.1 *satisfies*

$$\int_{\Omega} \exp\left(\frac{6C|x|^{2-\sigma}}{2b^2(2-\sigma)^2(1+t)^{\frac{3}{2}}}\right)\left[\left|u_t(t,x)\right|^2 + a^2\left|\nabla u(t,x)\right|^2 + \left(b^2-a^2\right)\left(\operatorname{div} u(t,x)\right)^2\right] dx \leqslant K_0. \tag{6.11}$$

Remark 6.8. By Theorem 6.7 we see again that only the case $\sigma = 2$ is excluded.

Proof of Theorem 6.7. This proof is analogous to that of the time-independent case. In fact, for the coefficient $V = V(t,x)$ we have the identity

$$\begin{aligned}
&e^{2\psi} u_t \cdot \left[u_{tt} - a^2 \Delta u - \left(b^2-a^2\right)\nabla(\operatorname{div} u) + V(t,x)u_t\right] \\
&\quad = \frac{\partial}{\partial t}\left[e^{2\psi} E(t,x)\right] - a^2 \sum_{j=1}^{N} \operatorname{div}\left(e^{2\psi} u_t^j \nabla u^j\right) - \left(b^2-a^2\right) \operatorname{div}\left[(\operatorname{div} u) e^{2\psi} u_t\right] \\
&\qquad - \frac{e^{2\psi}}{\psi_t}\left[a^2 \sum_{j=1}^{N} \left|\psi_t \nabla u^j - u_t^j \nabla\psi\right|^2 + \left(b^2-a^2\right)\left[(\operatorname{div} u)\psi_t - (\nabla\psi \cdot u_t)\right]^2\right] \\
&\qquad + \frac{e^{2\psi}}{\psi_t}\left[V(t,x)\psi_t |u_t|^2 + a^2 |\nabla\psi|^2 |u_t|^2 + \left(b^2-a^2\right)(\nabla\psi \cdot u_t)^2\right] - \psi_t |u_t|^2 e^{2\psi}.
\end{aligned} \tag{6.12}$$

Note that the weight function

$$\psi(t,x) = \frac{3C|x|^{2-\sigma}}{2b^2(2-\sigma)^2(1+t)^{\frac{3}{2}}} \tag{6.13}$$

satisfies the differential equation

$$V(t,x)\psi_t(t,x) + b^2\left|\nabla\psi(t,x)\right|^2 = 0, \tag{6.14}$$

and the condition

$$\psi_t(t,x) < 0. \tag{6.15}$$

At this point, using (6.12)–(6.15) and following the proof of Theorem 6.2 we can conclude the proof. □

Finally, we can observe that a similar result as in Corollary 6.4 holds for the time-dependent case. The result says that the solution decays very fast if $|x|^{2-\sigma}/t$ is large.

Corollary 6.9. *Let $n \geqslant 2$, $0 \leqslant \sigma < 2$ and $\eta > \frac{1}{2}$. Then it is true that*

$$\int_{|x| \geqslant (1+t)^{\frac{1+\eta}{2-\sigma}}} \left[\left|u_t(t,x)\right|^2 + a^2\left|\nabla u(t,x)\right|^2 + \left(b^2-a^2\right)\left(\operatorname{div} u(t,x)\right)^2\right] dx \leqslant K_0 \exp\left(-\frac{6C}{2b^2(2-\sigma)^2}(1+t)^{\eta-\frac{1}{2}}\right).$$

Remark 6.10. Similar comments as in Remark 6.5 hold when $0 \leqslant \sigma < \frac{1}{2}$ and $\frac{1}{2} < \eta < 1-\sigma$. However, the decay rate for the time-dependent case is weaker than that for the time-independent one.

Remark 6.11. All the results of Section 6 are also valid (see (6.8)–(6.9)) if we assume

$$V(t,x) \geqslant \frac{\delta}{|x|^{\sigma}}, \quad \delta > 0,\ \sigma \in [0,+\infty),\ \sigma \neq 2,$$

or

$$V(t,x) \geqslant \frac{C}{|x|^{\sigma}(1+t)^{\frac{1}{2}}}, \quad C > 0,\ \sigma \in [0,+\infty),\ \sigma \neq 2,$$

instead of (6.4) or (6.10), respectively.

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