J. Math. Anal. Appl. 387 (2012) 1033–1043

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Viscous limit to contact discontinuity for the 1-D compressible Navier–Stokes equations

Shixiang Ma [a,b,*]

[a] School of Mathematical Sciences, South China Normal University, Guang Zhou 510631, China
[b] The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

ARTICLE INFO

Article history:
Received 1 July 2011
Available online 8 October 2011
Submitted by D. Wang

Keywords:
Compressible Navier–Stokes equations
Compressible Euler system
Zero dissipation limit
Contact discontinuity

ABSTRACT

In this paper, we study the zero dissipation limit problem for the one-dimensional compressible Navier–Stokes equations. We prove that if the solution of the inviscid Euler equations is piecewise constants with a contact discontinuity, then there exist smooth solutions to the Navier–Stokes equations which converge to the inviscid solution away from the contact discontinuity at a rate of $\kappa^{\frac{3}{4}}$ as the heat-conductivity coefficient κ tends to zero, provided that the viscosity μ is higher order than the heat-conductivity κ or the same order as κ. Here we have no need to restrict the strength of the contact discontinuity to be small.

1. Introduction

Most of physical processes are modelled by the following one-dimensional onservation laws

$$\mathbf{u}_t + f(\mathbf{u})_x = 0, \quad x \in R,\ t \in R^+,\ \mathbf{u} \in R^n,\ f(\mathbf{u}) \in R^n, \tag{1.1}$$

when ignoring the small scale effects. At the next level of exactness, these small effects often make their appearance felt by the presence of higher-order derivatives multiplied by small coefficients in the equations such as

$$\mathbf{u}_t + f(\mathbf{u})_x = \varepsilon\big(B(\mathbf{u})\big)_{xx}, \tag{1.2}$$

where ε is the viscosity coefficient, and $B(\mathbf{u}) \in R^{n\times n}$ is called viscosity matrix. Then the consistency of the models would demand that solutions of the two sets of systems be "close" in some sense. It is of great importance to study the asymptotic equivalence between the viscous systems and the corresponding inviscid hyperbolic system in the limit of small dissipation. When viscosity matrix is positive definite, Bianchini and Bressan [1] considered the general solutions with the initial data having small total variations, they proved the convergence of the solutions for the viscous systems (1.2) to those for the associated hyperbolic systems (1.1) by establishing the uniform total variation estimates. Yet, the protypical example for conservation laws is the gas dynamics. The ideal fluid associated with (1.1) in Lagrangian coordinates is described by the following compressible Euler equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = 0, \quad x \in R,\ t > 0, \end{cases} \tag{1.3}$$

* Correspondence to: School of Mathematical Sciences, South China Normal University, Guang Zhou 510631, China.
E-mail address: mashx822@gmail.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.10.010

which is one of the most important nonlinear strictly hyperbolic systems of conservation laws. If adding the effects of the viscosity and thermal conductivity, corresponding to (1.2) the equations can be written as the following compressible Navier–Stokes equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu\left(\dfrac{u_x}{v}\right)_x, \\ \left(e + \dfrac{u^2}{2}\right)_t + (pu)_x = \kappa\left(\dfrac{\theta_x}{v}\right)_x + \mu\left(\dfrac{uu_x}{v}\right)_x, \quad x \in R,\ t > 0. \end{cases} \tag{1.4}$$

Here v, u, θ, p and e denote the specific volume, the velocity, the temperature, the pressure, and the internal energy, respectively, and μ, κ are the viscosity and heat-conductivity coefficients, respectively. x is the Lagrangian coordinate, so that $x = constant$ corresponds to a particle path. Here the viscosity matrix is only semi-positive definite and thus less dissipative, the method in [1] cannot be applied to the Navier–Stokes equations. This remains an important open problem. However, there are also many significant works on special solutions. For the case that the Euler flow contains a single shock, Hoff and Liu [4] studied the isentropic case, they established the limit process from the solutions of the compressible Navier–Stokes equations to the single shock-wave solution of the corresponding compressible Euler system (so-called p-system). They show that the solutions to the isentropic Navier–Stokes equations with shock data exist and converge to the inviscid shocks as the viscosity vanishes, uniformly away from the shocks. Ignoring the initial layers, Goodman and Xin [2] gave a very detailed description of the asymptotic behavior of solutions for the general viscous systems as the viscosity tends to zero, via a method of matching asymptotics. This method can be applied to the Navier–Stokes equations (1.4), such as [11,16,17]. Later Yu [19] revealed the rich structure of nonlinear wave interactions due to the presence of shocks and initial layers by a detailed pointwise analysis. As far as rarefaction wave is concerned, Xin in [18] has obtained that the solutions for the isentropic Navier–Stokes equations with weak centered rarefaction wave data exist for all time and converge to the weak centered rarefaction wave solution of the corresponding Euler system, as the viscosity tends to zero, uniformly away from the initial discontinuity. Moreover, in the case that either the initial layers are ignored or the rarefaction waves are smooth, he also obtains a rate of convergence which is valid uniformly for all time. Later Jiang et al. [8] improve the first part with weak centered rarefaction waves data and Zeng [20] improve the other results, respectively, in [18] to the full compressible Navier–Stokes equations, provided that the viscosity and heat-conductivity coefficients are in the same order. For composite wave, recently Huang et al. [7] study the case that the Riemann solution of the Euler system is a superposition of two rarefaction waves and a contact discontinuity. They obtain the corresponding convergence rate. Furthermore, by a spectral analysis and Evans function method, Kevin Zumbrun and his collaborators have obtained many important results even for large amplitude and multi-dimensional case [14,13,12,21,3], etc. Since the case that the solutions to the Euler system containing contact discontinuity is much more subtle, there are few results on this respect [10,7].

In this paper, we consider the case that the viscosity coefficient μ is higher order than the heat-conductivity coefficient κ or the same order as κ. We study the ideal polytropic gas, so that the pressure p and the internal energy e are related with v and θ by the following equations of state

$$p \equiv p(v,\theta) = R\theta/v, \qquad e \equiv e(\theta) = R\theta/(\gamma - 1) + constant, \tag{1.5}$$

where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent.

For the Riemann problem to the corresponding Euler system (1.3) with the Riemann initial data

$$(v,u,\theta)(x,0) = \begin{cases} (v_-, u_-, \theta_-), & \text{if } x < 0, \\ (v_+, u_+, \theta_+), & \text{if } x > 0. \end{cases} \tag{1.6}$$

A contact discontinuity takes the form

$$(\tilde V, \tilde U, \tilde\Theta)(x,t) = \begin{cases} (v_-, u_-, \theta_-), & \text{if } x < 0, \\ (v_+, u_+, \theta_+), & \text{if } x > 0, \end{cases} \tag{1.7}$$

provided that

$$u_- = u_+, \qquad p_- \equiv \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} \equiv p_+. \tag{1.8}$$

As in [6], in the setting of the compressible Navier–Stokes equations (1.4), the corresponding wave to the contact discontinuity becomes smooth and behaves as a diffusion wave due to the dissipation effect. We call this wave “viscous contact wave.” We now construct the viscous contact wave $(\bar v, \bar u, \bar\theta)$ as follows. Since the pressure of the profile $(\bar v, \bar u, \bar\theta)$ is expected to be almost constant, that is,

$$\bar p \equiv \frac{R\bar\theta}{\bar v} \approx p_+, \tag{1.9}$$

which indicates that when ignoring the higher order term the energy equation $(1.4)_3$ is

$$\frac{R}{\gamma-1}\theta_t + p_+ u_x = \kappa\left(\frac{\theta_x}{v}\right)_x. \tag{1.10}$$

Substituting (1.9) into (1.10) and using $(1.4)_1$ yield a nonlinear diffusion equation

$$\theta_t = a\kappa\left(\frac{\theta_x}{\theta}\right)_x, \qquad \theta(-\infty,t)=\theta_-, \qquad \theta(+\infty,t)=\theta_+, \qquad a=\frac{p_+(\gamma-1)}{\gamma R^2}>0, \tag{1.11}$$

which admits a unique self-similar solution $\Theta(x,t)=\Theta(\xi)$, $\xi=\frac{x}{\sqrt{1+t}}$ due to [5,15]. Furthermore, $\Theta(\xi)$ is a monotone function, increasing if $\theta_+>\theta_-$ and decreasing if $\theta_+<\theta_-$. Let $\delta=|\theta_+-\theta_-|$, then Θ satisfies

$$\left|\big(\kappa(1+t)\big)^{\frac{l}{2}}\partial_x^l\Theta\right| + |\Theta-\theta_\pm| \leqslant c_1\delta e^{-\frac{c_2x^2}{\kappa(1+t)}} \quad \text{as } |x|\to\infty,\ l\geqslant 1. \tag{1.12}$$

With Θ so determined, we can define the contact wave profile $(\bar v,\bar u,\bar\theta)$ as follows:

$$\begin{aligned}
&\bar v=\frac{R}{p_+}\Theta, \qquad \bar u = u_- + \frac{(\gamma-1)\kappa}{\gamma R}(\ln\Theta)_x,\\
&\bar\theta = \Theta - \frac{(\gamma-1)\kappa}{\gamma R p_+}\Theta_t + \frac{(\gamma-1)\mu\kappa}{\gamma R^2}(\ln\Theta)_{xx}.
\end{aligned} \tag{1.13}$$

Then $(\bar v,\bar u,\bar\theta)$ satisfies

$$\|\bar v-\tilde V,\bar u-\tilde U,\bar\theta-\tilde\Theta\|_{L^p} = O\big(\kappa^{1/(2p)}\big)(1+t)^{1/(2p)}, \quad p\geqslant 1, \tag{1.14}$$

and

$$\begin{cases}
\bar v_t-\bar u_x=0,\\
\bar u_t+\bar p_x=\mu\left(\dfrac{\bar u_x}{\bar v}\right)_x,\\
\left(\bar e+\dfrac{\bar u^2}{2}\right)_t + (\bar u\bar p)_x = \kappa\left(\dfrac{\bar\theta_x}{\bar v}\right)_x + \mu\left(\dfrac{\bar u\bar u_x}{\bar v}\right)_x + \bar R_1,
\end{cases} \tag{1.15}$$

where $\bar e=\frac{R\bar\theta}{\gamma-1}$ and

$$\begin{aligned}
\bar R_1 &= \frac{a\kappa^2}{p^+}\left\{\frac{(\gamma-1)}{\gamma}(\ln\Theta)_{xxt} + \frac{1}{\gamma}(\ln\Theta)_t(\ln\Theta)_{xx} + (\ln\Theta)_{tx}(\ln\Theta)_x\right\}\\
&\quad - \frac{\mu\kappa}{R}\left\{\frac{(\gamma-1)}{\gamma}(\ln\Theta)_{xxt} + (\ln\Theta)_t(\ln\Theta)_{xx} + (\ln\Theta)_{tx}(\ln\Theta)_x\right\}\\
&= O(\delta)\kappa^3\big(\kappa(1+t)\big)^{-2}e^{-\frac{c_2x^2}{\kappa(1+t)}} \quad \text{as } |x|\to\infty.
\end{aligned} \tag{1.16}$$

The main results of this paper are as follows:

Theorem 1.1. *For any given (v_-,u_-,θ_-), suppose that (v_+,u_+,θ_+) satisfies (1.8). Let $(\tilde V,\tilde U,\tilde\Theta)$ be a contact discontinuity solution of the form (1.7) with finite strength to the Euler system (1.3). Then, if the viscosity coefficient μ is higher order than κ or the same order as κ, there exists constant $\kappa_0>0$, such that for each $\kappa\in(0,\kappa_0]$, there is a smooth solution $(v^\kappa,u^\kappa,\theta^\kappa)$ to (1.4) on $R\times R^+$, still denoted by (v,u,θ), with the same initial data as $(\bar v,\bar u,\bar\theta)$. Moreover, for any arbitrarily large $T>0$ and small $h>0$, it holds that*

$$\sup_{0\leqslant t\leqslant T,\,|x|\geqslant h}\left|(v,u,\theta)(x,t)-(\tilde V,\tilde U,\tilde\Theta)(x,t)\right| \leqslant C\kappa^{\frac{3}{4}}, \tag{1.17}$$

where C is a positive constant independent of κ.

Remark 1.2. In this paper, we construct a new ansatz (see (1.13)), which gives better estimates for the error term $\bar R_1$ than the one which is in [10]. And thus we can obtain a higher convergence rate. Since we also consider the case that the viscosity coefficient μ can be higher order than the heat-conductivity κ, the term $\frac{\mu}{\kappa}\int\|\psi_y(\cdot,\tau)\|\,d\tau$ may be zero as κ tends to zero (see (3.7)). Here, to control the term $\int\|(\phi_y,\psi_y)(\cdot,\tau)\|\,d\tau$, we construct an explicit matrix S (see Section 4). From (4.7) we know that if $\int\|(\phi_y,\psi_y)(\cdot,\tau)\|\,d\tau$ can be better controlled, the optimal convergence rate may be obtained. We expect the explicit form of S will work on this respect. Yet it is regret that in this paper we have not done this, which will be left for future.

Notation. In this paper, $|a|=(\sum_{i=1}^n a_i^2)^{\frac{1}{2}}$ if $a=(a_i)$ is a vector in R^n and $|A|=(\sum_{i=1}^n\sum_{j=1}^n A_{ij}^2)^{\frac{1}{2}}$ if $A=(A_{ij})_{n\times n}$ is a matrix. We also use $H^l(l\geqslant 1)$ to denote the usual Sobolev space with the norm $\|\cdot\|_l$ and $\|\cdot\|=\|\cdot\|_0$ denotes the usual L^2-norm.

2. Reformulation of the problem

Due to the estimates (1.12) and (1.14), to prove the main theorem, it suffices to show that there exists an exact solution to (1.4) in a neighborhood of the approximate solution $\bar{U} \equiv (\bar{v}, \bar{u}, \bar{\theta})$, and that the asymptotic behavior of the solution to (1.4) is given by $\bar{U}$ for small heat-conductivity κ.

Suppose that $U \equiv (v, u, \theta)$ is the exact solution to (1.4) with the initial data $U(x,0) = \bar{U}(x,0)$. We decompose the solution as

$$\phi = v - \bar{v}, \qquad \psi = u - \bar{u}, \qquad \zeta = \theta - \bar{\theta}. \tag{2.1}$$

Then using the relation (1.15) for $\bar{U}$, we obtain that

$$\begin{cases} \phi_t - \psi_x = 0, \\ \psi_t + \left(\dfrac{R\zeta - \bar{p}\phi}{v}\right)_x = \mu\left(\dfrac{u_x}{v} - \dfrac{\bar{u}_x}{\bar{v}}\right)_x, \\ \dfrac{R}{\gamma - 1}\zeta_t + pu_x - \bar{p}\bar{u}_x = \kappa\left(\dfrac{\theta_x}{v} - \dfrac{\bar{\theta}_x}{\bar{v}}\right)_x + \mu\left(\dfrac{uu_x}{v} - \dfrac{\bar{u}\bar{u}_x}{\bar{v}}\right)_x - \bar{R}_1, \\ \phi(x,0) = \psi(x,0) = \zeta(x,0) = 0. \end{cases} \tag{2.2}$$

Using the following scalings,

$$y = \frac{x}{\kappa}, \qquad \tau = \frac{1+t}{\kappa}, \tag{2.3}$$

we transform (2.2) into

$$\begin{cases} \phi_\tau - \psi_y = 0, \\ \psi_\tau + \left(\dfrac{R\zeta - \bar{p}\phi}{v}\right)_y = \dfrac{\mu}{\kappa}\left(\dfrac{u_y}{v} - \frac{\bar{u}_y}{\bar{v}}\right)_y, \\ \dfrac{R}{\gamma - 1}\zeta_\tau + pu_y - \bar{p}\bar{u}_y = \left(\dfrac{\theta_y}{v} - \dfrac{\bar{\theta}_y}{\bar{v}}\right)_y + \dfrac{\mu}{\kappa}\left(\dfrac{uu_y}{v} - \dfrac{\bar{u}\bar{u}_y}{\bar{v}}\right)_y - R_1, \\ \phi(y,\tau_0) = \psi(y,\tau_0) = \zeta(y,\tau_0) = 0, \end{cases} \tag{2.4}$$

where $\tau_0 = 1/\kappa$, $R_1 = \kappa\bar{R}_1$ and

$$\left|\partial_y^l \Theta\right| \leqslant c_1\kappa^{\frac{l}{2}} e^{-\frac{c_2 y^2}{\tau}}, \quad l \geqslant 1; \qquad |R_1| \leqslant c_1\kappa^2 e^{-\frac{c_2 y^2}{\tau}}. \tag{2.5}$$

Set $\tau_1 = \frac{1+T}{\kappa}$. Then we only need to show that for suitably small κ, (2.4) has a unique "small" smooth solution on $R \times [\tau_0, \tau_1]$. By the standard existence and uniqueness theory, and the continuous induction argument for hyperbolic-parabolic equations [9], it suffices to close the following a priori estimate

$$N(\tau) \equiv \left\|(\phi, \psi, \zeta)(\cdot, \tau)\right\|_2 \leqslant \varepsilon, \tag{2.6}$$

where ε is a positive small constant depending on T, the initial data and the strength of the contact discontinuity. This is a consequence of a series of lemmas. We start with the lower order estimate.

3. Lower order estimate

Lemma 3.1. *Suppose that the Cauchy problem* (2.4) *has a solution* $(\phi, \psi, \zeta) \in C^1([\tau_0, \tau_2] : H^2(R^1))$ *for some* $\tau_0 < \tau_2 < \tau_1$. *Then there exist positive constants* ε_1, κ_1 *and* c, *which are independent of* κ *and* τ_2, *such that if* $0 < \varepsilon < \varepsilon_1$ *and* $\kappa \leqslant \kappa_1$, *we have*

$$\sup_{\tau_0 \leqslant \tau \leqslant \tau_1} \left\|(\phi, \psi, \zeta)(\cdot, \tau)\right\|^2 + \int_{\tau_0}^{\tau_1} \left\|\zeta_y(\cdot, \tau)\right\|^2 d\tau \leqslant c\kappa^{\frac{3}{2}}. \tag{3.1}$$

Proof. Similar to [6], we have

$$\left(\frac{1}{2}\psi^2 + R\bar{v}\Phi\left(\frac{v}{\bar{v}}\right) + \frac{R}{\gamma - 1}\bar{\theta}\Phi\left(\frac{\theta}{\bar{\theta}}\right)\right)_\tau + \frac{\mu}{\kappa}\frac{\psi_y^2}{v} + \frac{\zeta_y^2}{v\theta} + L_y + Q = 0, \tag{3.2}$$

where

$$\Phi(s) = s - 1 - \ln s, \tag{3.3}$$

$$L = R\left(\frac{\theta}{v} - \frac{\bar\theta}{\bar v}\right)\psi - \frac{\mu}{\kappa}\left\{\left(\frac{u_y}{v} - \frac{\bar u_y}{\bar v}\right)\psi + \left(\frac{uu_y}{v} - \frac{\bar u\bar u_y}{\bar v}\right)\frac{\zeta}{\theta}\right\} - \frac{\zeta}{\theta}\left(\frac{\theta_y}{v} - \frac{\bar\theta_y}{\bar v}\right), \tag{3.4}$$

and

$$Q = \bar p\Phi\left(\frac{\bar v}{v}\right)\bar u_y - \bar v\bar p_\tau\Phi\left(\frac{v}{\bar v}\right) + \frac{R}{\gamma-1}\Phi\left(\frac{\bar\theta}{\theta}\right)\bar\theta_\tau - \frac{\zeta}{\theta}(\bar p - p)\bar u_y + \frac{\bar\theta_y}{\theta}\left(\frac{1}{v} - \frac{1}{\bar v}\right)\zeta_y$$
$$- \frac{\zeta\theta_y}{\theta^2}\left(\frac{\theta_y}{v} - \frac{\bar\theta_y}{\bar v}\right) + \frac{\mu}{\kappa}\left\{\bar u_y\left(\frac{1}{v} - \frac{1}{\bar v}\right)\psi_y + \left(\frac{uu_y}{v} - \frac{\bar u\bar u_y}{\bar v}\right)\left(\frac{\zeta}{\theta}\right)_y\right\} + \frac{\zeta}{\theta}R_1, \tag{3.5}$$

satisfying

$$|Q| \leqslant (\varepsilon+\eta)\left(\frac{\mu}{\kappa}\psi_y^2 + \zeta_y^2\right) + c_\eta\left(|\Theta_{yy}| + |\Theta_y|^2\right)\left(\phi^2+\psi^2+\zeta^2\right) + \kappa\zeta^2 + C\kappa^{-1}R_1^2, \tag{3.6}$$

where $\eta > 0$ is a constant to be determined later. Then (3.2)–(3.6) and (2.5) yield that

$$\int\left(\frac{1}{2}\psi^2 + R\bar v\Phi\left(\frac{v}{\bar v}\right) + \frac{R}{\gamma-1}\bar\theta\Phi\left(\frac{\theta}{\bar\theta}\right)\right)dy + \int_{\tau_0}^{\tau}\int\left(\frac{\mu}{\kappa}\frac{\psi_y^2}{v} + \frac{1}{v\theta}\zeta_y^2\right)dy\,d\tau$$
$$= -\int_{\tau_0}^{\tau}\int Q\zeta\,dy\,d\tau \leqslant c(\varepsilon+\eta)\int_{\tau_0}^{\tau}\int\left(\frac{\mu}{\kappa}\psi_y^2 + \zeta_y^2\right)dy\,d\tau + c_\eta\kappa\int_{\tau_0}^{\tau}\int\left(\phi^2+\psi^2+\zeta^2\right)dy\,d\tau$$
$$+ \kappa^{-1}\int_{\tau_0}^{\tau}\int R_1^2\,dy\,d\tau. \tag{3.7}$$

Using (2.5) and taking ε and η to be sufficiently small, we obtain

$$\left\|(\phi,\psi,\zeta)(\cdot,\tau)\right\|^2 + \int_{\tau_0}^{\tau}\left\|\zeta_y(\cdot,\tau)\right\|^2 d\tau \leqslant c\kappa\int_{\tau_0}^{\tau}\left\|(\phi,\psi,\zeta)(\cdot,\tau)\right\|^2 + c\kappa^{\frac{3}{2}}. \tag{3.8}$$

And then we apply Gronwall's inequality to deduce that

$$\left\|(\phi,\psi,\zeta)(\cdot,\tau)\right\|^2 + \int_{\tau_0}^{\tau}\left\|\zeta_y(\cdot,\tau)\right\|^2 d\tau \leqslant c\kappa^{\frac{3}{2}}. \tag{3.9}$$

This finishes the proof of Lemma 3.1. □

4. Higher order estimates

Lemma 4.1. *Suppose that the conditions in Lemma* 3.1 *are satisfied. Then*

$$\left\|(\phi_y,\psi_y,\zeta_y)(\cdot,\tau)\right\|^2 + \int_{\tau_0}^{\tau}\left(\left\|(\phi_y,\psi_y)(\cdot,\tau)\right\|^2 + \left\|\zeta_{yy}(\cdot,\tau)\right\|^2\right)d\tau \leqslant c\kappa^{\frac{3}{2}}, \tag{4.1}$$

for all $\tau \in [\tau_0, \tau_2]$, where the constant c is independent of τ_2 and κ.

Proof. Step 1. Rewrite (1.4) in the following symmetric form

$$A^0(U)U_\tau + A(U)U_y = B(U)U_{yy} + g(U, U_y), \tag{4.2}$$

where $g(U,U_y) = (0, \frac{\mu}{\kappa}\theta(\frac{1}{v})_y u_y, (\frac{1}{v})_y\theta_y + \frac{\mu}{\kappa}\frac{u_y^2}{v})^t$, and

$$A^0(U) = \begin{pmatrix} -\theta p_v & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \frac{R}{\gamma-1}\end{pmatrix}, \qquad A(U) = \begin{pmatrix} 0 & \theta p_v & 0 \\ \theta p_v & 0 & p \\ 0 & p & 0\end{pmatrix}, \qquad B(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{\kappa}\frac{\theta}{v} & 0 \\ 0 & 0 & \frac{1}{v}\end{pmatrix}.$$

Consequently, the system (1.15) is transformed into

$$A^0(\bar U)\bar U_\tau + A(\bar U)\bar U_y = B(\bar U)\bar U_{yy} + g(\bar U, \bar U_y) + \bar F, \tag{4.3}$$

where $\bar F = (0, 0, R_1)^t$. Now we define a new matrix $\tilde A(U)$ as

$$\tilde A(U) = \begin{pmatrix} A_{11}(U) & A_{12}(\bar U) \\ A_{21}(\bar U) & 0 \end{pmatrix}, \tag{4.4}$$

where $A_{11}(U) = \begin{pmatrix} 0 & \theta p_v \\ \theta p_v & 0 \end{pmatrix}$ and $A_{12}(U) = \begin{pmatrix} 0 \\ p \end{pmatrix} = A_{21}(U)^t$. Set $W = U - \bar U$. (4.2)–(4.4) lead to

$$A^0(U)W_\tau + \tilde A(U)W_y = B(U)W_{yy} + \tilde g(U, U_y) + \big(\tilde A(U) - A(U)\big)W_y \tag{4.5}$$

where

$$\tilde g(U, U_y) = \big\{\big(A^0(\bar U) - A^0(U)\big)\bar U_\tau + \big(B(U) - B(\bar U)\big)\bar U_{yy}\big\} + \big(A(\bar U) - A(U)\big)\bar U_y + \big(g(U, U_y) - g(\bar U, \bar U_y)\big) - \bar F.$$

Differentiating (4.5) with respect to y, multiplying the resulting system by $\partial_y W$ and integrating on R, we obtain

$$\int \big\langle A^0(U)\partial_y W_\tau, \partial_y W\big\rangle dy + \int \big\langle \tilde A(U)\partial_y^2 W, \partial_y W\big\rangle dy = \int \big\langle B(U)\partial_y^3 W, \partial_y W\big\rangle dy + \int \langle \tilde H, \partial_y W\rangle\, dy. \tag{4.6}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on R^3, and

$$\begin{aligned}\tilde H = {} & A^0(U)\partial_y\big(A^0(U)^{-1}\tilde g\big) + A^0(U)\big[\partial_y, A^0(U)^{-1}B(U)\big]W_{yy} + A^0(U)\partial_y\big\{A^0(U)^{-1}\big(\tilde A(U) - A(U)\big)W_y\big\} \\ & - A^0(U)\big[\partial_y, A^0(U)^{-1}\tilde A(U)\big]W_y,\end{aligned}$$

where $[\cdot,\cdot]\cdot$ denotes the commutator. Next we will estimate the terms in (4.6) separately. First, using (1.13), (2.1), (2.5) and the system (2.4), we have

$$\begin{aligned}\int \big\langle A^0(U)\partial_y W_\tau, \partial_y W\big\rangle dy &= \frac{1}{2}\frac{d}{d\tau}\int \big\langle A^0(U)\partial_y W, \partial_y W\big\rangle dy - \frac{1}{2}\int \big\langle \partial_\tau A^0(U)\partial_y W, \partial_y W\big\rangle dy \\ &\geqslant \frac{1}{2}\frac{d}{d\tau}\int \big\langle A^0(U)\partial_y W, \partial_y W\big\rangle dy - c(\varepsilon + \kappa)\int \big(\phi_y^2 + \psi_y^2 + \zeta_{yy}^2\big)\, dy.\end{aligned}$$

Similarly, Sobolev's inequality and Young's inequality yield

$$\begin{aligned}-\int \big\langle \tilde A(U)\partial_y^2 W, \partial_y W\big\rangle dy &= \frac{1}{2}\int \big\langle \partial_y \tilde A(U)\partial_y W, \partial_y W\big\rangle dy \\ &\leqslant c\int \big(|W_y| + |\bar U_y|\big)|W_y|^2\, dy \\ &\leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}}\big)\int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\, dy.\end{aligned}$$

By a direct calculation, the third term is estimated as

$$\begin{aligned}\int \big\langle B(U)\partial_y^3 W, \partial_y W\big\rangle dy &= \int \left(\frac{\mu}{\kappa}\frac{\theta}{v}\partial_y^3\psi\,\partial_y\psi + \frac{1}{v}\partial_y^3\zeta\,\partial_y\zeta\right) dy \\ &\leqslant -\int \left(\frac{\mu}{\kappa}\frac{\theta}{v}\psi_{yy}^2 + \frac{1}{v}\zeta_{yy}^2\right) dy + c\int \big(|W_y| + |\bar U_y|\big)\left(\frac{\mu}{\kappa}|\psi_y||\psi_{yy}| + |\zeta_y||\zeta_{yy}|\right) dy \\ &\leqslant -\int \left(\frac{\mu}{\kappa}\frac{\theta}{v}\psi_{yy}^2 + \frac{1}{v}\zeta_{yy}^2\right) dy + c\big(\varepsilon + \kappa^{\frac{1}{2}}\big)\int \left(\frac{\mu}{\kappa}\psi_{yy}^2 + \zeta_{yy}^2 + \psi_y^2 + \zeta_y^2\right) dy.\end{aligned}$$

Finally,

$$\begin{aligned}\int \langle \tilde H, \partial_y W\rangle\, dy = {} & \int \big\langle A^0(U)\partial_y\big(A^0(U)^{-1}\tilde g\big), \partial_y W\big\rangle dy + \int \big\langle A^0(U)\big[\partial_y, A^0(U)^{-1}B(U)\big]W_{yy}, \partial_y W\big\rangle dy \\ & + \int \big\langle A^0(U)\partial_y\big\{A^0(U)^{-1}\big(\tilde A(U) - A(U)\big)W_y\big\} - A^0(U)\big[\partial_y, A^0(U)^{-1}\tilde A(U)\big]W_y, \partial_y W\big\rangle dy.\end{aligned}$$

We denote the terms on the right in order by I, II, III, which can be estimated separately below.

$$I = \int \big\langle A^0(U)\partial_y\big(A^0(U)^{-1}\big)\tilde g, \partial_y W\big\rangle dy + \int \langle \partial_y \tilde g, \partial_y W\rangle\, dy \equiv I_1 + I_2.$$

Using the estimates in (2.5) and Lemma 3.1, we have

$$\begin{aligned} I_1 &\leqslant c\int\big(|W_y|+|\bar U_y|\big)\big\{|W_y|+\big(|\bar U_\tau|+|\bar U_{yy}|+|\bar U_y|\big)|W|+|\bar F|\big\}|W_y|\,dy\\ &\leqslant \big(\varepsilon+\kappa^{\frac12}+\eta\big)\int|W_y|^2\,dy+c_\eta\kappa^2\int\big|W(\cdot,\tau)\big|^2\,dy+c\kappa^{\frac12}\int|R_1|^2\,dy\\ &\leqslant \big(\varepsilon+\kappa^{\frac12}+\eta\big)\int\big(\phi_y^2+\psi_y^2+\zeta_y^2\big)\,dy+c_\eta\kappa^{\frac72}, \end{aligned}$$

provided that $\|W(\cdot,\tau)\|_{L^\infty}$ is bounded, where η is a small constant to be determined later.

$$\begin{aligned} I_2 &= \int\big\langle\partial_y\big\{\big(A^0(\bar U)-A^0(U)\big)\bar U_\tau+\big(B(U)-B(\bar U)\big)\bar U_{yy}+\big(A(\bar U)-A(U)\big)\bar U_y\big\},\partial_y W\big\rangle\,dy\\ &\quad+\int\big\langle\partial_y\big(g(U,U_y)-g(\bar U,\bar U_y)\big),\partial_y W\big\rangle\,dy-\int\langle\partial_y\bar F,\partial_y W\rangle\,dy\\ &\equiv\sum_{j=1}^{3}I_{2j}. \end{aligned}$$

By the definition of $\bar U$ and the estimates (2.5) and Lemma 3.1 again, we get

$$\begin{aligned} I_{21} &\leqslant c\int\big(|\bar U_\tau|+|\bar U_{yy}|+|\bar U_y|\big)\big(|W_y|+|\bar U_y||W|\big)|W_y|\,dy+c\int\big(|\bar U_{\tau y}|+\big|\partial_y^3\bar U\big|+\big|\partial_y^2\bar U\big|\big)|W||W_y|\,dy\\ &\leqslant\big(\kappa^{\frac12}+\eta\big)\int|W_y|^2\,dy+c_\eta\kappa^2\int\big|W^2\big|\,dy\\ &\leqslant\big(\kappa^{\frac12}+\eta\big)\int\big(\phi_y^2+\psi_y^2+\zeta_y^2\big)\,dy+c_\eta\kappa^{\frac72}. \end{aligned}$$

I_{22} is estimated as follows:

$$\begin{aligned} I_{22} &= \int\frac{\mu}{\kappa}\partial_y\left\{\theta\left(\frac1v\right)_y u_y-\bar\theta\left(\frac1{\bar v}\right)_y\bar u_y\right\}\psi_y+\partial_y\left\{\left(\frac1v\right)_y\theta_y-\left(\frac1{\bar v}\right)_y\bar\theta_y\right\}\zeta_y+\frac{\mu}{\kappa}\partial_y\left\{\frac{u_y^2}{v}-\frac{\bar u_y^2}{\bar v}\right\}\zeta_y\,dy\\ &=-\int\left\{\frac{\mu}{\kappa}\left(\theta\left(\frac1v\right)_y u_y-\bar\theta\left(\frac1{\bar v}\right)_y\bar u_y\right)\psi_{yy}+\left(\left(\frac1v\right)_y\theta_y-\left(\frac1{\bar v}\right)_y\bar\theta_y+\frac{\mu}{\kappa}\left(\frac{u_y^2}{v}-\frac{\bar u_y^2}{\bar v}\right)\right)\zeta_{yy}\right\}dy\\ &\leqslant c\int\big\{|W_y|^2+|\bar U_y||W_y|+|\bar U_y|^2|W|\big\}|\left(\frac{\mu}{\kappa}|\psi_{yy}|+|\zeta_{yy}|\right)dy\\ &\leqslant\big(\varepsilon+\kappa^{\frac12}+\eta\big)\int\left(\phi_y^2+\psi_y^2+\zeta_y^2+\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa^2\int|W|^2\,dy\\ &\leqslant\big(\varepsilon+\kappa^{\frac12}+\eta\big)\int\left(\phi_y^2+\psi_y^2+\zeta_y^2+\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa^{\frac72}. \end{aligned}$$

Finally, Young's inequality and the estimates in (2.5) yield that

$$I_{23}\leqslant c\kappa\int|\partial_y W|^2\,dy+c\kappa^{-1}\int|\partial_y R_1|^2\,dy\leqslant c\kappa\int\big(\phi_y^2+\psi_y^2+\zeta_y^2\big)\,dy+c\kappa^{\frac72}.$$

Consequently,

$$I_2=\sum_{j=1}^{3}I_{2j}\leqslant\big(\varepsilon+\kappa^{\frac12}+\eta\big)\int\left(\phi_y^2+\psi_y^2+\zeta_y^2+\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa^{\frac72}.$$

And then

$$I=I_1+I_2\leqslant\big(\varepsilon+\kappa^{\frac12}+\eta\big)\int\left(\phi_y^2+\psi_y^2+\zeta_y^2+\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa^{\frac72}.$$

We continue to estimate the terms *II* and *III*.

$$
\begin{aligned}
II &= \int \langle A^0(U)\partial_y\big(A^0(U)^{-1}B(U)\big)W_{yy}, W_y\rangle\,dy \\
&= \int \left(\frac{\mu}{\kappa}\theta\partial_y\left(\frac{1}{v}\right)\psi_y\psi_{yy} + \partial_y\left(\frac{1}{v}\right)\zeta_y\zeta_{yy}\right)dy \\
&\leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}}\big)\int\left(\psi_y^2 + \zeta_y^2 + \frac{\mu}{\kappa}\psi_{yy}^2 + \zeta_{yy}^2\right)dy.
\end{aligned}
$$

Differentiating directly shows that

$$
\begin{aligned}
III &= \int \langle A^0(U)\partial_y\big(A^0(U)^{-1}\big)\big(\tilde{A}(U) - A(U)\big)W_y, W_y\rangle\,dy \\
&\quad + \int \langle \partial_y\big\{\big(\tilde{A}(U) - A(U)\big)W_y\big\} - A^0(U)\partial_y\big\{\big(A^0(U)^{-1}\big)\tilde{A}(U)\big\}W_y, W_y\rangle\,dy \\
&\equiv III_1 + III_2.
\end{aligned}
$$

First, Sobolev's inequality gives

$$
III_1 \leqslant c\int \big(|W_y| + |\bar{U}_y|\big)|W||W_y|^2\,dy \leqslant c\varepsilon\int |W_y|^2,
$$

provided that $\|W_y(\cdot,\tau)\|_{L^\infty}$ is bounded. Using integration by parts, we have

$$
\begin{aligned}
III_2 &= \int \big\{\psi_y\partial_y\big((\bar{p} - p)\zeta_y\big) + \zeta_y\partial_y\big((\bar{p} - p)\psi_y\big)\big\}\,dy - \int \left\{\theta\psi_y\left(\phi_y\partial_y p_v + \zeta_y\partial_y\left(\frac{\bar{p}}{\theta}\right)\right) + \partial_y\bar{p}\psi_y\zeta_y\right\}dy \\
&= \int \psi_y\partial_y(\bar{p} - p)\zeta_y\,dy - \int \left\{\theta\psi_y\left(\phi_y\partial_y p_v + \zeta_y\partial_y\left(\frac{\bar{p}}{\theta}\right)\right) + \partial_y\bar{p}\psi_y\zeta_y\right\}dy \\
&\leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}}\big)\int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\,dy.
\end{aligned}
$$

Hence it follows that

$$
III \leqslant c\big(\varepsilon + \kappa^{\frac{1}{2}}\big)\int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\,dy.
$$

And then

$$
\int \langle \tilde{H}, \partial_y W\rangle\,dy = I + II + III \leqslant \big(\varepsilon + \kappa^{\frac{1}{2}} + \eta\big)\int\left(\phi_y^2 + \psi_y^2 + \zeta_y^2 + \frac{\mu}{\kappa}\psi_{yy}^2 + \zeta_{yy}^2\right)dy + c_\eta\kappa^{\frac{7}{2}}.
$$

Collecting all the estimates we have obtained so far, after choosing ε and κ to be sufficiently small, we get

$$
\frac{d}{d\tau}\int \langle A^0(U)\partial_y W, \partial_y W\rangle\,dy + \int\left(\frac{\mu}{\kappa}\frac{\theta}{v}\psi_{yy}^2 + \frac{1}{v}\zeta_{yy}^2\right)dy \leqslant \big(\varepsilon + \kappa^{\frac{1}{2}} + \eta\big)\int \big(\phi_y^2 + \psi_y^2 + \zeta_y^2\big)\,dy + c_\eta\kappa^{\frac{7}{2}}.
$$

Integrating this inequality with respect to τ and using Lemma 3.1, we arrive that

$$
\begin{aligned}
&\big\|(\phi_y, \psi_y, \zeta_y)(\cdot,\tau)\big\|^2 + \int_{\tau_0}^{\tau}\left(\frac{\mu}{\kappa}\big\|\psi_{yy}(\cdot,\tau)\big\|^2 + \big\|\zeta_{yy}(\cdot,\tau)\big\|^2\right)d\tau \\
&\quad \leqslant \big(\varepsilon + \kappa^{\frac{1}{2}} + \eta\big)\int_{\tau_0}^{\tau}\big\|(\phi_y, \psi_y)(\cdot,\tau)\big\|^2\,d\tau + c_\eta\kappa^{\frac{5}{2}}.
\end{aligned} \tag{4.7}
$$

Step 2. In this step, we will estimate $\int_{\tau_0}^{\tau}\|(\phi_y, \psi_y)(\cdot,\tau)\|^2\,d\tau$. First, linearizing (4.2) at $\bar{U}$, and then subtracting (4.3) from the resulting system, one gets that

$$
A^0(\bar{U})W_\tau + A(\bar{U})W_y = B(\bar{U})W_{yy} + H, \tag{4.8}
$$

where

$$
\begin{aligned}
H = A^0(\bar{U})\big\{&A^0(U)^{-1}g(U, U_y) - A^0(\bar{U})^{-1}g(\bar{U}, \bar{U}_y) - \big(A^0(U)^{-1}A(U) - A^0(\bar{U})^{-1}A(\bar{U})\big)W_y \\
&- \big(A^0(U)^{-1}A(U) - A^0(\bar{U})^{-1}A(\bar{U})\big)\bar{U}_y + \big(A^0(U)^{-1}B(U) - A^0(\bar{U})^{-1}B(\bar{U})\big)W_{yy} \\
&+ \big(A^0(U)^{-1}B(U) - A^0(\bar{U})^{-1}B(\bar{U})\big)\bar{U}_{yy}\big\} - \bar{F}.
\end{aligned} \tag{4.9}
$$

We take $S(U)=\begin{pmatrix} 0 & -p & 0 \\ v & 0 & 2\theta \\ 0 & -\frac{2R}{\gamma-1} & 0 \end{pmatrix}$ and multiply (4.8) by $W_y^t S(\bar U)$, and then integrate with respect to y on R to obtain

$$\int \langle S(\bar U)A^0(\bar U)W_\tau, W_y\rangle dy + \int \langle S(\bar U)A(\bar U)W_y, W_y\rangle dy$$
$$= \int \langle S(\bar U)B(\bar U)W_{yy}, W_y\rangle dy + \int \langle S(\bar U)H, W_y\rangle dy. \tag{4.10}$$

Since $SA^0=\begin{pmatrix} 0 & -\theta p & 0 \\ \theta p & 0 & \frac{2R\theta}{\gamma-1} \\ 0 & -\frac{2R\theta}{\gamma-1} & 0 \end{pmatrix}$ is skew-symmetric, the first term on the left of (4.10) can be written as

$$\begin{aligned}
\int \langle S(\bar U)A^0(\bar U)W_\tau, W_y\rangle dy &= \frac{1}{2}\biggl\{\frac{d}{d\tau}\int \langle S(\bar U)A^0(\bar U)W, W_y\rangle dy - \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_\tau W, W_y\bigr\rangle dy \\
&\quad + \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, W_\tau\bigr\rangle dy\biggr\} \\
&= \frac{1}{2}\biggl\{\frac{d}{d\tau}\int \langle S(\bar U)A^0(\bar U)W, W_y\rangle dy - \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_\tau W, W_y\bigr\rangle dy \\
&\quad - \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}A(U)W_y\bigr\rangle dy \\
&\quad + \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}B(U)W_{yy}\bigr\rangle dy \\
&\quad + \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}\tilde g(U,U_y)\bigr\rangle dy\biggr\},
\end{aligned}$$

where the system (4.5) has been used. Substitute this into (4.10) to get

$$\begin{aligned}
\int \langle S(\bar U)A(\bar U)W_y, W_y\rangle dy &= -\frac{1}{2}\frac{d}{d\tau}\int \langle S(\bar U)A^0(\bar U)W, W_y\rangle dy + \frac{1}{2}\biggl\{\int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_\tau W, W_y\bigr\rangle dy \\
&\quad + \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}A(U)W_y\bigr\rangle dy\biggr\} \\
&\quad - \frac{1}{2}\int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}B(U)W_{yy}\bigr\rangle dy \\
&\quad - \frac{1}{2}\int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}\tilde g(U,U_y)\bigr\rangle dy \\
&\quad + \int \langle S(\bar U)B(\bar U)W_{yy}, W_y\rangle dy + \int \langle S(\bar U)H, W_y\rangle dy.
\end{aligned} \tag{4.11}$$

Next we estimate all of the terms above separately. First,

$$\begin{aligned}
\int \langle S(\bar U)A(\bar U)W_y, W_y\rangle dy &= \int \left(\frac{\bar p^3}{R}\phi_y^2 + \bar\theta\bar p\psi_y^2 + \frac{3-\gamma}{\gamma-1}\bar p^2\phi_y\zeta_y - \frac{2R}{\gamma-1}\bar p\zeta_y^2\right) dy \\
&\geqslant \int \left(\frac{\bar p^3}{2R}\phi_y^2 + \bar\theta\bar p\psi_y^2\right) dy - c\int \zeta_y^2\, dy.
\end{aligned}$$

Using Young's inequality and Lemma 3.1, we have

$$\begin{aligned}
&\int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_\tau W, W_y\bigr\rangle dy + \int \bigl\langle \bigl(S(\bar U)A^0(\bar U)\bigr)_y W, A^0(U)^{-1}A(U)W_y\bigr\rangle dy \\
&\quad \leqslant c\int \bigl(|\bar U_\tau| + |\bar U_y|\bigr)|W||W_y|\, dy \\
&\quad \leqslant \eta\int |W_y|^2\, dy + c_\eta\kappa\int |W|^2\, dy \\
&\quad \leqslant \eta\int |W_y|^2\, dy + c_\eta\kappa^{\frac{5}{2}},
\end{aligned}$$

where $\eta > 0$ is a constant to be determined later. Due to the form of B, direct calculations and Young's inequality lead to

$$
\begin{aligned}
&-\frac{1}{2}\int\big\langle\big(S(\bar{U})A^0(\bar{U})\big)_y W, A^0(U)^{-1}B(U)W_{yy}\big\rangle dy+\int\big\langle S(\bar{U})B(\bar{U})W_{yy}, W_y\big\rangle dy\\
&\quad\leqslant c\int|\bar{U}_y||W|\left(\frac{\mu}{\kappa}|\psi_{yy}|+|\zeta_{yy}|\right)dy+c\int|W_y|\left(\frac{\mu}{\kappa}|\psi_{yy}|+|\zeta_{yy}|\right)dy\\
&\quad\leqslant \eta\int|W_y|^2\,dy+c_\eta\int\left(\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c\kappa\int|W|^2\,dy\\
&\quad\leqslant \eta\int|W_y|^2\,dy+c_\eta\int\left(\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c\kappa^{\frac{5}{2}}.
\end{aligned}
$$

It follows from the definition of $\tilde{g}$ and H that

$$
\begin{aligned}
&-\int\big\langle\big(S(\bar{U})A^0(\bar{U})\big)_y W, A^0(U)^{-1}\tilde{g}(U,U_y)\big\rangle dy\\
&\quad\leqslant c\int|\bar{U}_y||W|\big\{\big(|\bar{U}_\tau|+|\bar{U}_{yy}|+|\bar{U}_y|\big)|W|+|W_y|^2+|\bar{U}_y||W_y|+|\bar{U}_y|^2|W|+R_1\big\}\,dy\\
&\quad\leqslant c(\varepsilon+\kappa)\int|W_y|^2\,dy+c\kappa\int|W|^2\,dy+c\int|R_1|^2\,dy\\
&\quad\leqslant c(\varepsilon+\kappa)\int|W_y|^2\,dy+c\kappa^{\frac{5}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\int\big\langle S(\bar{U})H, W_y\big\rangle dy\leqslant c\int\bigg\{&\big(|W_y|^2+|\bar{U}_y||W_y|+|\bar{U}_y|^2|W|\big)+|W||W_y|+|\bar{U}_y||W|\\
&+|W|\left(\frac{\mu}{\kappa}|\psi_{yy}|+|\zeta_{yy}|\right)+|\bar{U}_{yy}||W|\bigg\}|W_y|\,dy\\
\leqslant{}& c\big(\varepsilon+\kappa^{\frac{1}{2}}+\eta\big)\int|W_y|^2\,dy+c\varepsilon\int\left(\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa\int|W|^2\,dy\\
\leqslant{}& c\big(\varepsilon+\kappa^{\frac{1}{2}}+\eta\big)\int|W_y|^2\,dy+c\varepsilon\int\left(\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c_\eta\kappa^{\frac{5}{2}}.
\end{aligned}
$$

Collecting all the estimates we have obtained, by choosing ε,κ and η to be sufficiently small, we get

$$
\int\left(\frac{\bar{p}^3}{R}\phi_y^2+\bar{\theta}\bar{p}\psi_y^2\right)dy\leqslant-\frac{d}{d\tau}\int\big\langle S(\bar{U})A^0(\bar{U})W, W_y\big\rangle dy+c\int\left(\zeta_y^2+\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy+c\kappa^{\frac{5}{2}}.
$$

Integrating this inequality with respect to τ and using Cauchy–Schwatz inequality and Lemma 3.1, we may conclude that

$$
\int_{\tau_0}^{\tau}\int\big(\phi_y^2+\psi_y^2\big)\,dy\,d\tau\leqslant c\int\big(\phi_y^2+\psi_y^2+\zeta_y^2\big)\,dy+c\int_{\tau_0}^{\tau}\int\left(\frac{\mu}{\kappa}\psi_{yy}^2+\zeta_{yy}^2\right)dy\,d\tau+c\kappa^{\frac{3}{2}}. \tag{4.12}
$$

Inserting (4.12) into (4.7) and then taking ε,κ and η to be sufficiently small, we can obtain the estimate (4.1), which completes the proof of Lemma 4.1. For the second order derivatives, one has the following estimate. □

Lemma 4.2. *Suppose that the conditions in Lemma* 3.1 *are satisfied. Then*

$$
\big\|\big(\partial_y^2\phi,\partial_y^2\psi,\partial_y^2\zeta\big)(\cdot,\tau)\big\|^2+\int_{\tau_0}^{\tau}\big(\big\|\big(\partial_y^2\phi,\partial_y^2\psi\big)(\cdot,\tau)\big\|^2+\big\|\partial_y^3\zeta(\cdot,\tau)\big\|^2\big)\,d\tau\leqslant c\kappa^{\frac{3}{2}}, \tag{4.13}
$$

for all $\tau\in[\tau_0,\tau_2]$, where the constant c is independent of τ_2 and κ.

The proof is similar to the proof of Lemma 4.1. Hence we omit it.

5. Proof of Theorem 1.1

Combining the results of Lemma 3.1 and Lemmas 4.1–4.2 leads to

Proposition 5.1. *There exist positive constants κ_0 and C, which are independent of κ such that if $0 < \kappa < \kappa_0$, then for any $T > 0$, the Cauchy problem (2.4) has a unique solution $(\phi, \psi, \zeta) \in C^1([\frac{1}{\kappa}, \frac{1+T}{\kappa}] : H^2(R^1))$. Furthermore, the following inequality holds*

$$\sup_{\frac{1}{\kappa} \leqslant \tau \leqslant \frac{1+T}{\kappa}} \left\|(\phi, \psi, \zeta)(\cdot, \tau)\right\|_2^2 + \int_{\frac{1}{\kappa}}^{\frac{1+T}{\kappa}} \left(\left\|(\phi_y, \psi_y)(\cdot, \tau)\right\|_1^2 + \left\|\zeta_y(\cdot, \tau)\right\|_2^2\right) d\tau \leqslant C\kappa^{\frac{3}{2}}. \tag{5.1}$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For any $T > 0$, in view of (5.1), we have

$$\left\|(v - \bar{v}, u - \bar{u}, \theta - \bar{\theta})(\cdot, t)\right\|_{L^\infty} \leqslant C\left\|(\phi, \psi, \zeta)(\cdot, t)\right\|^{\frac{1}{2}} \left\|(\phi_y, \psi_y, \zeta_y)(\cdot, t)\right\|^{\frac{1}{2}} \leqslant C\kappa^{\frac{3}{4}}, \quad \forall t \in [0, T].$$

This, together with (1.12), yields (1.17).

Hence we have completed the proof of the Theorem 1.1. □

Acknowledgments

This research was partially done when I was visiting the Institute of Mathematical Sciences of The Chinese University of Hong Kong. The financial support from the IMS and the help from the staffs are appreciated greatly. I also thank Dr. Wang Yi for the fruitful discussions. The author is supported by Tianyuan Foundation (No. 11026093) and the National Natural Science Foundation of China (Nos. 11101162, 11071086).

References

[1] Stefano Bianchini, Alberto Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. (2) 161 (1) (2005) 223–342.
[2] Jonathan Goodman, Zhou-Ping Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, Arch. Ration. Mech. Anal. 121 (3) (1992) 235–265.
[3] Olivier Guès, Guy Métivier, Mark Williams, Kevin Zumbrun, Existence and stability of multidimensional shock fronts in the vanishing viscosity limit, Arch. Ration. Mech. Anal. 175 (2) (2005) 151–244.
[4] David Hoff, Tai-Ping Liu, The inviscid limit for the Navier–Stokes equations of compressible, isentropic flow with shock data, Indiana Univ. Math. J. 38 (4) (1989) 861–915.
[5] Ling Hsiao, Tai-Ping Liu, Nonlinear diffusive phenomena of nonlinear hyperbolic systems, Chin. Ann. Math. Ser. B 14 (4) (1993) 465–480, a Chinese summary appears in Chinese Ann. Math. Ser. A 14 (6) (1993) 740.
[6] Feimin Huang, Akitaka Matsumura, Zhouping Xin, Stability of contact discontinuities for the 1-D compressible Navier–Stokes equations, Arch. Ration. Mech. Anal. 179 (1) (2006) 55–77.
[7] Feimin Huang, Yi Wang, Tong Yang, Fluid dynamic limit to the Riemann solutions of Euler equations: I. Superposition of rarefaction waves and contact discontinuity, Kinet. Relat. Models 3 (4) (2010) 685–728.
[8] Song Jiang, Ni. Guoxi, Wenjun Sun, Vanishing viscosity limit to rarefaction waves for the Navier–Stokes equations of one-dimensional compressible heat-conducting fluids, SIAM J. Math. Anal. 38 (2) (2006) 368–384 (electronic).
[9] Shuichi Kawashima, Systems of hyperbolic–parabolic composite type, with applications to the equations of magneto-hydrodynamics, Doctoral thesis, Kyoto Univ., 1983.
[10] Ma Shixiang, Zero dissipation limit to strong contact discontinuity for the 1-D compressible Navier–Stokes equations, J. Differential Equations 248 (1) (2010) 95–110.
[11] Ma Shixiang, The inviscid limit for an inflow problem of compressible viscous gas in presence of both shocks and boundary layers, J. Math. Anal. Appl. 378 (2011) 268–288.
[12] Corrado Mascia, Kevin Zumbrun, Stability of large-amplitude viscous shock profiles of hyperbolic–parabolic systems, Arch. Ration. Mech. Anal. 172 (1) (2004) 93–131.
[13] Corrado Mascia, Kevin Zumbrun, Stability of large-amplitude shock profiles of general relaxation systems, SIAM J. Math. Anal. 37 (3) (2005) 889–913 (electronic).
[14] Anders Szepessy, Kevin Zumbrun, Stability of rarefaction waves in viscous media, Arch. Ration. Mech. Anal. 133 (3) (1996) 249–298.
[15] C.J. van Duyn, L.A. Peletier, A class of similarity solutions of the nonlinear diffusion equation, Nonlinear Anal. 1 (3) (1976/1977) 223–233.
[16] Wang Huiying, Viscous limits for piecewise smooth solutions of the p-system, J. Math. Anal. Appl. 299 (2) (2004) 411–432.
[17] Yi Wang, Zero dissipation limit of the compressible heat-conducting Navier–Stokes equations in the presence of the shock, Acta Math. Sci. Ser. B Engl. Ed. 28 (4) (2008) 727–748.
[18] Zhou-Ping Xin, Zero dissipation limit to rarefaction waves for the one-dimensional Navier–Stokes equations of compressible isentropic gases, Comm. Pure Appl. Math. 46 (5) (1993) 621–665.
[19] Yu Shih-Hsien, Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws, Arch. Ration. Mech. Anal. 146 (4) (1999) 275–370.
[20] Huihui Zeng, Asymptotic behavior of solutions to fluid dynamical equations, Doctoral thesis, The Chinese University of Hong Kong.
[21] Kevin Zumbrun, Peter Howard, Pointwise semigroup methods and stability of viscous shock waves, Indiana Univ. Math. J. 47 (3) (1998) 741–871.