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The complete asymptotic expansion for Bernstein operators

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ABSTRACT

In this paper we study the asymptotic behavior of the classical Bernstein operators, applied to q-times continuously differentiable functions. Our main results extend the results of S.N. Bernstein and R.G. Mamedov for all q-odd natural numbers and thus generalize the theorem of E.V. Voronovskaja. The exact degree of approximation is also proved.

1. Introduction

For every function $f \in C[0,1]$ the Bernstein polynomial operator is given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

For this operator the theorem of Voronovskaja was first proved in [13] and is given in the book of DeVore and Lorentz [2] as follows:

Theorem A. *If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ for some $x \in [0,1]$, then*

$$\lim_{n\to\infty} n \cdot \left[B_n(f,x) - f(x)\right] = \frac{x(1-x)}{2} \cdot f''(x).$$

If $f \in C^2[0,1]$, the convergence is uniform.

This result has attracted the attention of many authors in the last 80 years. Inspired by the result of Voronovskaja, her scientific advisor S. Bernstein generalized Theorem A, showing in [1] the asymptotic expansion of Bernstein operator for $f \in C^q[0,1]$ for q-even as follows:

Theorem B. *If $q \in N$ is even, $f \in C^q[0,1]$, then uniformly in $x \in [0,1]$,*

$$n^{\frac{q}{2}} \cdot \left[B_n(f,x) - f(x) - \sum_{r=1}^{q} B_n\left((e_1 - x)^r, x\right) \cdot \frac{f^{(r)}(x)}{r!}\right] \to 0, \quad n \to \infty.$$

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Here and to the end of the paper $e_1 : t \to t$, $t \in [0,1]$ is the monomial function. In [9] Mamedov considered also the case $f \in C^q[0,1]$, q-even, namely:

Theorem C. *Let $q \in N$ be even, $f \in C^q[0,1]$, and $L_n : C[0,1] \to C[0,1]$ be a sequence of positive linear operators such that*

$$L_n(e_0, x) = 1, \quad x \in [0,1];$$
$$\lim_{n\to\infty} \frac{L_n((e_1-x)^{q+2j}, x)}{L_n((e_1-x)^q, x)} = 0$$

for at least one $j \in \{1, 2, \ldots\}$. Then

$$\frac{1}{L_n((e_1-x)^q, x)} \cdot \left[L_n(f,x) - f(x) - \sum_{r=1}^{q} L_n\big((e_1-x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0,$$

when $n \to \infty$.

A complete asymptotic expansion in quantitative form was already given some 30 years ago by Sikkema and van der Meer in [11]:

Theorem D. *Let $WC^q[0,1]$ denote the set of all functions on $[0,1]$ whose q-th derivative is piecewise continuous, $q \geqslant 0$. Moreover, let (L_n) be a sequence of positive linear operators $L_n : WC^q[0,1] \to C[0,1]$ satisfying $L_n(e_0, x) = 1$. Then for all $f \in WC^q[0,1]$, $q \in N_0$, $x \in [0,1]$, $n \in N$ and $\delta > 0$ one has*

$$\left| L_n(f,x) - f(x) - \sum_{r=1}^{q} L_n\big((e_1-x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!} \right| \leqslant c_{n,q}(x,\delta) \cdot \omega\big(f^{(q)}, \delta\big).$$

Here

$$c_{n,q}(x,\delta) = \delta^q \cdot L_n\left(s_{q,\mu}\left(\frac{e_1 - x}{\delta}\right), x\right),$$
$$\mu = \frac{1}{2}, \quad \text{if } L_n\big((e_1-x)^q, x\big) \geqslant 0,$$
$$\mu = -\frac{1}{2}, \quad \text{if } L_n\big((e_1-x)^q, x\big) < 0,$$
$$s_{q,\mu}(u) = \frac{1}{q!}\left(\frac{1}{2}|u|^q + \mu \cdot u^q\right) + \frac{1}{(q+1)!}\big(b_{q+1}\big(|u|\big) - b_{q+1}\big(|u| - \big[|u|\big]\big)\big).$$

b_{q+1} is the Bernoulli polynomial of degree $q+1$ and $[t] = \max\{z \in Z:\ z \leqslant t\}$. Moreover, the functions $c_{n,q}(x,\delta)$ are best possible for each $f \in C^q[0,1]$, $x \in [0,1]$, $n \in N$, and $\delta > 0$.

It is hardly possible to list all known results, concerning pointwise estimates in Voronovskaja-type theorems, but we end this brief history mentioning the result of Videnskij, published in [12]: For $f \in C^2[0,1]$ we have

$$\left| n\big[B_n(f,x) - f(x)\big] - \frac{x(1-x)f''(x)}{2n} \right| \leqslant x(1-x) \cdot \omega\left(f'', \sqrt{\frac{2}{n}}\right).$$

Inspired by the results of Bernstein and Mamedov, very recently H. Gonska proved more general asymptotic statement in a quantitative form. This is given (see [4, Theorem 3.2])

Theorem E. *Let $q \in N_0$, $f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left| L(f,x) - \sum_{r=0}^{q} L\big((e_1-x)^r, x\big) \cdot \frac{f^{(r)}}{r!} \right| \leqslant \frac{L(|e_1-x|^q, x)}{q!} \cdot \tilde{\omega}\left(f^{(q)}, \frac{1}{q+1} \cdot \frac{L((e_1-x)^{q+1}, x)}{L(|e_1-x|^q, x)}\right). \tag{1.1}$$

We use Theorem E in the proof of our results. Further results as continuation of [4] are obtained by Gonska and Raşa in [5–7]. Now we formulate our main statements.

Theorem 1 *(Extension of Mamedov's result). Let $q \in N$ be odd, $f \in C^q[0,1]$, and $L_n : C[0,1] \to C[0,1]$ be a sequence of positive linear operators such that*

$$L_n(e_0, x) = 1, \quad x \in [0,1];$$
$$\lim_{n\to\infty} \frac{L_n((e_1 - x)^{l+2j}, x)}{L_n((e_1 - x)^l, x)} = 0$$

for $l \in N$ – any even number and for at least one $j \in \{1, 2, \ldots\}$. Then if $n \to \infty$, we have

$$\frac{1}{L_n(|e_1 - x|^q, x)} \cdot \left[L_n(f, x) - f(x) - \sum_{r=1}^{q} L_n\big((e_1 - x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0. \tag{1.2}$$

Theorem 2 *(Extension of Bernstein's result). If $q \in N$ is odd, $f \in C^q[0,1]$, then uniformly in $x \in [0,1]$, when $n \to \infty$ we have*

$$n^{\frac{q}{2}} \cdot \left[B_n(f, x) - f(x) - \sum_{r=1}^{q} B_n\big((e_1 - x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0. \tag{1.3}$$

Theorem 3 *(Optimal rate of convergence). Let $q \in N$ and $f \in C^q[0,1]$. There exists no constant $\beta > 0$, such that the uniform convergence (w.r.t. $x \in [0,1]$)*

$$n^{\frac{q}{2}+\beta} \cdot \left[B_n(f, x) - f(x) - \sum_{r=1}^{q} B_n\big((e_1 - x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!} \right] \to 0, \quad n \to \infty. \tag{1.4}$$

In Section 2 we give the proofs of Theorems 1–3 and make some remarks and formulate a conjecture, concerning the moments of Bernstein operator.

2. Proofs

Proof of Theorem 1. We apply Theorem E. Our aim is to show that the argument of the least concave majorant of the modulus of continuity $\tilde{\omega}(f^{(q)}, \cdot)$ in (1.1) tends to 0, when $n \to \infty$, namely

$$\lim_{n\to\infty} \frac{L((e_1 - x)^{q+1}, x)}{L(|e_1 - x|^q, x)} = 0. \tag{2.1}$$

Using the Cauchy–Schwarz inequality for positive linear functionals repeatedly we obtain

$$\frac{L((e_1 - x)^{q+1}, x)}{L(|e_1 - x|^q, x)} = \frac{L_n(|e_1 - x|^{\frac{q}{2}} \cdot |e_1 - x|^{\frac{q}{2}+1}, x)}{L(|e_1 - x|^q, x)} \leqslant \sqrt{\frac{L(|e_1 - x|^{q+2}, x)}{L(|e_1 - x|^q, x)}} \leqslant \sqrt{\frac{L(|e_1 - x|^{q+2}, x)}{L(|e_1 - x|^{q+1}, x)}}. \tag{2.2}$$

The last inequality follows from the positivity of L_n. We observe that now $l = q + 1$ is an even number and the assumptions, made for L_n are fulfilled. Consequently the Cauchy–Schwarz inequality implies

$$\sqrt{\frac{L(|e_1 - x|^{q+2}, x)}{L(|e_1 - x|^{q+1}, x)}} \leqslant \sqrt[4]{\frac{L(|e_1 - x|^{q+3}, x)}{L(|e_1 - x|^{q+1}, x)}} \leqslant \cdots \leqslant \sqrt[4j]{\frac{L(|e_1 - x|^{q+1+2j}, x)}{L(|e_1 - x|^{q+1}, x)}}. \tag{2.3}$$

From (2.2) and (2.3) it follows that (2.1) holds true, since according to Mamedov assumptions on L_n, for $l = q + 1$ the last quantity in (2.3) tends to 0 for at least one $j \in N$. The proof is completed. $\square$

Proof of Theorem 2. We apply Theorem E for $L = B_n$ – the Bernstein operator. First we establish upper bound for $B_n(|e_1 - x|^q, x)$ again using the Cauchy–Schwarz inequality. Hence

$$B_n\big(|e_1 - x|^q, x\big) \leqslant \sqrt{B_n\big(|e_1 - x|^{2q}, x\big)}. \tag{2.4}$$

If we denote by

$$T_{n,s} := \sum_{k=0}^{n} (k - nx)^s \cdot p_{n,k}(x), \quad n = 1, 2, \ldots, \ s = 0, 1, \ldots$$

then for each $s = 0, 1, \ldots$ there is a constant A_s such that

$$0 \leqslant T_{n,2s}(x) \leqslant A_s \cdot n^s. \tag{2.5}$$

The last inequality is the well-known bound for the moments of the Bernstein operator and can be found in Ch. 10 in [2]. Obviously

$$B_n\big((e_1 - x)^{2q}, x\big) = n^{-2q} \cdot T_{n,2q}(x). \tag{2.6}$$

Therefore (2.4), (2.5) and (2.6) lead to

$$B_n\big(|e_1 - x|^q, x\big) \leqslant \sqrt{A_q} \cdot n^{-\frac{q}{2}}. \tag{2.7}$$

Our next step is to show that the argument of the least concave majorant in Theorem E uniformly on [0, 1] tends to 0, when $n \to \infty$ and q is an odd number. Some suggestions in this direction are made by Gonska in [4], but at the moment it is not clear how to establish this fact for the case of Bernstein operator. According to this problem we formulate a conjecture at the end of the paper. We choose another method, namely the use of K-functional's technique. Let $I = [a, b]$ be a compact interval of the real axis and $f \in C(I)$. In [10] the following result for the least concave majorant is proved for $t \geqslant 0$:

$$K\left(\frac{t}{2}, f; C[a,b], C^1[a,b]\right) := \inf_{g \in C^2(I)}\left(\|f - g\|_\infty + \frac{t}{2}\|g'\|_\infty\right) = \frac{1}{2}\tilde{\omega}(f, t). \tag{2.8}$$

For arbitrary $g \in C^1[0,1]$ we apply (2.8) and estimate the right-hand side of the inequality in Theorem E as follows

$$\begin{aligned}
&\frac{B_n(|e_1 - x|^q, x)}{q!} \cdot \tilde{\omega}\left(f^{(q)}, \frac{1}{q+1} \cdot \frac{B_n((e_1 - x)^{q+1}, x)}{B_n(|e_1 - x|^q, x)}\right) \\
&\quad \leqslant \frac{B_n(|e_1 - x|^q, x)}{q!} \cdot 2\big\|f^{(q)} - g\big\|_\infty + \frac{B_n(|e_1 - x|^{q+1}, x)}{(q+1)!}\big\|g'\big\|_\infty \\
&\quad \leqslant \frac{\sqrt{B_n(|e_1 - x|^{2q}, x)}}{q!} \cdot 2\big\|f^{(q)} - g\big\|_\infty + \frac{\sqrt{B_n(|e_1 - x|^{2q}, x)} \cdot \sqrt{B_n(|e_1 - x|^2, x)}}{(q+1)!}\big\|g'\big\|_\infty \\
&\quad \leqslant \frac{\sqrt{B_n(|e_1 - x|^{2q}, x)}}{q!} 2\left[\big\|f^{(q)} - g\big\|_\infty + \frac{\sqrt{B_n(|e_1 - x|^2, x)}}{2(q+1)}\big\|g'\big\|_\infty\right].
\end{aligned} \tag{2.9}$$

In the last inequality we applied (2.4) and also Cauchy–Schwarz inequality to estimate the moments of B_n of order q and $q + 1$. We take the infimum over all $g \in C^1[0,1]$ in (2.9) and (2.8) implies

$$\begin{aligned}
&\frac{B_n(|e_1 - x|^q, x)}{q!} \cdot \tilde{\omega}\left(f^{(q)}, \frac{1}{q+1} \cdot \frac{B_n((e_1 - x)^{q+1}, x)}{B_n(|e_1 - x|^q, x)}\right) \\
&\quad \leqslant \frac{\sqrt{B_n(|e_1 - x|^{2q}, x)}}{q!} \cdot \tilde{\omega}\left(f^{(q)}, \frac{1}{q+1} \cdot \sqrt{B_n\big(|e_1 - x|^2, x\big)}\right).
\end{aligned} \tag{2.10}$$

It is known that

$$B_n\big(|e_1 - x|^2, x\big) = \frac{x(1-x)}{n}. \tag{2.11}$$

Now (2.5), (2.6), (2.10), (2.11) and Theorem E imply

$$n^{\frac{q}{2}} \cdot \left[B_n(f, x) - f(x) - \sum_{r=1}^{q} B_n\big((e_1 - x)^r, x\big) \cdot \frac{f^{(r)}(x)}{r!}\right] \leqslant \frac{\sqrt{A_q}}{q!} \cdot \tilde{\omega}\left(f^{(q)}, \frac{1}{q+1} \cdot \sqrt{\frac{x(1-x)}{n}}\right). \tag{2.12}$$

The right-hand side of (2.12) uniformly tends to 0 when $n \to \infty$. The proof is completed. □

Proof of Theorem 3. For any $q \in N$ we define

$$f_0(x) := \left|x - \frac{1}{2}\right|^{q+\alpha}, \tag{2.13}$$

where $0 < \alpha < 2\beta$, $x \in [0,1]$. We suppose that (1.3) holds uniformly in $x \in [0,1]$ for the function f_0. Let $x_0 = \frac{1}{2}$. Then

$$\begin{aligned}
&B_n(f_0, x_0) - f_0(x_0) - \sum_{r=1}^{q} B_n\big((e_1 - x_0)^r, x_0\big) \cdot \frac{f_0^{(r)}(x_0)}{r!} \\
&\quad = B_n(f_0, x_0) - f_0(x_0) = \|B_n f_0 - f_0\|_\infty \geqslant c\omega_\varphi^2\left(f_0, \frac{1}{\sqrt{n}}\right)
\end{aligned}$$

$$\geqslant c \sup_{0<h\leqslant\frac{1}{\sqrt{n}}} \left| f_0\left(\frac{1}{2}+h\varphi\left(\frac{1}{2}\right)\right) - 2f_0\left(\frac{1}{2}\right) + f_0\left(\frac{1}{2}-h\varphi\left(\frac{1}{2}\right)\right)\right|$$

$$= c \sup_{0<h\leqslant\frac{1}{\sqrt{n}}} 2\left(h\varphi\left(\frac{1}{2}\right)\right)^{q+\alpha} = c2^{1-q-\alpha}\left(\frac{1}{\sqrt{n}}\right)^{q+\alpha}. \tag{2.14}$$

In the second line of (2.14) we used the remarkable strong converse inequality for Bernstein operator, established first from Knoop and Zhou in [8], which states that for any $f \in C[0,1]$ and second order Ditzian–Totik modulus of smoothness $\omega_2^{\varphi}(f_0, \frac{1}{\sqrt{n}})$ (for definition, properties and many applications of this constructive function characteristic see [3]) there exist positive constants c, C, independent of n and f, such that

$$c\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right) \leqslant \|B_n f - f\|_{C[0,1]} \leqslant C\omega_2^{\varphi}\left(f, \frac{1}{\sqrt{n}}\right).$$

From (2.14) we obtain

$$n^{\frac{q}{2}+\beta} \cdot \left[B_n(f_0, x_0) - f_0(x_0) - \sum_{r=1}^{q} B_n\big((e_1-x_0)^r, x_0\big) \cdot \frac{f_0^{(r)}(x_0)}{r!}\right] \geqslant C(q,\alpha)\cdot n^{\beta-\frac{\alpha}{2}} \to \infty, \quad n\to\infty,$$

which contradicts our supposition. The proof of Theorem 3 is completed. □

Remark 1. To prove Theorem 2 we may proceed in a different way, using the results of Sikkema and van der Meer. To this aim we need to estimate the terms $c_{n,q}(x,\delta)$ for the case of the Bernstein operator, which seems to be very complicated.

Remark 2. We may prove Theorem 2 also if we can show that the argument in the modulus in Theorem E uniformly tends to 0, when q is odd number.

Therefore we formulate the following

Conjecture. *Prove that for all natural numbers q the following holds true*

$$\lim_{n\to\infty} \frac{B_n((e_1-x)^{q+1}, x)}{B_n(|e_1-x|^q, x)} = 0. \tag{2.15}$$

It is easy to verify, that (2.15) *is true for* $q = 0, 1, 2$.

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