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One-parameter Lie groups and inverse integrating factors of n-th order autonomous systems ☆

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ABSTRACT

The method of obtaining inverse integrating factors of n-th order autonomous systems using one-parameter Lie groups admitted by the systems is given. By describing n-th order autonomous systems with differential form systems, the properties of inverse integrating factors of the n-th order autonomous systems are presented.

1. Introduction

We consider n-th order autonomous systems of differential equations of the following form

$$\frac{dx_i}{dt} = P_i(x), \quad i = 1, 2, \ldots, n, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n) \in D \subset \mathbf{R}^n$ (or $\mathbf{C}^n$), $P_i : D \to \mathbf{R}$ (or $\mathbf{C}$), $P_i \in C_\infty(D)$ and $t \in \mathbf{R}$ (or $\mathbf{C}$). Associated to system (1) there is the differential operator

$$L_x = \sum_{i=1}^{n} P_i(x) \frac{\partial}{\partial x_i}. \tag{2}$$

Sometimes, the vector field of system (1) will be denoted simply by $P = (P_1, P_2, \ldots, P_n)$. The divergence of the vector field P is defined by

$$div(P) = div(P_1, P_2, \ldots, P_n) = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}.$$

Differential equations are mainly used to describe the change of quantities or the behavior of certain systems in the time, such as those systems governed by Newton's laws in physics. Usually, explicit solutions of the differential equations cannot be found, so we must look for other methods to study properties of the systems. Some of the most interesting questions

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are related to the so-called qualitative properties and the integrability [1–3]. Among them, the inverse integrating factors play some important roles. A real (complex) C^1 function $\mu(x_1, x_2, \ldots, x_n) \in D$, that satisfies the equation

$$P_i \sum_{i=1}^{n} \frac{\partial \mu}{\partial x_i} = \mu \left(\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} \right)$$

or

$$L_x \mu = \mu \, div \, P,$$

is called an inverse integrating factor of system (1). Obviously, if $\mu(x_1, x_2, \ldots, x_n)$ is an inverse integrating factor of system (1), then $\frac{1}{\mu}$ $(\mu \neq 0)$ is an integrating factor of system (1), and

$$L_x \frac{1}{\mu} = -\frac{1}{\mu} \, div \, P.$$

Inverse integrating factors of a differential equations system can reveal some qualitative properties of the system [4,5]. Some elegant results on integrating factors for polynomial differential systems are presented in [6,7], and the relation between Darboux first integrals of the system and the inverse integrating factors is showed. In [8], the integrability of planar quasihomogeneous systems is considered, and a method for calculating the inverse integrating factors of such a system is proposed. In [9,10], we show methods to get inverse integrating factors of some system by Lie group or invariant manifolds of the system. As we know, searching for first integrals and integrating factors (inverse integrating factors) of a system plays a very important role for integrating system. By using the Jacobi Theorem [11], an n-th order autonomous system can be integrated if one can get an integrating factor under knowing $n-2$ first integrals. For a second order autonomous system, we can get an integrating factor if we know a one-parameter Lie group admitted by the system [2]. For n-th order autonomous systems, in order to obtain integrating factors, a multiple-parameter Lie group admitted by the systems is required in the traditional theory of Lie group [2]. But, it is difficult to obtain a multiple-parameter Lie group admitted by the systems. In [12], the spacial structure spanned by generators of one-parameter Lie groups admitted by a higher order autonomous system is studied. But, how to obtain integrating factors concretely when one has known more one-parameter Lie groups admitted by the system is still an issue needing to study. We have considered to get an inverse integrating factor by generators of one-parameter Lie groups admitted by 3-rd order autonomous systems in [13]. In this paper, we still consider the issue concretely for n-order autonomous systems, and give the approach to obtaining inverse integrating factors by using the generators of one-parameter Lie groups admitted by the systems.

2. Obtaining an inverse integrating factor using one-parameter Lie groups

Now, we consider system (1), and let

$$V_i = \sum_{j=1}^{n} \xi_j^i(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_j} \quad (i = 1, 2, \ldots, n-1)$$

be the generators of $n-1$ independent one-parameter Lie groups G_i $(i = 1, 2, \ldots, n-1)$ admitted by system (1). System (1) admits $n-1$ independent one-parameter Lie groups G_i $(i = 1, 2, \ldots, n-1)$, that is, the transformations of G_i leave system (1) invariant.

Lemma 1. *(See [12].) If there exist functions $A_i(x_1, x_2, \ldots, x_n) \in C_\infty$ $(i = 1, 2, \ldots, n-1)$ satisfying*

$$[V_i, L_x] = A_i L_x \quad (i = 1, 2, \ldots, n-1),$$

then system (1) admits the Lie groups G_i $(i = 1, 2, \ldots, n-1)$ generated by V_i $(i = 1, 2, \ldots, n-1)$.

We can construct a function $\mu(x_1, x_2, \ldots, x_n)$ of the following determinant form,

$$\mu(x_1, x_2, \ldots, x_n) = \begin{vmatrix} P_1 & P_2 & \ldots & P_n \\ \xi_1^1 & \xi_2^1 & \ldots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \ldots & \xi_n^{n-1} \end{vmatrix}.$$

Since V_i $(i = 1, 2, \ldots, n-1)$ are independent, $\mu(x_1, x_2, \ldots, x_n) \neq 0$ holds.

Theorem 1. *$\mu(x_1, x_2, \ldots, x_n)$ is an inverse integrating factor of system (1).*

Proof.

$$L_x\mu = \sum_{i=1}^{n} P_i \frac{\partial \mu}{\partial x_i} = P_1 \begin{vmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_n}{\partial x_1} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_1 \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \xi_1^{n-1}}{\partial x_1} & \frac{\partial \xi_2^{n-1}}{\partial x_1} & \cdots & \frac{\partial \xi_n^{n-1}}{\partial x_1} \end{vmatrix}$$

$$+ P_2 \begin{vmatrix} \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_n}{\partial x_2} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_2 \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \xi_1^{n-1}}{\partial x_2} & \frac{\partial \xi_2^{n-1}}{\partial x_2} & \cdots & \frac{\partial \xi_n^{n-1}}{\partial x_2} \end{vmatrix} + \cdots$$

$$+ P_n \begin{vmatrix} \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \cdots & \frac{\partial P_n}{\partial x_n} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_n \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \xi_1^{n-1}}{\partial x_n} & \frac{\partial \xi_2^{n-1}}{\partial x_n} & \cdots & \frac{\partial \xi_n^{n-1}}{\partial x_n} \end{vmatrix}.$$

From the above determinants, one can get the following formulas,

$$L_x\mu = \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \sum_{i=1}^n P_i \frac{\partial \xi_1^1}{\partial x_i} & \sum_{i=1}^n P_i \frac{\partial \xi_2^1}{\partial x_i} & \cdots & \sum_{i=1}^n P_i \frac{\partial \xi_n^1}{\partial x_i} \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \sum_{i=1}^n P_i \frac{\partial \xi_1^2}{\partial x_i} & \sum_{i=1}^n P_i \frac{\partial \xi_2^2}{\partial x_i} & \cdots & \sum_{i=1}^n P_i \frac{\partial \xi_n^2}{\partial x_i} \\ \xi_1^3 & \xi_2^3 & \cdots & \xi_n^3 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} P_1 & P_2 & \cdots & P_n \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \cdots & \xi_n^{n-2} \\ \sum_{i=1}^n P_i \frac{\partial \xi_1^{n-1}}{\partial x_i} & \sum_{i=1}^n P_i \frac{\partial \xi_2^{n-1}}{\partial x_i} & \cdots & \sum_{i=1}^n P_i \frac{\partial \xi_n^{n-1}}{\partial x_i} \end{vmatrix} + P_1 \begin{vmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_n}{\partial x_1} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix}$$

$$+ P_2 \begin{vmatrix} \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_n}{\partial x_2} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_n \begin{vmatrix} \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \cdots & \frac{\partial P_n}{\partial x_n} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix}. \tag{3}$$

Because system (1) admits the independent Lie groups G_i $(i = 1, 2, \ldots, n-1)$ generated by V_i $(i = 1, 2, \ldots, n-1)$ respectively, there exist functions $A_i(x_1, x_2, \ldots, x_n)$ to satisfy

$$[P, V_i] = A_i P \quad (i = 1, 2, \ldots, n-1)$$

that is,

$$PV_i = V_i P + A_i P \quad (i = 1, 2, \ldots, n-1).$$

By the property of the differential operator, one can get the following formula from the above equations,

$$\sum_{j=1}^{n} P_j \frac{\partial \xi_k^i}{\partial x_j} \frac{\partial}{\partial x_k} = \sum_{j=1}^{n} \xi_j^i \frac{\partial P_k}{\partial x_j} \frac{\partial}{\partial x_k} + A_i P_k \frac{\partial}{\partial x_k} \quad (i = 1, 2, \ldots, n-1)\ (k = 1, 2, \ldots, n).$$

So

$$\sum_{j=1}^{n} P_j \frac{\partial \xi_k^i}{\partial x_j} = \sum_{j=1}^{n} \xi_j^i \frac{\partial P_k}{\partial x_j} + A_i P_k \quad (i = 1, 2, \ldots, n-1)\ (k = 1, 2, \ldots, n).$$

Substituting this equation to the ahead $n-1$ determinants of (3), the following formula is easy to be obtained,

$$\begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \sum_{j=1}^n \xi_j^1 \frac{\partial P_1}{\partial x_j} + A_1 P_1 & \sum_{j=1}^n \xi_j^1 \frac{\partial P_2}{\partial x_j} + A_1 P_2 & \dots & \sum_{j=1}^n \xi_j^1 \frac{\partial P_n}{\partial x_j} + A_1 P_n \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix}$$

$$+ \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \sum_{j=1}^n \xi_j^2 \frac{\partial P_1}{\partial x_j} + A_2 P_1 & \sum_{j=1}^n \xi_j^2 \frac{\partial P_2}{\partial x_j} + A_2 P_2 & \dots & \sum_{j=1}^n \xi_j^2 \frac{\partial P_n}{\partial x_j} + A_2 P_n \\ \xi_1^3 & \xi_2^3 & \dots & \xi_n^3 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \dots & \xi_n^{n-2} \\ \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_1}{\partial x_j} + A_{n-1} P_1 & \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_2}{\partial x_j} + A_{n-1} P_2 & \dots & \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_n}{\partial x_j} + A_{n-1} P_n \end{vmatrix}. \tag{4}$$

By the properties of the determinant, (4) can be changed to the following expression,

$$\begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \sum_{j=1}^n \xi_j^1 \frac{\partial P_1}{\partial x_j} & \sum_{j=1}^n \xi_j^1 \frac{\partial P_2}{\partial x_j} & \dots & \sum_{j=1}^n \xi_j^1 \frac{\partial P_n}{\partial x_j} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \sum_{j=1}^n \xi_j^2 \frac{\partial P_1}{\partial x_j} & \sum_{j=1}^n \xi_j^2 \frac{\partial P_2}{\partial x_j} & \dots & \sum_{j=1}^n \xi_j^2 \frac{\partial P_n}{\partial x_j} \\ \xi_1^3 & \xi_2^3 & \dots & \xi_n^3 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \dots & \xi_n^{n-2} \\ \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_1}{\partial x_j} & \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_2}{\partial x_j} & \dots & \sum_{j=1}^n \xi_j^{n-1} \frac{\partial P_n}{\partial x_j} \end{vmatrix}$$

$$= \xi_1^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \dots & \frac{\partial P_n}{\partial x_1} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \xi_2^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \dots & \frac{\partial P_n}{\partial x_2} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ \xi_n^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \dots & \frac{\partial P_n}{\partial x_n} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots + \xi_n^{n-1} \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \dots & \xi_n^{n-2} \\ \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \dots & \frac{\partial P_n}{\partial x_n} \end{vmatrix}.$$

Substituting the above formula to the ahead $n-1$ determinants of (3), we have

$$L_x\mu = \xi_1^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \dots & \frac{\partial P_n}{\partial x_1} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \xi_2^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \dots & \frac{\partial P_n}{\partial x_2} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ \xi_n^1 \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \dots & \frac{\partial P_n}{\partial x_n} \\ \xi_1^2 & \xi_2^2 & \dots & \xi_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots + \xi_n^{n-1} \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-2} & \xi_2^{n-2} & \dots & \xi_n^{n-2} \\ \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \dots & \frac{\partial P_n}{\partial x_n} \end{vmatrix}$$

$$+ P_1 \begin{vmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \dots & \frac{\partial P_n}{\partial x_1} \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + P_2 \begin{vmatrix} \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \dots & \frac{\partial P_n}{\partial x_2} \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots$$

$$+ P_n \begin{vmatrix} \frac{\partial P_1}{\partial x_n} & \frac{\partial P_2}{\partial x_n} & \dots & \frac{\partial P_n}{\partial x_n} \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix}. \tag{5}$$

Simplifying the first determinant of every classes of the above determinants, we can get the following formula,

$$\begin{aligned} L_x\mu &= \frac{\partial P_1}{\partial x_1} \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \frac{\partial P_2}{\partial x_2} \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} + \cdots + \frac{\partial P_n}{\partial x_n} \begin{vmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix} \\ &= \mu\, div(P), \end{aligned} \tag{6}$$

that is

$$L_x\mu = \mu\, div(P).$$

So $\mu(x_1, x_2, \dots, x_n)$ is an inverse integrating factor of system (1). □

Actually, Theorem 1 provides a method of obtaining an inverse integrating factor of system (1). But we need previously know $n-1$ independent one-parameter Lie groups admitted by system (1).

Example 1. Consider the following autonomous system,

$$\begin{cases} \dfrac{dx}{dt} = y^2 + z^2, \\ \dfrac{dy}{dt} = -xy - iz\sqrt{x^2 + y^2 + z^2}, \\ \dfrac{dz}{dt} = -xz + iy\sqrt{x^2 + y^2 + z^2}, \\ \dfrac{du}{dt} = 0. \end{cases} \tag{7}$$

Its corresponding operator is

$$L_x = \left(y^2 + z^2\right)\frac{\partial}{\partial x} + \left(-xy - iz\sqrt{x^2 + y^2 + z^2}\right)\frac{\partial}{\partial y} + \left(-xz + iy\sqrt{x^2 + y^2 + z^2}\right)\frac{\partial}{\partial z}.$$

It is easy to check that this system admits Lie groups with the following generators,

$$V_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$
$$V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$
$$V_3 = \frac{\partial}{\partial u}.$$

Based on Theorem 1,

$$\begin{vmatrix} y^2+z^2 & -xy - iz\sqrt{x^2+y^2+z^2} & -xz + iy\sqrt{x^2+y^2+z^2} & 0 \\ 0 & z & -y & 0 \\ x & y & z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \left(y^2+z^2\right)\left(x^2+y^2+z^2\right)$$

is an inverse integrating factor of system (7).

3. The properties of inverse integrating factors

From system (1), without loss of generality, assume $P_1(x_1, x_2, \ldots, x_n) \neq 0$, then system (1) can be rewritten as

$$\frac{dx_i}{dx_1} = \frac{P_i(x_1, x_2, \ldots, x_n)}{P_1(x_1, x_2, \ldots, x_n)}, \quad i = 2, \ldots, n. \tag{8}$$

The corresponding differential operator is

$$\tilde{L}_x = \frac{\partial}{\partial x_1} + \frac{P_2}{P_1}\frac{\partial}{\partial x_2} + \cdots + \frac{P_n}{P_1}\frac{\partial}{\partial x_n}.$$

Theorem 2. *System* (8) *admits the Lie groups G_i $(i = 1, 2, \ldots, n-1)$ generated by V_i $(i = 1, 2, \ldots, n-1)$ if and only if the Lie groups G_i $(i = 1, 2, \ldots, n-1)$ are admitted by system* (1).

Proof. Let V_i $(i = 1, 2, \ldots, n-1)$ be generators of Lie groups admitted by system (1). By using Lemma 1, we get

$$[V_i, L_x] = A_i L_x, \quad i = 1, 2, \ldots, n-1.$$

Because

$$\begin{aligned} [V_i, \tilde{L}_x] &= V_i\left(\frac{1}{P_1}L_x\right) - \left(\frac{1}{P_1}L_x\right)V_i = \left(V_i\frac{1}{P_1}\right)L_x + \frac{1}{P_1}V_i L_x - \frac{1}{P_1}L_x V_i \\ &= -\frac{1}{P_1^2}(V_i P_1)L_x + \frac{1}{P_1}A_i L_x \\ &= \left(-\frac{1}{P_1}V_i P_1 + A_i\right)\left(\frac{1}{P_1}L_x\right) \\ &= \left(-\frac{1}{P_1}V_i P_1 + A_i\right)\tilde{L}_x, \end{aligned}$$

system (8) admits Lie group G_i $(i = 1, 2, \ldots, n-1)$ generators by V_i $(i = 1, 2, \ldots, n-1)$ from Lemma 1.

Let V_i $(i = 1, 2, \ldots, n-1)$ be generators of Lie groups admitted by system (8). Similarly, one have

$$[V_i, \tilde{L}_x] = \tilde{A}_i \tilde{L}_x, \quad i = 1, 2, \ldots, n-1.$$

Therefore,

$$\begin{aligned} [V_i, L_x] &= V_i(P_1\tilde{L}_x) - P_1\tilde{L}_x V_i = P_1 V_i \tilde{L}_x + \tilde{L}_x V_i P_1 - P_1 \tilde{L}_x V_i \\ &= P_1 \tilde{A}_i \tilde{L}_x + \tilde{L}_x V_i P_1 \\ &= \tilde{A}_i P_1 \tilde{L}_x + \frac{1}{P_1}(V_i P_1) P_1 \tilde{L}_x \\ &= \left(\tilde{A}_i + \frac{1}{P_1}V_i P_1\right)L_x. \end{aligned}$$

So system (1) admits Lie group G_i $(i = 1, 2, \ldots, n-1)$ generators by V_i $(i = 1, 2, \ldots, n-1)$ from Lemma 1. □

We construct the following determinant,

$$\mu^*(x_1, x_2, \dots, x_n) = \begin{vmatrix} 1 & \frac{P_2}{P_1} & \dots & \frac{P_n}{P_1} \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix}.$$

Thus, we can obtain easily the following result.

Lemma 2. $P_1 \mu^* = \mu$.

Theorem 3. *μ^* is an inverse integrating factor of system* (8).

Proof. It is easy to finish the proof by using Lemma 2. $\square$

System (8) is equivalent to a system of differential forms

$$\begin{cases} P_1\,dx_2 - P_2\,dx_1 = 0, \\ P_1\,dx_3 - P_3\,dx_1 = 0, \\ \vdots \\ P_1\,dx_n - P_n\,dx_1 = 0. \end{cases} \tag{9}$$

For system (9), if there exists a matrix function M, satisfying $|M| \neq 0$ for all $x \in D$,

$$M = \begin{bmatrix} M_{(1)}^{(1)}(x) & M_{(2)}^{(1)}(x) & \dots & M_{(n-1)}^{(1)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ M_{(1)}^{(n-1)}(x) & M_{(2)}^{(n-1)}(x) & \dots & M_{(n-1)}^{(n-1)}(x) \end{bmatrix},$$

such that

$$M \begin{bmatrix} P_1\,dx_2 - P_2\,dx_1 \\ \vdots \\ P_1\,dx_n - P_n\,dx_1 \end{bmatrix} = \begin{bmatrix} d\phi_1 \\ \vdots \\ d\phi_{n-1} \end{bmatrix},$$

where $\phi_1, \phi_2, \dots, \phi_{n-1}$ are $n-1$ differentiable functions, then system (9) obviously is integrable.

In [14], there exists the following result about the inverse matrix M^{-1} of matrix M and the generators V_i ($i = 1, 2, \dots, n-1$) of Lie groups admitted by system (8).

Theorem 4. *(See [14].) $(M^{-1})_{(j)}^{(i)}(x) = \Sigma_{k=1}^{n} \xi_k^j a_k^i$, where a_k^i is the function ahead dx_k at the left side of i-th equation of system* (9).

Based on Theorem 4, we have

$$\begin{aligned} & a_1^1 = -P_2, \quad a_2^1 = P_1, \quad a_3^1 = \dots = 0; \\ & a_1^2 = -P_3, \quad a_3^2 = P_1, \quad a_2^2 = a_4^2 = \dots = 0; \\ & a_1^3 = -P_4, \quad a_4^3 = P_1, \quad a_2^3 = a_3^3 = \dots = 0; \\ & \dots \\ & a_1^{n-1} = -P_n, \quad a_n^{n-1} = P_1, \quad a_2^n = \dots = a_{n-1}^n = 0. \end{aligned}$$

So

$$M^{-1} = \begin{bmatrix} -\xi_1^1 P_2 + \xi_2^1 P_1 & -\xi_1^2 P_2 + \xi_2^2 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ -\xi_1^1 P_3 + \xi_3^1 P_1 & -\xi_1^2 P_3 + \xi_3^2 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots \\ -\xi_1^1 P_n + \xi_n^1 P_1 & -\xi_1^2 P_n + \xi_n^2 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{bmatrix}.$$

Theorem 5. $|M^{-1}| = P_1^{n-1} \mu^*$.

Proof. On the one hand, by the properties of the determinant, one has

$$\begin{aligned}
|M^{-1}| &= -\xi_1^1 \begin{vmatrix} P_2 & -\xi_1^2 P_2 + \xi_2^2 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ P_3 & -\xi_1^2 P_3 + \xi_3^2 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots \\ P_n & -\xi_1^2 P_n + \xi_n^2 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{vmatrix} \\
&\quad + P_1 \begin{vmatrix} \xi_2^1 & -\xi_1^2 P_2 + \xi_2^2 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ \xi_3^1 & -\xi_1^2 P_3 + \xi_3^2 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & -\xi_1^2 P_n + \xi_n^2 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{vmatrix} \\
&= -\xi_1^1 P_1 \begin{vmatrix} P_2 & \xi_2^2 & -\xi_1^3 P_2 + \xi_2^3 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ P_3 & \xi_3^2 & -\xi_1^3 P_3 + \xi_3^3 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_n & \xi_n^2 & -\xi_1^3 P_n + \xi_n^3 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{vmatrix} \\
&\quad + P_1(-\xi_1^2) \begin{vmatrix} \xi_2^1 & P_2 & -\xi_1^3 P_2 + \xi_2^3 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ \xi_3^1 & P_3 & -\xi_1^3 P_3 + \xi_3^3 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & P_n & -\xi_1^3 P_n + \xi_n^3 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{vmatrix} \\
&\quad + P_1^2 \begin{vmatrix} \xi_2^1 & \xi_2^2 & -\xi_1^3 P_2 + \xi_2^3 P_1 & \dots & -\xi_1^{n-1} P_2 + \xi_2^{n-1} P_1 \\ \xi_3^1 & \xi_3^2 & -\xi_1^3 P_3 + \xi_3^3 P_1 & \dots & -\xi_1^{n-1} P_3 + \xi_3^{n-1} P_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & -\xi_1^3 P_n + \xi_n^3 P_1 & \dots & -\xi_1^{n-1} P_n + \xi_n^{n-1} P_1 \end{vmatrix}.
\end{aligned}$$

Expanding the above every determinants by the properties of determinants and simplifying them, one can get

$$\begin{aligned}
|M^{-1}| &= -\xi_1^1 P_1^{n-2} \begin{vmatrix} P_2 & \xi_2^2 & \xi_2^3 & \dots & \xi_2^{n-1} \\ P_3 & \xi_3^2 & \xi_3^3 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_n & \xi_n^2 & \xi_n^3 & \dots & \xi_n^{n-1} \end{vmatrix} + \left(-\xi_1^2\right) P_1^{n-2} \begin{vmatrix} \xi_2^1 & P_2 & \xi_2^3 & \dots & \xi_2^{n-1} \\ \xi_3^1 & P_3 & \xi_3^3 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & P_n & \xi_n^3 & \dots & \xi_n^{n-1} \end{vmatrix} \\
&\quad + \left(-\xi_1^3\right) P_1^{n-2} \begin{vmatrix} \xi_2^1 & \xi_2^2 & P_2 & \xi_2^4 & \dots & \xi_2^{n-1} \\ \xi_3^1 & \xi_3^2 & P_3 & \xi_3^4 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & P_n & \xi_n^4 & \dots & \xi_n^{n-1} \end{vmatrix} + \dots + \left(-\xi_1^{n-1}\right) P_1^{n-2} \begin{vmatrix} \xi_2^1 & \xi_2^2 & \dots & \xi_2^{n-2} & p_2 \\ \xi_3^1 & \xi_3^2 & \dots & \xi_3^{n-2} & p_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & \dots & \xi_n^{n-2} & p_n \end{vmatrix} \\
&\quad + p_1^{n-1} \begin{vmatrix} \xi_2^1 & \xi_2^2 & \dots & \xi_2^{n-1} \\ \xi_3^1 & \xi_3^2 & \dots & \xi_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \xi_n^1 & \xi_n^2 & \dots & \xi_n^{n-1} \end{vmatrix}. \qquad (10)
\end{aligned}$$

On the other hand, expanding the determinant μ^* in the first column of it, we have

$$P_1^{n-1} \mu^* = P_1^{n-1} \begin{vmatrix} 1 & \frac{P_2}{P_1} & \dots & \frac{P_n}{P_1} \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{vmatrix}$$

$$= P_1^{n-1}\begin{vmatrix} \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + P_1^{n-1}(-1)^{1+2}\xi_1^1\begin{vmatrix} \frac{P_2}{P_1} & \cdots & \frac{P_n}{P_1} \\ \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix}$$

$$+ P_1^{n-1}(-1)^{1+3}\xi_1^2\begin{vmatrix} \frac{P_2}{P_1} & \cdots & \frac{P_n}{P_1} \\ \xi_2^1 & \cdots & \xi_n^1 \\ \xi_2^3 & \cdots & \xi_n^3 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_1^{n-1}(-1)^{1+n}\xi_1^{n-1}\begin{vmatrix} \frac{P_2}{P_1} & \cdots & \frac{P_n}{P_1} \\ \xi_2^1 & \cdots & \xi_n^1 \\ \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-2} & \cdots & \xi_n^{n-2} \end{vmatrix}.$$

Multiplying the first low of latter $n-1$ determinants by P_1 in their coefficients, we can have the following expressions,

$$P_1^{n-1}\mu^* = P_1^{n-1}\begin{vmatrix} \xi_2^1 & \cdots & \xi_n^1 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + P_1^{n-2}(-1)^{1+2}\xi_1^1\begin{vmatrix} P_2 & \cdots & P_n \\ \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix}$$

$$+ P_1^{n-2}(-1)^{1+3}\xi_1^2\begin{vmatrix} P_2 & \cdots & P_n \\ \xi_2^1 & \cdots & \xi_n^1 \\ \xi_2^3 & \cdots & \xi_n^3 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{vmatrix} + \cdots + P_1^{n-2}(-1)^{1+n}\xi_1^{n-1}\begin{vmatrix} P_2 & \cdots & P_n \\ \xi_2^1 & \cdots & \xi_n^1 \\ \xi_2^2 & \cdots & \xi_n^2 \\ \vdots & \vdots & \vdots \\ \xi_2^{n-2} & \cdots & \xi_n^{n-2} \end{vmatrix}. \tag{11}$$

The transposing determinant and the original determinant are equal. Any two columns are interchanged, the sign of the determinant is changed. So, we can know the right side of the equal sign of (10) and that of (11) being equal, and hence $|M^{-1}| = P_1^{n-1}\mu^*$. This complete the proof. □

Example 2. Consider the following autonomous system,

$$\begin{cases} \frac{dx}{dt} = y, \frac{dy}{dt} = yz, \frac{dz}{dt} = -z^2. \end{cases} \tag{12}$$

The system admits two Lie groups generated by

$$V_1 = \frac{\partial}{\partial x}$$

and

$$V_2 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}.$$

We have

$$\mu = \begin{vmatrix} y & yz & -z^2 \\ 1 & 0 & 0 \\ x & 0 & -z \end{vmatrix} = yz^2, \qquad \mu^* = \begin{vmatrix} 1 & z & -\frac{z^2}{y} \\ 1 & 0 & 0 \\ x & 0 & -z \end{vmatrix} = z^2 = \frac{\mu}{P_1}$$

and

$$\left|M^{-1}\right| = \begin{vmatrix} -yz & -xyz \\ z^2 & -yz + xz^2 \end{vmatrix} = y^2z^2.$$

Obviously,

$$\left|M^{-1}\right| = P_1^2\mu^*.$$

4. Conclusion

In this paper, we have considered the relation between one-parameter Lie groups admitted by n-th order autonomous systems and inverse integrating factors of the systems. From the above investigation, the generators of one-parameter Lie groups admitted by n-th order autonomous systems can help us get an inverse integrating factor of the systems. Specially, we have proposed a method of obtaining an inverse integrating factor of the systems under knowing $n-1$ one-parameter Lie groups admitted by the systems. Moreover, by describing the n-th order autonomous systems with differential form systems, some properties of inverse integrating factors are presented.

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