J. Math. Anal. Appl. 385 (2012) 559–571

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The boundedness of Riesz transforms for Hermite expansions on the Hardy spaces ☆

Huang Jizheng [a,b,*]

[a] College of Sciences, North China University of Technology, Beijing 100144, China
[b] CEMA, Central University of Finance and Economics, Beijing 100081, China

ARTICLE INFO

Article history:
Received 4 November 2010
Available online 1 July 2011
Submitted by R.H. Torres

Keywords:
Hermite expansions
Lusin area integral
Riesz transform
Hardy space

ABSTRACT

Let $L = -\Delta + |x|^2$ be a Hermite operator, where Δ is the Laplacian on $\mathbb{R}^d$. In this paper, we first characterize the Hardy spaces $H_L^1(\mathbb{R}^d)$ associated with L by a new version of area integral. Then, we use it to prove the boundedness of Riesz transforms R_j^L, $j = 1, 2, \ldots$ for L on $H_L^1(\mathbb{R}^d)$. Moreover, we characterize the Hardy space $H_L^1(\mathbb{R}^d)$ by R_j^L. This gives a negative answer to a question asked by Thangavelu.

1. Introduction

The Hermite polynomial on the real line is defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}\left(e^{-x^2}\right)e^{x^2}, \quad k = 0, 1, 2, \ldots.$$

Then, the Hermite function is defined by

$$h_k(x) = \left(\pi^{1/2}2^k k!\right)^{-1/2} H_k(x)\exp\left(-x^2/2\right), \quad k = 0, 1, \ldots.$$

In the d-dimensional Euclidean space $\mathbb{R}^d$, the Hermite functions are defined as follows. For any multi-index α and $x \in \mathbb{R}^d$, we define

$$h_\alpha(x) = \prod_{j=1}^{d} h_{\alpha_j}(x_j),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, \ldots\}$, $x = (x_1, \ldots, x_d)$. The system $\{h_\alpha\}$ is a complete orthonormal basis for $L^2(\mathbb{R}^d)$. They are eigenfunctions of the d-dimensional Hermite operator

$$L = -\Delta + |x|^2.$$

☆ Supported by National Natural Science Foundation of China under Grant #11001002, the Beijing Foundation Program under Grants #201010009009, #2010D005002000002.

* Correspondence to: College of Sciences, North China University of Technology, Beijing 100144, China.
E-mail address: hjzheng@163.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.075

Moreover, we have

$$Lh_\alpha = \big(2|\alpha| + d\big)h_\alpha,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The operator L is positive and symmetric in $L^2(\mathbb{R}^d)$. For more about Hermite functions one can refer to [13].

The heat semigroup $\{T_t^L\}_{t\geqslant 0}$ associated to $\mathcal{L}$ is defined by

$$T_t^L f = e^{-t\mathcal{L}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)} \mathcal{P}_n f,$$

where $f \in L^2(\mathbb{R}^d)$ and

$$\mathcal{P}_n f = \sum_{|\alpha|=n} < f, \quad h_\alpha > h_\alpha.$$

It is well known that this semigroup is a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d)$. The Poisson semigroup $\{P_t^L\}_{t\geqslant 0}$ associated to $\mathcal{L}$ is defined by

$$P_t^L f = e^{-t\mathcal{L}^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \mathcal{P}_n f, \quad f \in L^2\big(\mathbb{R}^d\big).$$

The Poisson semigroup is also a strongly continuous semigroup of contractions on $L^2(\mathbb{R}^d)$. By the principle of subordination, we have

$$P_t^L f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} \exp\big(-t^2/4s\big) T_s^L f(x)\, ds. \tag{1}$$

Since

$$L = -\frac{1}{2}\big[(\nabla + x)\cdot(\nabla - x) + (\nabla - x)\cdot(\nabla + x)\big],$$

we can define the following version of Riesz transforms R_j^L, $j = 1, 2, \ldots, d$

$$R_j^L = \left(\frac{\partial}{\partial x_j} + x_j\right) L^{-1/2}.$$

The definition was first suggested by Thangavelu in [12]. If $f \in L^2(\mathbb{R}^d)$, then

$$R_j^L f = \sum_\alpha \left(\frac{2\alpha_j}{2|\alpha| + d}\right)^{1/2} \langle f, h_\alpha\rangle h_{\alpha - e_j} = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left(\frac{2\alpha_j}{2|\alpha| + d}\right)^{1/2} \langle f, h_\alpha\rangle h_{\alpha - e_j}, \tag{2}$$

where e_j are the coordinate vectors in $\mathbb{R}^d$.

The kernel $R_j^L(x, y)$ of R_j^L is defined by

$$R_j^L(x, y) = \frac{1}{\Gamma(1/2)}\left(\frac{\partial}{\partial x_j} + x_j\right) \int_0^\infty G_t(x, y) t^{-1/2}\, dt = \frac{1}{\Gamma(1/2)} \int_0^\infty \left(\frac{\partial}{\partial x_j} + x_j\right) G_t(x, y) t^{-1/2}\, dt, \tag{3}$$

where

$$G_t^L(x, y) = (2\pi \sinh 2t)^{-d/2} \exp\left(-\frac{1}{2}|x - y|^2 \coth 2t - x\cdot y \tanh t\right).$$

Stempak and Torrea proved that (cf. [11])

Proposition 1. *The Riesz operators R_j^L, initially defined on $L^2(\mathbb{R}^d)$ by* (2)*, are Calderón–Zygmund operators associated with the kernels $R_j^L(x, y)$, given by* (3)*, which satisfy*

$$\big|R_j^L(x, y)\big| \leqslant \frac{C}{|x - y|^d},$$

and

$$\left|\nabla_x R_j^L(x,y)\right| + \left|\nabla_y R_j^L(x,y)\right| \leqslant \frac{C}{|x-y|^{d+1}}.$$

In [13], the author proves that the Riesz transforms R_j^L are bounded on $h^1(\mathbb{R}^d)$, where $h^1(\mathbb{R}^d)$ is the local Hardy space defined by Goldberg in [7]. Thangavelu also asks whether we can characterize $h^1(\mathbb{R}^d)$ by R_j^L, i.e., whether the equality

$$h^1(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d):\ R_j^L f \in L^1(\mathbb{R}^d),\ j = 1, 2, \ldots, n \right\}$$

is true.

In this paper, we will consider the boundedness of R_j^L on Hardy spaces $H_L^1(\mathbb{R}^d)$, $d \geqslant 3$, where $H_L^1(\mathbb{R}^d)$ is the Hardy space associated to L (cf. [4]). Moreover, we will characterize $H_L^1(\mathbb{R}^d)$ by R_j^L. This gives a negative answer to the question asked by Thangavelu.

Throughout the article, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leqslant \frac{B_1}{B_2} \leqslant C$.

We define the Hardy space $H_L^1(\mathbb{R}^d)$, $d \geqslant 3$, associated to L as follows

$$H_L^1(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d):\ \mathcal{M}_L f \in L^1(\mathbb{R}^d) \right\},$$

where $\mathcal{M}_L f(x) = \sup_{t>0} |T_t^L f(x)|$.

Let

$$\rho(x) = \frac{1}{1+|x|}, \tag{4}$$

and $B(x,r)$ be a ball in $\mathbb{R}^d$ with the center at x and radius r, we say a function $a(x)$ is an $H_L^{1,q}$-atom associate to a ball $B(x_0,r)$ for the space $H_L^1(\mathbb{R}^d)$, if

(1) $\operatorname{supp} a \subset B(x_0,r)$,
(2) $\|a\|_{L^\infty} \leqslant |B(x_0,r)|^{-1}$,
(3) if $r < \rho(x_0)$, then $\int a(x)\,dx = 0$.

The atomic quasi-norm in $H_L^1(\mathbb{R}^d)$ is defined by

$$\|f\|_{L\text{-atom}} = \inf\left\{ \sum |c_j| \right\},$$

where the infimum is taken over all decompositions $f = \sum c_j a_j$, where a_j are H_L^1-atoms.

In [4], the authors proved the following result.

Proposition 2. *The norms $\|f\|_{H_L^1}$ and $\|f\|_{L\text{-atom}}$ are equivalent, that is, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H_L^1} \leqslant \|f\|_{L\text{-atom}} \leqslant C\|f\|_{H_L^1}.$$

We define the following Lusin area integral

$$A_L f(x) = \left(\int_0^\infty \int_{|x-y|<t} \left|D_t^L f(y)\right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2},$$

where $D_t^L f(x) = t(\partial_t P_t^L f)(x)$.

Then, we can prove (cf. [8])

Proposition 3. *A function $f \in H_L^1(\mathbb{R}^d)$ if and only if its area integral $A_L f \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H_L^1} \sim \|A_L f\|_{L^1} + \|f\|_{L^1}.$$

In this paper, we will consider a general version of Lusin area integral

$$S_L f(x) = \left(\int_0^\infty \int_{|x-y|<t} t^2 \left| \nabla_L P_t^L f(y) \right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2},$$

where $\nabla_L = (\partial_t, \delta_1, \ldots, \delta_d)$ and $\delta_i = \frac{\partial}{\partial x_i} + x_i$, $i = 1, 2, \ldots, d$.

The main results of this paper are the following theorems.

Theorem 1. *A function $f \in H_L^1(\mathbb{R}^d)$ if and only if its area integral $S_L f \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover,*

$$\|f\|_{H_L^1} \sim \|S_L f\|_{L^1} + \|f\|_{L^1}.$$

Theorem 2. *The operators R_j^L are bounded on $H_L^1(\mathbb{R}^d)$ for all $j = 1, 2, \ldots, d$, i.e., there exists a constant $C > 0$ such that*

$$\left\| R_j^L f \right\|_{H_L^1} \leqslant C \|f\|_{H_L^1}.$$

The paper is organized as follows. In Section 2, we give some basic facts about the heat-diffusion semigroup and the Poisson semigroup associate to L. In Section 3, we prove Theorem 1. The proof of Theorem 2 will be given in Section 4.

2. Preliminaries

In this section, we give some basic facts about the heat-diffusion semigroup and the Poisson semigroup associate to L. Let $G_t^L(x, y)$ be the heat kernel of $\{T_t^L\}$. Then by the Feynman–Kac formula, we get

$$G_t^L(x, y) \leqslant W_t(x - y),$$

where

$$W_t(x) = (4\pi t)^{-d/2} \exp\left(-|x|^2/(4t)\right)$$

is the usual Gauss–Weierstrass kernel on $\mathbb{R}^d$.

The proof of the following proposition can be found in [5].

Proposition 4. (a) *For every N, there is a constant $C_N > 0$ such that*

$$0 \leqslant G_t^L(x, y) \leqslant C_N t^{-\frac{d}{2}} e^{-(5t)^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}. \tag{5}$$

(b) *There exist $0 < \delta < 1$ and $C > 0$ such that for every $N > 0$, there is a constant $C_N > 0$ so that, for all $|h| \leqslant \sqrt{t}$,*

$$\left| G_t^L(x + h, y) - G_t^L(x, y) \right| \leqslant C_N \left(\frac{|h|}{\sqrt{t}} \right)^\delta t^{-\frac{d}{2}} e^{-At^{-1}|x-y|^2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N}. \tag{6}$$

Remark 1. By part (a) of Proposition 4, it is easy to see that we can replace the condition $|h| \leqslant \sqrt{t}$ by $|h| \leqslant \frac{|x-y|}{2}$ in part (b) of Proposition 4.

By the subordination formula, we get

$$f(t, x) = \frac{t}{\sqrt{4\pi}} \int_{\mathbb{R}^d} \int_0^\infty G_s^L(x, y) s^{-3/2} e^{-t^2/4s}\, ds\, f(y)\, dy = \int_{\mathbb{R}^d} P_t^L(x, y) f(y)\, dy,$$

where

$$P_t^L(x, y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty G_s^L(x, y) s^{-3/2} e^{-t^2/4s}\, ds$$

is the Poisson kernel associated to L.

By the subordination formula and Proposition 4, we can prove

Proposition 5. (a) *For every N, there is a constant $C_N > 0$ such that*

$$0 \leqslant P_t^L(x,y) \leqslant C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{7}$$

(b) *Let $0 < \delta < 1$ and $|h| < \frac{|x-y|}{2}$. Then for any $N > 0$, there exist constants $C > 0$, $C_N > 0$, such that*

$$\left|P_t^L(x+h,y) - P_t^L(x,y)\right| \leqslant C_N \left(\frac{|h|}{\sqrt{t}}\right)^{\delta} \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}. \tag{8}$$

Let $D_t^L(x,y) = t\partial_t P_t^L(x,y)$, by Proposition 5, we can prove

Proposition 6. *Let $D_t^L(x,y)$ be the integral kernel of the operator D_t^L. Then there exist constants C, $0 < \delta' < \delta$, such that for every N, there is a constant C_N, so that*

(a) $|D_t^L(x,y)| \leqslant C_N \frac{t}{(t^2+C|x-y|^2)^{(d+1)/2}} (1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)})^{-N}$;
(b) $|D_t^L(x+h,y) - D_t^L(x,y)| \leqslant C_N (\frac{|h|}{t})^{\delta'} \frac{t}{(t^2+C|x-y|^2)^{(d+1)/2}} (1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)})^{-N}$, *for all* $|h| \leqslant t$;
(c) $|\int_{\mathbb{R}^d} D_t^L(x,y)\,dy| \leqslant C_N \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}$.

We have the following property about $\rho(x)$ (cf. [10, Lemma 1.4]).

Proposition 7. *There exist $C, k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \frac{\rho(y)}{\rho(x)} \leqslant C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}.$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

3. An area integral characterization of $H_L^1(\mathbb{R}^d)$

In this section, we give an area integral characterization of $H_L^1(\mathbb{R}^d)$. We will divide it into several steps and our proof is motivated by [1].

3.1. From Hardy spaces to maximal Hardy spaces

For $f \in L_{loc}^1(\mathbb{R}^d)$ and $|y|^{-d-1} f(y) \in L^1(\mathbb{R}^d)$, we define

$$f_L^*(x) = \sup_{\{(t,y)\in\mathbb{R}_+\times\mathbb{R}^d:\ |x-y|<t\}} \left|P_t^L f(y)\right|.$$

If $f \in L^1(\mathbb{R}^d)$ and $f_L^* \in L^1(\mathbb{R}^d)$. We will say $f \in H_{\max,L}^1(\mathbb{R}^d)$. Define

$$\|f\|_{H_{\max,L}^1(\mathbb{R}^d)} = \left\|f_L^*\right\|_{L^1(\mathbb{R}^d)}.$$

Lemma 1. *Let f be a locally integrable function on $\mathbb{R}^d$, then we have*

$$\|f\|_{H_{\max,L}^1} \leqslant C\|f\|_{H_L^1}.$$

Proof. Let $f \in H_L^1(\mathbb{R}^d)$. Then by Proposition 1, f can be decomposed into $H_L^{1,\infty}$-atoms. Let $a(x)$ be an $H_L^{1,\infty}$-atom, we will prove

$$\|a\|_{H_{\max,L}^1} \leqslant C, \tag{9}$$

where C is a positive constant and independent of a.

Assume $\operatorname{supp} a \subset B(y_0, r)$. We will consider two cases.

Case 1: $r < \rho(y_0)$, $a(x)$ satisfies the moment condition. Then, we can prove (9) in a standard fashion (cf. [1] or [6]).

Case 2: $r \geqslant \rho(y_0)$, $a(x)$ doesn't satisfy the moment condition. We have

$$P_t^L a(x) = \int_B P_t^L(x, y) a(y)\, dy$$

and

$$\int_{\mathbb{R}^d} \left|a_L^*(x)\right| dx = \int_{B^*} \left|a_L^*(x)\right| dx + \int_{(B^*)^c} \left|a_L^*(x)\right| dx = I_1 + I_2,$$

where $B^* = B(y_0, 2r)$. For I_1, we have

$$I_1 \leqslant \left|B^*\right|^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \left|M(a)(x)\right|^2 dx\right)^{1/2} \leqslant C|B|^{\frac{1}{2}} |B|^{-\frac{1}{2}} = C, \tag{10}$$

where M is the Hardy–Littlewood maximal function.

For I_2, we will first prove that

$$a_L^*(x) \leqslant C|B|^{\frac{1}{d}} |x - y_0|^{-d-1} \tag{11}$$

for any x such that $|x - y_0| > 2r$. When $|x - y_0| > 2r$, $|x - y| < t$ and $|y_0 - z| < r$, we have

$$t + |y - z| \geqslant t + |x - y_0| - |x - y| - |y_0 - z| \geqslant |x - y_0| - r \geqslant \frac{|x - y_0|}{2}.$$

By Proposition 7, we know $\rho(z) \leqslant Cr$ for any $z \in B(y_0, r)$. Therefore, by Proposition 5

$$\begin{aligned} \left|\int_B P_t^L(y, z) a(z)\, dz\right| &\leqslant C \int_B \frac{t}{(t^2 + A|y - z|^2)^{(d+1)/2}} \left(\frac{\rho(z)}{t}\right) \left|a(z)\right| dz \\ &\leqslant Cr \int_B \frac{1}{(t + |y - z|)^{d+1}} \left|a(z)\right| dz \\ &\leqslant C|B|^{\frac{1}{d}} |x - y_0|^{-d-1}. \end{aligned} \tag{12}$$

Taking supremum in $\{(y, t) \in \mathbb{R}^d \times (0, \infty)\colon\ |x - y| < t\}$, we get (11). Then

$$I_2 = \int_{(B^*)^c} \left|a_L^*(x)\right| dx \leqslant C|B|^{\frac{1}{d}} \int_{(B^*)^c} \frac{1}{|x - y_0|^{(d+1)}}\, dx \leqslant C|B|^{\frac{1}{d}} |B|^{-\frac{1}{d}} = C.$$

This completes the proof of Lemma 1. □

3.2. From maximal functions to area integral functions

We first prove the following version of Caccioppoli inequality associated to L, which is very important for our proof (cf. [9]).

Lemma 2. *Assume $u \in L^2_{loc}(B((x_0, t_0), 4r))$ and is a weak solution of $-\Delta u + \partial_t^2 u + |x|^2 u = 0$ in the ball $B((x_0, t_0), 4r)$, then we have*

$$\int_{B((x_0,t_0),r)} \left(\left|\partial_t u(x, t)\right|^2 + \sum_{i=1}^d \left|\delta_i u(x, t)\right|^2\right) dx\, dt \leqslant \frac{C}{r^2} \int_{B((x_0,t_0),2r)} \left|u(x, t)\right|^2 dx\, dt.$$

Proof. Let $\eta \in C_0^\infty(B((x_0, t_0), 2r))$ and satisfy $0 \leqslant \eta \leqslant 1$, $\eta(y, t) = 1$ for $(y, t) \in B((x_0, t_0), r)$ and $|\partial_t \eta|^2 + \sum_{i=1}^d |\partial_{x_i} \eta|^2 \leqslant \frac{C}{r^2}$.

By

$$\left|\partial_t u(x, t)\right|^2 + \sum_{i=1}^d \left|\delta_i u(x, t)\right|^2 \leqslant C\left(\left|\partial_t u(x, t)\right|^2 + \sum_{i=1}^d \left|\partial_i u(x, t)\right|^2 + |x|^2 \left|u(x, t)\right|^2\right),$$

we get

$$\int_{B((x_0,t_0),r)} \left(\left|\partial_t u(x,t)\right|^2 + \sum_{i=1}^{d}\left|\delta_i u(x,t)\right|^2\right) dx\,dt \leqslant C \int_{B((x_0,t_0),r)} \left(\left|\partial_t u(x,t)\right|^2 + \sum_{i=1}^{d}\left|\partial_i u(x,t)\right|^2 + |x|^2\left|u(x,t)\right|^2\right) dx\,dt$$
$$= C \int_{B((x_0,t_0),r)} \left(\left|\overline{\nabla} u(x,t)\right|^2 + |x|^2\left|u(x,t)\right|^2\right) dx\,dt,$$

where $|\overline{\nabla} u(x,t)|^2 = |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2$.

In the following, we will prove

$$\int_{B((x_0,t_0),r)} \left(\left|\overline{\nabla} u\right|^2 + |x|^2\left|u(x,t)\right|^2\right) dx\,dt \leqslant \frac{C}{r^2} \int_{B((x_0,t_0),2r)} \left|u(x,t)\right|^2 dx\,dt. \tag{13}$$

For simplicity, we use $B(2r)$ to denote $B((x_0,t_0),2r)$.

$$\begin{aligned}
0 &= \int_{B(2r)} \overline{\nabla} u(x,t)\cdot\overline{\nabla}\big(u\eta^2\big)(x,t) + |x|^2 u\big(u\eta^2\big)(x,t)\,dx\,dt \\
&= \int_{B(2r)} \overline{\nabla} u(x,t)\cdot\big(\eta^2\overline{\nabla} u(x,t) + 2\eta u\overline{\nabla}\eta(x,t)\big)\,dx\,dt + \int_{B(2r)} |x|^2\left|u^2\eta^2\right|(x,t)\,dx\,dt \\
&= \int_{B(2r)} \eta(x,t)\overline{\nabla} u(x,t)\cdot\big(\eta\overline{\nabla} u(x,t) + 2u\overline{\nabla}\eta(x,t)\big)\,dx\,dt + \int_{B(2r)} |x|^2\left|u^2\eta^2\right|(x,t)\,dx\,dt \\
&= \int_{B(2r)} \big(\overline{\nabla}(u\eta)(x,t) - u\overline{\nabla}\eta(x,t)\big)\cdot\big(\overline{\nabla}(u\eta)(x,t) - u\overline{\nabla}\eta(x,t) + 2u\overline{\nabla}\eta(x,t)\big)\,dx\,dt + \int_{B(2r)} |x|^2\left|u^2\eta^2\right|(x,t)\,dx\,dt \\
&= \int_{B(2r)} \left|\overline{\nabla}(u\eta)(x,t)\right|^2 dx\,dt - \int_{B(2r)} \left|u(x,t)\right|^2\left|\overline{\nabla}\eta(x,t)\right|^2 dx\,dt + \int_{B(2r)} |x|^2\left|u^2\eta^2\right|(x,t)\,dx\,dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{B(r)} \left|\overline{\nabla} u(x,t)\right|^2 + |x|^2\left|u(x,t)\right|^2 dx\,dt &\leqslant \int_{B(2r)} \left|\overline{\nabla}(u\eta)(x,t)\right|^2 + |x|^2\left|u^2\eta^2\right|(x,t)\,dx\,dt \\
&= \int_{B(2r)} \left|u(x,t)\right|^2\left|\overline{\nabla}\eta(x,t)\right|^2 dx\,dt \leqslant \frac{C}{r^2}\int_{B(2r)} \left|u(x,t)\right|^2 dx\,dt.
\end{aligned}$$

This gives the proof of (13) and then Lemma 2 is proved. □

Let

$$S_{L,\alpha} f(x) = \left(\int_{\Gamma_\alpha(x)} t^2\left|\nabla_L P_t^L f(y)\right|^2 \frac{dy\,dt}{t^{d+1}}\right)^{1/2}$$

and

$$S_{L,\alpha}^{\epsilon,R} f(x) = \left(\int_{\Gamma_\alpha^{\epsilon,R}(x)} t^2\left|\nabla_L P_t^L f(y)\right|^2 \frac{dy\,dt}{t^{d+1}}\right)^{1/2},$$

where $\Gamma_\alpha(x) = \{(y,t)\in\mathbb{R}^d\times\mathbb{R}^+\colon\ |y-x|<\alpha t\}$ and $\Gamma_\alpha^{\epsilon,R}(x) = \{(y,t)\in\mathbb{R}^d\times(\epsilon,R)\colon\ |y-x|<\alpha t\}$.

By Lemma 2, we can prove the following lemma (cf. [1]).

Lemma 3. *If $\alpha<1$, then, for $f\in L^2(\mathbb{R}^d)$, we have*

$$S_{L,\alpha}^{\epsilon,R} f(x) \leqslant C_\alpha\big(1+\left|\ln(R/\epsilon)\right|\big)^{1/2} f_L^*(x)$$

for some $C_\alpha>0$ that is independent of f.

Let

$$\widetilde{S}_{\alpha}^{\epsilon,R} f(x) = \left(\int_1^2 \int_{\Gamma_{\alpha/a}^{a\epsilon,aR}(x)} t^{1-d} \left| \nabla_L P_t^L f(y) \right|^2 dy\, dt\, da \right)^{1/2}.$$

It is easy to see that

$$S_{\alpha/2}^{2\epsilon,R} f(x) \leqslant \widetilde{S}_{\alpha}^{\epsilon,R} f(x) \leqslant S_{2\alpha}^{\epsilon,2R} f(x). \tag{14}$$

Moreover, we have

Lemma 4. *There exists $C > 0$ such that, for all $0 < \gamma < 1$, $\lambda > 0$, $0 < \epsilon < R < \infty$ and $f \in H_{\max,L}^1 \cap L^2(\mathbb{R}^d)$, we have*

$$\left| \left\{ x \in \mathbb{R}^d \colon \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda,\ f_L^*(x) \leqslant \gamma\lambda \right\} \right| \leqslant C\gamma^2 \left| \left\{ x \in \mathbb{R}^d \colon \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda \right\} \right|.$$

Proof. Let ϵ, R and λ be fixed. Then for $f \in H_{\max,L}^p \cap L^2(\mathbb{R}^d)$, define $O = \{x \in \mathbb{R}^d \colon \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda\}$. We assume that $O \neq \mathbb{R}^d$, otherwise, there is nothing prove. Let $O = \bigcup_k Q_k$ be a Whitney decomposition of O by dyadic cubes, so that, for all k, $2Q_k \subset O \subset \mathbb{R}^d$, but $4Q_k$ intersects O^c. Since

$$\left\{ x \in \mathbb{R}^d \colon \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda \right\} \subset \left\{ x \in \mathbb{R}^d \colon \widetilde{S}_{1/2}^{\epsilon,R} f(x) > \lambda \right\},$$

it is sufficient to prove

$$\left| \left\{ x \in Q_k \colon \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda,\ f_L^*(x) \leqslant \gamma\lambda \right\} \right| \leqslant C\gamma^2 |Q_k|. \tag{15}$$

Fixing k and let l be the side length of Q_k. If $x \in Q_k$, we have

$$\widetilde{S}_{1/20}^{\sup\{10l,\epsilon\},R} f(x) \leqslant \lambda.$$

In fact, choose $x_k \in 4Q_k$ and $x_k \notin O$. If $|x - y| < \frac{t}{20}$ and $t \geqslant \sup\{10l, \epsilon\}$, then we have

$$|x_k - y| < |x - y| + |x - x_k| < \frac{t}{20} + 4l < \frac{t}{2}.$$

Therefore,

$$\widetilde{S}_{1/20}^{\sup\{10l,\epsilon\},R} f(x) \leqslant \widetilde{S}_{1/2}^{\sup\{10l,\epsilon\},R} f(x_k) \leqslant \lambda.$$

If $\epsilon \geqslant 10l$, then $\{x \in Q_k \colon \widetilde{S}_{1/20}^{\epsilon,R} f(x) > 2\lambda\} = \phi$. So (15) holds.

If $\epsilon < 10l$, then

$$\widetilde{S}_{1/20}^{\epsilon,R} f(x) \leqslant \widetilde{S}_{1/20}^{\epsilon,10l} f(x) + \widetilde{S}_{1/20}^{10l,R} f(x).$$

Therefore, we only need to prove

$$\left| \left\{ x \in Q_k \colon \widetilde{S}_{1/20}^{\epsilon,10l} f(x) > \lambda,\ f_L^*(x) \leqslant \gamma\lambda \right\} \right| \leqslant C\gamma^2 |Q_k|. \tag{16}$$

For simplicity, we denote

$$g(x) = \widetilde{S}_{1/20}^{\epsilon,10l} f(x), \quad \text{and} \quad F = \left\{ x \in \mathbb{R}^d \colon f_L^*(x) \leqslant \gamma\lambda \right\}.$$

As $(x,t) \to u_t(x) = P_t^L f(x)$ is a continuous function, we know F is a closed subset of $\mathbb{R}^d$. Then, (16) follows from

$$\left| \left\{ x \in Q_k \colon g(x) > \lambda \right\} \right| \leqslant \frac{1}{\lambda^2} \int_{Q_k} \left| g(x) \right|^2 dx,$$

and

$$\int_{Q_k} \left| g(x) \right|^2 dx \leqslant C\gamma^2\lambda^2 |Q_k|. \tag{17}$$

If $5l \leqslant \epsilon$, then by Lemma 3, we obtain

$$\int_{Q_k} \left|g(x)\right|^2 dx \leqslant C \int_{Q_k} f_L^*(x)^2\, dx \leqslant C\gamma^2\lambda^2|Q_k|.$$

If $\epsilon < 5l$, then

$$\int_{Q_k} \left|g(x)\right|^2 dx \leqslant C \int_1^2 \int_{\xi_a} t\left|\nabla_L P_t^L f(y)\right|^2 dy\, dt\, da,$$

where $\xi_a = \{(y,t) \in \mathbb{R}^d \times (a\epsilon, 10al)\colon\ a\psi(y) < t\}$ and $\psi(y) = 20d(y, Q_k \cap F)$.

It is obvious that $\xi_a = \{(y, at)\colon\ (y,t) \in \xi_1\}$. Let $E = \{y\colon\ (y,t) \in \xi_1\}$. Then E is an open subset of $\mathbb{R}^d$. For any connected component G of E, we let $\mathcal{L}_a = \{(y,t) \in \xi_a\colon\ y \in G\}$. It is sufficient to prove that

$$\int_1^2 \int_{\mathcal{L}_a} t\left|\nabla_L u_t(y)\right|^2 dy\, dt\, da \leqslant c\lambda^2\gamma^2|G|. \tag{18}$$

In fact, if we can prove (18), by summing over all the connected components of E, then we get

$$\int_1^2 \int_{\xi_a} t\left|\nabla_L u_t(y)\right|^2 dy\, dt\, da \leqslant c\lambda^2\gamma^2|E|.$$

If $y \in E$, then there is a point $(y,t) \in \xi_1$. Therefore, there exists $x \in Q_k$ such that $|x-y| < \frac{t}{20}$. As $t < 10l$, we have $|x-y| < \frac{l}{2}$, hence $E \subset 2Q_k$, which shows that (18) holds.

In the following, we prove (18). We fix a connected component G of E, consider $a \in (1,2)$ and note that $\mathcal{L}_a$ is connected and has a Lipschitz boundary. By

$$-\left(-\Delta + |x|^2\right)u_t(x) + \partial_t^2 u_t(x) = 0$$

in the weak sense on $\mathbb{R}^d \times (0,\infty)$, we get

$$\left|\nabla_L u_t(x)\right|^2 = \frac{1}{2}\left(\Delta u_t^2(x) + \partial_t^2 u_t^2(x)\right).$$

Therefore

$$\left|\nabla_L u_t(x)\right|^2 \leqslant \frac{1}{2}\Delta u_t^2(x) + \frac{1}{2}\partial_t^2 u_t^2(x).$$

Then by the Green formula, we have

$$\begin{aligned}
\int_{\mathcal{L}_a} t\left|\nabla_L u_t(y)\right|^2 dy\, dt &\leqslant \frac{1}{2}\int_{\mathcal{L}_a} t\Delta u_t^2(y) + t\partial_t^2 u_t^2(y)\, dy\, dt \\
&= \int_{\partial\mathcal{L}_a} t u_t(y)\overline{\nabla} u_t(y) \cdot N_a(y,t)\, d\sigma_a(y,t) + \frac{1}{2}\int_{\partial\mathcal{L}_a} u_t^2(y) N_a(y,t) \cdot (0,\ldots,0,1)\, d\sigma_a(y,t),
\end{aligned} \tag{19}$$

where $N_a(y,t)$ is the unit normal vector outward $\mathcal{L}_a$ and $d\sigma_a(y,t)$ is the surface measure over $\partial\mathcal{L}_a$.

In the following, we show that $y \in 2Q_k \subset \mathbb{R}^d$ and $(y,t) \in \xi_1$ for $(y,t) \in \overline{\mathcal{L}_a}$. In fact, by the definition of $\mathcal{L}_a$ and note that F is a closed subset of $\mathbb{R}^d$, we know there exists $x \in Q_k \cap F$ such that $|x-y| \leqslant \frac{t}{20a}$. Since $t < 10la$, we have $|x-y| < \frac{l}{2}$, then we prove that $y \in 2Q_k$. By $|x-y| \leqslant \frac{t}{20a} < t$, we get $(y,t) \in \xi_1$. Therefore, $\overline{\mathcal{L}_a}$ remains far from the boundary of $\mathbb{R}^d \times (0,\infty)$, so that we do not care about the boundary values of $u_t(y)$ and $|u_t(y)| \leqslant \gamma\lambda$ on $\overline{\mathcal{L}_a}$.

By (19), we can obtain

$$\int_{\mathcal{L}_a} t\left|\overline{\nabla} u_t(y)\right|^2 dy\, dt \leqslant C\int_{\partial\mathcal{L}_a} t\left|u_t(y)\right|\left|\overline{\nabla} u_t(y)\right| d\sigma_a(y,t) + \int_{\partial\mathcal{L}_a} \left|u_t(y)\right|^2 d\sigma_a(y,t).$$

Since $|u_t(y)| \leqslant \gamma\lambda$ on $\partial\mathcal{L}_a$, we have

$$\int_1^2 \int_{\partial\mathcal{L}_a} \left|u_t(y)\right|^2 d\sigma_a(y,t)\,da \leqslant \gamma^2\lambda^2 \int_1^2 \int_{\partial\mathcal{L}_a} d\sigma_a(y,t)\,da.$$

We will show that

$$\int_1^2 \int_{\partial\mathcal{L}_a} d\sigma_a(y,t)\,da \leqslant C|G|. \tag{20}$$

We have

$$\int_1^2 \int_{\partial\mathcal{L}_a} d\sigma_a(y,t)\,da \leqslant C \int_{\mathcal{G}} \frac{dz\,ds}{s},$$

where $\mathcal{G}$ is the union of the sets $\partial\mathcal{L}_a$ for $1 < a < 2$, then

$$\mathcal{G} = \big\{(z,s)\colon\ z \in G \text{ and } \epsilon < s < 2\epsilon \text{ or } \psi(z) < s < 2\psi(z) \text{ or } 10l < s < 20l\big\}.$$

Therefore

$$\int_1^2 \int_{\partial\mathcal{L}_a} d\sigma_a(y,t)\,da \leqslant C \int_{\mathcal{G}} \frac{dz\,ds}{s} \leqslant C|G|.$$

It remains to prove

$$\int_1^2 \int_{\partial\mathcal{L}_a} t\left|u_t(y)\right|\left|\overline{\nabla}u_t(y)\right| d\sigma_a(y,t)\,da \leqslant C\gamma^2\lambda^2|G|. \tag{21}$$

Let $\mathcal{G}$ be the same set as above, then

$$\int_1^2 \int_{\partial\mathcal{L}_a} t\left|u_t(y)\right|\left|\overline{\nabla}u_t(y)\right| d\sigma_a(y,t)\,da \leqslant C\gamma\lambda \int_{\mathcal{G}} \left|\overline{\nabla}u_t(y)\right| dy\,dt.$$

Let $B_j = B((x_j,t_j), \frac{\epsilon t_j}{20})$ be a covering of $\mathcal{G}$ with bounded overlap. Noting that $(x,t) \in B_j$ implies $t \sim t_j \sim r_j$, the radius of B_j. Then by Hölder's inequality and Caccioppoli's inequality, we get

$$\begin{aligned}
\int_{\mathcal{G}} \left|\overline{\nabla}u_t(y)\right| dy\,dt &\leqslant C \sum_j \int_{B_j} \left|\overline{\nabla}u_t(y)\right| dy\,dt \\
&\leqslant C \sum_j |B_j|^{1/2} \bigg(\int_{B_j} \left|\overline{\nabla}u_t(y)\right|^2 dy\,dt\bigg)^{1/2} \\
&\leqslant C \sum_j |B_j|^{1/2} r_j^{-1} \bigg(\int_{2B_j} \left|u_t(y)\right|^2 dy\,dt\bigg)^{1/2} \\
&\leqslant C\gamma\lambda \sum_j |B_j| r_j^{-1} \leqslant C\gamma\lambda \int_{\widetilde{\mathcal{G}}} \frac{dz\,ds}{s},
\end{aligned}$$

where $\widetilde{\mathcal{G}}$ is a set like $\mathcal{G}$ but slightly enlarged: it is contained in the set of points (z,s) with $z \in G$ and $\epsilon/2 < s < 4\epsilon$ or $\psi(z)/2 < s < 4\psi(z)$ or $5l < s < 40l$.

This proves that (21) holds and Lemma 4 is proved. $\square$

The proof of the following lemma can be found in [2].

Lemma 5. *For $\alpha, \beta > 0$ and $0 < \epsilon < R < \infty$, we have*

$$\left\| S_\alpha^{\epsilon,R} \right\|_{L^1(\mathbb{R}^d)} \sim \left\| S_\beta^{\epsilon,R} \right\|_{L^1(\mathbb{R}^d)},$$

where the implicit constants do not dependent on ϵ, R, f.

Now, we can prove

Lemma 6. *Let f be a locally integrable function on $\mathbb{R}^d$ such that $\|\widetilde{S}_{1/20}^{\epsilon,R} f\|_{L^1(\mathbb{R}^d)} < \infty$. Then we have*

$$\|S_L f\|_{L^1(\mathbb{R}^d)} \leqslant C \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)}.$$

Proof. Firstly, let $f \in H^1_{\max,L} \cap L^2(\mathbb{R}^d)$. By Lemma 4, we get

$$\left\|\widetilde{S}_{1/20}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} \leqslant C\gamma^{-1} \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)} + C\gamma^2 \left\|\widetilde{S}_{1/2}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)}. \tag{22}$$

By Lemma 5 and (14), we know

$$\left\| S_1^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} \leqslant C \left\| S_{1/40}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} \leqslant C \left\|\widetilde{S}_{1/20}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)}.$$

Then by Lemma 3,

$$\begin{aligned}
\left\|\widetilde{S}_{1/2}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} &\leqslant C \left\| S_1^{\epsilon/2,2R} f\right\|_{L^1(\mathbb{R}^d)} \\
&\leqslant C \left\| S_1^{\epsilon/2,\epsilon} f\right\|_{L^1(\mathbb{R}^d)} + \left\| S_1^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} + \left\| S_1^{R,2R} f\right\|_{L^1(\mathbb{R}^d)} \\
&\leqslant C \left\| S_1^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} + C \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)} \\
&\leqslant C \left\|\widetilde{S}_{1/20}^{\epsilon,R} f\right\|_{L^1(\mathbb{R}^d)} + C \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

We can choose a proper γ in (22) to get Lemma 6.

Now we relax the assumption $f \in L^2(\mathbb{R}^d)$. By $f_L^* \in L^1(\mathbb{R}^d)$, we get $P_s^L f \in L^1(\mathbb{R}^d)$ for any $s > 0$. Moreover, for any $s > 0$ and x, we have

$$\left| P_s^L f(x) \right| \leqslant f_L^*(y), \quad y \in B(x,s).$$

Therefore, by Proposition 5, we can get

$$P_s^L f(x) \leqslant \frac{1}{|B(x,s)|} \int\limits_{B(x,s)} f_L^*(y)\, dy \leqslant \frac{c}{s^d} \int\limits_{\mathbb{R}^d} f_L^*(y)\, dy.$$

This gives $P_s^L f \in L^\infty(\mathbb{R}^d)$ for any $s > 0$. Then, we can get $P_s^L f \in L^2(\mathbb{R}^d)$ for any $s > 0$. The proof of the above gives

$$\|S_L f_s\|_{L^1(\mathbb{R}^d)} \leqslant C \left\| (f_s)_L^* \right\|_{L^1(\mathbb{R}^d)},$$

where $f_s = P_s^L f$. As

$$(f_s)_L^*(x) = \sup_{|x-y|<t} \left| P_t^L f_s(y) \right| = \sup_{|x-y|<t} \left| P_{t+s}^L f(y) \right| \leqslant \sup_{|x-y|<t+s} \left| P_{t+s}^L f(y) \right| = f_L^*(x),$$

we have

$$\|S_L f_s\|_{L^1(\mathbb{R}^d)} \leqslant C \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)}.$$

Let $s \to 0$, by monotone theorem, we get

$$\|S_L f\|_{L^1(\mathbb{R}^d)} \leqslant C \left\| f_L^* \right\|_{L^1(\mathbb{R}^d)}.$$

This gives the proof of Lemma 6. $\square$

3.3. From area integral functions to Hardy spaces

By $\|A_L f\|_{L^1(\mathbb{R}^d)} \leqslant \|S_L f\|_{L^1(\mathbb{R}^d)}$ and Proposition 3, we know

Lemma 7. *If $f \in L^1(\mathbb{R}^d)$ and $S_L f \in L^1(\mathbb{R}^d)$, then $f \in H^1_L(\mathbb{R}^d)$. Moreover, we have*

$$\|f\|_{H^1_L(\mathbb{R}^d)} \leqslant C\|S_L\|_{L^1(\mathbb{R}^d)} + \|f\|_{L^1}.$$

Proof of Theorem 1. Theorem 1 follows from Lemmas 1, 6 and 7. □

4. Boundedness of R^L_j on $H^1_L(\mathbb{R}^d)$

In this section, we prove that the operators R^L_j, $j = 1, 2, \ldots$ are bounded on $H^1_L(\mathbb{R}^d)$.
We first prove the following lemma.

Lemma 8. *If $f \in L^2(\mathbb{R}^d)$, then*

$$D_t^{L+2}\big(R^L_j f\big) = -t\delta_j e^{-tL^{1/2}} f$$

for all $j = 1, 2, \ldots, d$.

Proof. By Lemma 4.2 in [11], we have

$$e^{-t(L+2)^{1/2}}\big(R^L_j f\big)(x) = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \sum_{|\alpha|=n} \left(\frac{2\alpha_i}{2|\alpha|+d}\right)^{1/2} \langle f, h_\alpha\rangle h_{\alpha-e_j}(x).$$

Therefore

$$\partial_t e^{-t(L+2)^{1/2}}\big(R^L_j f\big)(x) = -\sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \sum_{|\alpha|=n} (2\alpha_i)^{1/2}\langle f, h_\alpha\rangle h_{\alpha-e_j}(x). \tag{23}$$

Similarly, by

$$e^{-tL^{1/2}} f(x) = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \sum_{|\alpha|=n} \langle f, h_\alpha\rangle h_\alpha(x)$$

and (3), we get

$$\delta_j e^{-tL^{1/2}} f(x) = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \sum_{|\alpha|=n} (2\alpha_i)^{1/2}\langle f, h_\alpha\rangle h_{\alpha-e_j}(x). \tag{24}$$

Lemma 8 follows from (23) and (24). □

In order to get our result, we need the following lemma.

Lemma 9. *If $f \in L^1(\mathbb{R}^d)$, then $f \in H^1_L(\mathbb{R}^d)$ if and only if $f \in H^1_{L+2}(\mathbb{R}^d)$.*

Proof. When $f \in H^1_L(\mathbb{R}^d)$, then $\mathcal{M}_L f \in L^1(\mathbb{R}^d)$. Therefore, $f \in H^1_{L+2}(\mathbb{R}^d)$ follows from $\mathcal{M}_{L+2} f \leqslant \mathcal{M}_L f$.
When $f \in H^1_{L+2}(\mathbb{R}^d)$, by Proposition 2

$$f(x) = \sum_{i=1}^{\infty} \lambda_i a_i(x),$$

where $a_i(x)$, $i = 1, 2, \ldots$ are H^1_{L+2}-atoms. It is easy to prove

$$\rho(x) \sim \rho\big(x, |x|^2 + 2\big). \tag{25}$$

Then, by Proposition 6, we can prove that there exists a constant $C > 0$ such that (cf. Lemma 6 in [3])

$$\big\|A_L(a_i)\big\| \leqslant C.$$

Therefore

$$\|f\|_{H_L^1} \leqslant \|f\|_{H_{L+2}^1}. \tag{26}$$

Then Lemma 9 is proved. □

Now, we can prove the main result of this paper.

Proof of Theorem 2. As $H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $H_L^1(\mathbb{R}^d)$ (see [8]), we can assume $f \in H_L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. By Lemma 9, it is sufficient to prove $R_j^L f \in H_{L+2}^1(\mathbb{R}^d)$. Following from Proposition 3, Theorem 1 and Lemma 8,

$$\begin{aligned}
\|R_j^L f\|_{H_{L+2}^1(\mathbb{R}^d)} &\leqslant C\|A_{L+2}R_j^L f\|_{L^1(\mathbb{R}^d)} \\
&= C\left\| \left(\int_0^\infty \int_{|x-y|<t} \left|D_t^{L+2}\left(R_j^L f\right)(y)\right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
&= C\left\| \left(\int_0^\infty \int_{|x-y|<t} \left|t\delta_j e^{-tL^{1/2}} f(y)\right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2} \right\|_{L^1(\mathbb{R}^d)} \\
&\leqslant C\|S_L f\|_{L^1(\mathbb{R}^d)} \leqslant C\|f\|_{H_L^1(\mathbb{R}^d)}.
\end{aligned}$$

This completes the proof of Theorem 2. □

By Theorem 2, we can prove

Corollary 1. *For $f \in L^1(\mathbb{R}^d)$, $f \in H_L^1(\mathbb{R}^d)$ if and only if $R_j^L f \in L^1(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$ i.e.,*

$$H_L^1(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d):\ R_j^L f \in L^1(\mathbb{R}^d),\ j = 1, 2, \ldots, d \right\}.$$

Moreover, we have

$$\|f\|_{H_L^1(\mathbb{R}^d)} \sim \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^d \|R_j^L f\|_{L^1(\mathbb{R}^d)}.$$

Proof. By Theorem 2, we know $R_j^L f \in L^1(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$ for $f \in H_L^1(\mathbb{R}^d)$.

For the reverse, by $R_j^L f \in L^1(\mathbb{R}^d)$ and $x_j f \in L^1(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$ (cf. Lemma 0.9 in [10]), we get $\partial_j L^{-\frac{1}{2}} f \in L^1(\mathbb{R}^d)$, $j = 1, 2, \ldots, d$. Then Corollary 1 follows from Theorem 2 in [4]. □

Acknowledgments

The author expresses his appreciations of the reviewer's valued suggestions to make the paper more comfortable. He is also very grateful to his advisor professor Heping Liu for his encouragements.

References

[1] P. Auscher, E. Russ, Hardy spaces and divergence operators on strongly Lipschitz domain of $\mathbb{R}^n$, J. Funct. Anal. 201 (2003) 148–184.
[2] R.R. Coifman, Y. Meyer, E.M. Stein, Some new function spaces and their applications to harmonic analysis, J. Funct. Anal. 62 (1985) 304–335.
[3] J. Dziubański, G. Garrigós, T. Martínez, J.L. Torrea, J. Zienkiewicz, BMO spaces related to Schrödinger operator with potential satisfying reverse Hölder inequality, Math. Z. 249 (2005) 329–356.
[4] J. Dziubański, J. Zienkiewicz, The Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iberoam. 15 (1999) 279–296.
[5] J. Dziubański, J. Zienkiewicz, H^p spaces for Schrödinger operators, in: Fourier Analysis and Related Topics, in: Banach Center Publ., vol. 56, 2002, pp. 45–53.
[6] J. Garcia-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Math. Stud., vol. 116, North-Holland Publishing Co., Amsterdam, 1985.
[7] D. Goldberg, A local version of real Hardy spaces, Duke Math. J. 46 (1) (1979) 27–42.
[8] S. Hofmann, G.Z. Lu, D. Mitres, L.X. Yan, Hardy spaces associated with nonnegative self-adjoint operators satisfying Davies–Gafney estimates, preprint, 2008.
[9] H. Li, Estimations L^p des opérateurs de Schrödinger sur les groupes nilpotents, J. Funct. Anal. 161 (1999) 152–218.
[10] Z. Shen, L^p estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble) 45 (1995) 513–546.
[11] K. Stempak, J.L. Torrea, Poisson integrals and Riesz transforms for the Hermite function expansions with weights, J. Funct. Anal. 202 (2003) 443–472.
[12] S. Thangavelu, Riesz transforms and the wave equation for the Hermite operator, Comm. Partial Differential Equations 15 (1990) 1199–1215.
[13] S. Thangavelu, Lectures on Hermite and Laguerre Expansions, Math. Notes, vol. 42, Princeton University Press, Princeton, NJ, 1993.