Variation of normal derivatives of Green functions

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We derive a variational formula for the outward normal derivative of the Green function for the Schrödinger and Laplace–Beltrami operators, viewed as perturbations of the Laplacian. As an application we begin to characterize elliptic growth—the growth of a domain pushed outward by its own Green function.

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1. Introduction

The outward normal derivative of the Green function of a domain is of central importance to boundary-value problems associated to the underlying elliptic operator. Given a domain $D \subset \mathbb{C}$, a point $w \in D$, and an elliptic differential operator L, we define the Green function $g_w : \overline{D} - \{w\} \to \mathbb{R}$ as the solution (if it exists) to

$$\begin{cases} Lg_w = \delta_w & \text{in } D \\ g_w = 0 & \text{on } \partial D, \end{cases}$$

where δ_w denotes the Dirac delta supported at w. Occasionally we will denote $g_w(z)$ by $g(z, w)$, emphasizing the functional dependence upon both inputs. The existence of a Green function for a domain is tied to the regularity of the boundary. We will work only with smooth and analytic boundaries, which guarantees existence of Green functions and allows use of Green's theorem.

The Green function is used to solve the inhomogeneous boundary-value problem for L with zero boundary data. That is, a solution to

$$\begin{cases} Lu = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

is given by

$$u(w) = \int_D g_w f \, dA.$$

Of equal importance is the homogeneous boundary-value problem

$$\begin{cases} Lu = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{cases} \tag{1}$$

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In some situations, the Green function can be used to extract a solution to this problem as well. In the case of the Laplacian, $L = \Delta$, a solution to (1) is given by

$$u(w) = \int_{\partial D} f \partial_n g_w \, ds,$$

where ∂_n denotes the outward normal derivative. The integral kernel $P_\zeta(w) = \partial_n g_w(\zeta)$ is called the Poisson kernel of the region D, and for simplicity of notation we sometimes use $P_\zeta(w)$ in place of $\partial_n g_w(\zeta)$, especially when we wish to view ζ as a fixed parameter.

If instead we consider an operator in divergence form, $L = \nabla \lambda \nabla$, then a solution to (1) is given by

$$u(w) = \int_{\partial D} f \lambda \partial_n g_w \, ds.$$

The normal derivative of the Green function also appears in the dynamics of moving boundaries. In [1] the authors define a process wherein a domain grows with a boundary velocity determined by the outward derivative of its own Green function. This so-called elliptic growth is used to model real processes such as electrodeposition, crystal formation, and quasi-static fluid flowing through a porous medium. When the underlying operator is the Laplacian the growth is commonly called Laplacian growth or Hele–Shaw flow—see [2] for a survey. When the underlying operator is more general however, much less work has been done. It is known for example that lemniscates cannot survive a Laplacian growth process (see [3]), but a carefully-chosen elliptic growth might preserve them. To approach such problems we need an understanding of how the Green function depends upon the operator.

Compared to that of a general elliptic operator, the Green function of the Laplacian is well-understood. In [4] the author develops formulas for Green functions under various perturbations of the Laplacian; here we follow the same paradigm, studying the normal derivative of the Green function under a perturbation of the operator.

2. Variation of the normal derivative

2.1. Schrödinger

Our starting point is the variational formula derived in [4]. Consider a bounded domain D with smooth, analytic boundary, a positive function $u \in C^\infty(\overline{D})$ and $z, w \in D$. For $\epsilon > 0$ the Green function g_w^* of $\Delta - \epsilon u$ satisfies

$$g_w^*(z) = g_w(z) + \epsilon \int_D u(\xi) g(z, \xi) g(w, \xi) \, dA(\xi) + o(\epsilon) \tag{2}$$

as $\epsilon \to 0$, where g_w denotes the Green function for the Laplacian and the error term converges uniformly in z for each fixed w. Furthermore, a full series expansion is given by

$$g_w^* = \sum_{n=0}^{\infty} \epsilon^n (TM)^n g_w,$$

where we have used the operators $M : \varphi \mapsto u\varphi$ and

$$T : \varphi \mapsto \int_D g_w \varphi \, dA.$$

To derive a similar formula for $\partial_n g_w^*$ our major tool is the following lemma.

Lemma 1. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and $f \in C^1(\overline{D}) \cap C^2(D)$. Suppose further that $f = 0$ on ∂D. Then for $\zeta \in \partial D$ we have*

$$\frac{\partial f}{\partial n}(\zeta) = \int_D \Delta f(z) P(z, \zeta) \, dA(z),$$

where $P(z, \zeta) = \partial_n g_z(\zeta)$ is the Poisson kernel of D.

Proof. Let $A \subset \partial D$ be measurable with respect to arc length on ∂D. We define an auxiliary function u which solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \chi_A & \text{on } \partial D \end{cases}$$

where χ_A denotes the characteristic function of A. From the Green identity

$$\int_{\partial D} \left(u \frac{\partial f}{\partial n} - f \frac{\partial u}{\partial n} \right) ds = \int_D (u \Delta f - f \Delta u) \, dA \tag{3}$$

we use the assumptions on u and f to find

$$\int_A \frac{\partial f}{\partial n}\, ds = \int_D u \Delta f\, dA. \tag{4}$$

Given $z \in D$ we return to the Green identity (3) with the functions g_z and u. It follows that

$$u(z) = \int_{\partial D} u \frac{\partial g_z}{\partial n}\, ds = \int_A \frac{\partial g_z}{\partial n}\, ds.$$

Inserting this into (4) and swapping the order of integration gives

$$\int_A \frac{\partial f}{\partial n}\, ds = \int_A \left[\int_D \Delta f(z) \frac{\partial g}{\partial n}(z, \zeta)\, dA(z) \right] ds(\zeta).$$

Since A is arbitrary, we conclude

$$\frac{\partial f}{\partial n}(\zeta) = \int_D \Delta f(z) \frac{\partial g}{\partial n}(z, \zeta)\, dA(z)$$

almost everywhere with respect to arc length. Our assumptions on f imply that each side of this equation is a continuous function of ζ. Hence the equation in fact holds everywhere. □

Before we can proceed to the main result, we need to ascertain the regularity of the relevant Green functions. To use Lemma 1 we need to be sure the Green function for Schrödinger operators are continuously differentiable in some neighborhood of the closure of the domain.

Lemma 2. *Suppose that $D \subset \mathbb{C}$ is a bounded domain with smooth, analytic boundary. Then the Green function of the Laplacian on D is analytic in a neighborhood of $\overline{D}$ except at the singularity.*

Proof. Fix $w \in D$. Since ∂D is smooth analytic, for each $\zeta \in \partial D$ there is a neighborhood N_ζ of ζ and a conformal map $\varphi : N_\zeta \to \mathbb{D}$ so that $\varphi(N_\zeta \cap D) = \mathbb{D}^+$, the upper half-disk. Taking the union of all such N_ζ and D gives $\tilde{D}$, a neighborhood of $\overline{D}$. We can extend g_w to $\tilde{D}$ as follows.

Given $\zeta \in \partial D$ find the corresponding neighborhood N_ζ and conformal map φ. Since $g_w \circ \varphi^{-1}$ is harmonic on the upper-half disk, 0 on the interval $(-1, 1)$, and continuous up to the real axis, the reflection principle for harmonic functions allows us to extend $g_w \circ \varphi^{-1}$ to a harmonic function on $\mathbb{D}$ via the definition

$$g_w \circ \varphi^{-1}(z) = -g_w \circ \varphi^{-1}(\overline{z})$$

whenever $z \in \mathbb{D}$ is below the real axis. Then $g_w = (g_w \circ \varphi^{-1}) \circ \varphi$ extends to a harmonic function on $D \cup N_\zeta$. Now suppose that $z \in \mathbb{C} - \overline{D}$ is a point in both N_ζ and N_ξ; we need to show that the extensions arising from ζ and ξ agree at z. Consider the conformal maps $\varphi : N_\zeta \to \mathbb{D}$ and $\psi : N_\xi \to \mathbb{D}$. Then $\psi \circ \varphi^{-1}$ is a map preserving the unit disk and the real line; hence for some real $a \in (-1, 1)$ we can define $M_a : \mathbb{D} \to \mathbb{D}$ as

$$M_a(t) = \frac{t - a}{1 - at}$$

so that $\psi \circ \varphi^{-1} = M_a$. We can rewrite the extension of g_w resulting from N_ξ as

$$\begin{aligned}
(g_w \circ \psi^{-1})(\overline{\psi(z)}) &= (g_w \circ \varphi^{-1} \circ M_a^{-1})(\overline{M_a(\varphi(z))}) \\
&= (g_w \circ \varphi^{-1} \circ M_a^{-1})(M_a(\overline{\varphi(z)})) \\
&= (g_w \circ \varphi^{-1})(\overline{\varphi(z)}),
\end{aligned}$$

as desired. Since g_w is harmonic away from the singularity, it is analytic there. □

Lemma 3. *Suppose that $D \subset \mathbb{C}$ is a bounded domain with smooth, analytic boundary. Let u be a positive, smooth function in a neighborhood of $\overline{D}$. Then the Green function of D for the operator $\Delta - u$ is smooth in a neighborhood of $\overline{D}$ except at the singularity. In particular, the Green function satisfies the conditions of Lemma 1.*

Proof. Fix a point $w \in D$. We can define

$$T : \varphi \mapsto \int_D g_z \varphi\, dA$$

which as a left inverse to the Laplacian is a map from the Sobolev space $H^k(D)$ to $H^{k+2}(D)$ for any $k \geq 0$. Define M to be the multiplication operator arising from u. Since u is smooth in a neighborhood of $\overline{D}$, the operator M maps $H^k(D)$ into itself. It was shown in [4] that

$$g_w^* = \sum_{n=0}^{\infty} \epsilon^n (TM)^n g_w \tag{5}$$

holds throughout D. Since g_w is smooth in $D - \{w\}$, so is g_w^*. Furthermore, since g_w extends smoothly into a neighborhood of $\overline{D}$ excluding w, Eq. (5) gives a smooth extension of g_w^* into the same domain. $\quad\square$

Theorem 4. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and fix $w \in D$. Suppose that $u \in C^\infty(\overline{D})$ is a positive function. The outward normal derivative of the Green function g^* of the Schrödinger operator $\Delta - \epsilon u$ satisfies*

$$\partial_n g_w^*(\zeta) = \partial_n g_w(\zeta) + \epsilon \int_D u(z) g(z, w) P(z, \zeta) \, dA(z) + o(\epsilon)$$

as $\epsilon \to 0$, where the convergence of $o(\epsilon)$ is uniform in ζ for each fixed w.

Proof. Notice that g_w^* satisfies

$$\begin{cases} \Delta g_w^* = \delta_w + \epsilon u g_w^* & \text{in } D \\ g_w^* = 0 & \text{on } \partial D. \end{cases}$$

Using Lemma 1 gives

$$\begin{aligned} \partial_n g_w^*(\zeta) &= \int_D \left(\delta_w + \epsilon u(z) g_w^*(z) \right) P(z, \zeta) \, dA(z) \\ &= \partial_n g(\zeta, w) + \epsilon \int_D u(z) g_w^*(z) P(z, \zeta) \, dA(z), \end{aligned} \tag{6}$$

where we have used the fact that $P(z, \zeta) = \partial_n g_z(\zeta)$. Inserting the perturbation formula of Eq. (2) into (6) gives

$$\partial_n g_w^*(\zeta) = \partial_n g(\zeta, w) + \epsilon \int_D u(z) g_w(z) P(z, \zeta) \, dA + o(\epsilon). \quad\square$$

We remark that this formula can be easily derived from formal manipulation; beginning with Eq. (2) take the outward normal derivative in the z variable at a point ζ on the boundary. Assuming $o(\epsilon)$ remains small and the derivative commutes with the integral, we produce the result. The above proof actually produces an expression for the second variation of $\partial_n g_w^*$, which we give here as a corollary.

Corollary 5. *Under the same conditions as the previous theorem, the second variation of $\partial_n g_w^*$ is given by*

$$\delta^2 \partial_n g_w^*(\zeta) = \int_D \int_D u(z) u(\xi) g(\xi, w) g(\xi, z) P(z, \zeta) \, dA(\xi) dA(z).$$

As a further corollary, we can deduce a monotonicity result.

Corollary 6. *Under the same conditions as the previous theorem, the first variation $\delta \partial_n g_w^*(\zeta)$ is nonpositive at each point of ∂D.*

Proof. For any $z, w \in D$ and $\zeta \in \partial D$ we have $P(z, \zeta), u(z) \geq 0$ and $g(z, w) \leq 0$. Thus

$$\int_D u(z) g(z, w) P(z, \zeta) \, dA(z) \leq 0.$$

The result follows from Theorem 4. $\quad\square$

2.2. Laplace–Beltrami

Having completed the analysis of the Schrödinger case, we turn to a different perturbation of the Laplacian. As before, consider a bounded domain $D \subset \mathbb{C}$ with smooth, analytic boundary and $u \in C^\infty(\overline{D})$. Given $\epsilon > 0$ define $\lambda = 1 + \epsilon u$ and the Laplace–Beltrami operator $L = \nabla \lambda \nabla$. Let g_w^* denote the Green function for L with singularity at $w \in D$; we wish to derive a perturbation formula for $\partial_n g_w^*(\zeta)$ when ϵ is close to 0. The result we obtained for the Schrödinger operator turned out to be

the obvious one that results from formal manipulation. In this vein, we can anticipate a Laplace–Beltrami analogue by using the following perturbation formula from [4]:

$$g_w^*(z) = g_w(z) + \epsilon \int_D u \nabla g_z \cdot \nabla g_w \, dA + o(\epsilon) \tag{7}$$

as $\epsilon \to 0$, where $w, z \in D$ and $o(\epsilon)$ converges uniformly in z for each fixed w. Let us take a normal derivative in z at the point $\zeta \in \partial D$, assuming that $o(\epsilon)$ remains small and derivative commutes with integral. We are lead to the formula

$$\partial_n g_w^*(\zeta) = \partial_n g_w(\zeta) + \epsilon \int_D u \nabla P_\zeta \cdot \nabla g_w \, dA + o(\epsilon).$$

Unfortunately this integral might diverge, but formal manipulation and integration by parts lead us to the following theorem.

Theorem 7. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and fix $w \in D$. Suppose that $u \in C^\infty(\overline{D})$ is a positive function and define $\lambda(z) = 1 + \epsilon u(z)$ for $\epsilon > 0$. The outward normal derivative of the Green function g^* of the Laplace–Beltrami operator $L = \nabla \lambda \nabla$ satisfies*

$$\partial_n g_w^*(\zeta) = \partial_n g_w(\zeta) + \frac{\epsilon}{2} \left[\int_D \Delta u g_w P_\zeta \, dA - \partial_n g_w(\zeta) \left[u(\zeta) + u(w) \right] \right] + o(\epsilon)$$

as $\epsilon \to 0$, where the convergence of $o(\epsilon)$ is uniform in ζ for each fixed w.

Proof. For a rigorous proof, we use a well-known change of variables to reduce the problem to that of the Schrödinger operator. Since $u \in C^\infty(\overline{D}) \subset L^\infty(D)$, we consider only those ϵ satisfying $0 \le \epsilon \le \epsilon_0 < \|u\|_\infty^{-1}$. This ensures that $\lambda \ge 0$ and that all of the following linear approximations have uniformly small error terms. If we define the auxiliary functions $G_w^* = g_w^* \sqrt{\lambda}$ and

$$V = \frac{\Delta \sqrt{\lambda}}{\sqrt{\lambda}},$$

then we claim $\sqrt{\lambda(w)} G_w^*$ is the Green function for $\Delta - V$. Verification is a straight forward computation which we omit; a full derivation is given in [4]. Furthermore, the dependence of V upon ϵ can be seen to be $V = \epsilon \Delta u/2 + o(\epsilon)$. Using Theorem 4 we obtain

$$\partial_n \left[\sqrt{\lambda(z)\lambda(w)} g_w^*(z) \right]_{z=\zeta} = \partial_n g_w(\zeta) + \frac{\epsilon}{2} \int_D \Delta u g_w P_\zeta \, dA + o(\epsilon). \tag{8}$$

Using $g_w^*(\zeta) = 0$ the left-hand side of this equation simplifies to

$$\partial_n \left[\sqrt{\lambda(z)\lambda(w)} g_w^*(z) \right]_{z=\zeta} = g_w^*(\zeta) \partial_n \left[\sqrt{\lambda(z)\lambda(w)} \right]_{z=\zeta} + \sqrt{\lambda(\zeta)\lambda(w)} \partial_n g_w^*(\zeta)$$

$$= \sqrt{\lambda(\zeta)\lambda(w)} \partial_n g_w^*(\zeta).$$

We can rewrite (8) as

$$\partial_n g_w^*(\zeta) = (\lambda(\zeta)\lambda(w))^{-1/2} \left[\partial_n g_w(\zeta) + \frac{\epsilon}{2} \int_D \Delta u g_w P_\zeta \, dA + o(\epsilon) \right]. \tag{9}$$

We only want first-order dependence upon ϵ, so we can expand

$$(\lambda(\zeta)\lambda(w))^{-1/2} = 1 - \epsilon \left(\frac{u(\zeta) + u(w)}{2} \right) + o(\epsilon)$$

so that (9) becomes

$$\partial_n g_w^*(\zeta) = \left[1 - \epsilon \left(\frac{u(\zeta) + u(w)}{2} \right) + o(\epsilon) \right] \left[\partial_n g_w(\zeta) + \frac{\epsilon}{2} \int_D \Delta u g_w P_\zeta \, dA + o(\epsilon) \right]$$

$$= \partial_n g_w(\zeta) + \frac{\epsilon}{2} \left[\int_D \Delta u g_w P_\zeta \, dA - \partial_n g_w(\zeta) \left[u(\zeta) + u(w) \right] \right] + o(\epsilon),$$

as desired. $\quad \square$

As we did for the Schrödinger operator, we can deduce another monotonicity result.

Corollary 8. *Under the same conditions as the previous theorem, the variation $\delta \partial_n g_w^*(\zeta)$ satisfies*

$$\int_{\partial D} \delta \partial_n g_w^* \, ds \leq 0.$$

Proof. From the previous theorem,

$$\int_{\partial D} \delta \partial_n g_w^* \, ds = \int_D \Delta u g_w \, dA - u(w) - \int_{\partial D} u \partial_n g_w \, ds,$$

where we have swapped integrals and used the fact that

$$\int_{\partial D} \partial_n g_w \, ds = \int_{\partial D} P_\zeta(z) \, ds(\zeta) = 1$$

for each $w, z \in D$. The integral swap is valid since all integrands are of constant sign. Green's identity allows us to write

$$\int_{\partial D} \delta \partial_n g_w^* \, ds = -2 \int_{\partial D} u \partial_n g_w \, ds \leq 0,$$

since $u, \partial_n g_w \geq 0$ everywhere on ∂D. □

3. Applications

3.1. The Dirichlet problem

For an elementary application of the Green variation, consider solving the Dirichlet problem on a domain D:

$$\begin{cases} (\Delta - \epsilon u)\varphi_\epsilon = 0 & \text{in } D \\ \varphi_\epsilon = f & \text{on } \partial D \end{cases} \tag{10}$$

where f is a bounded Borel function, $u \geq 0$ is smooth in a neighborhood of $\overline{D}$ and $\epsilon \geq 0$ is small. If we ignored the ϵu term in the problem – that is, assume ϵ is zero – we can estimate the error with the following theorem and corollary.

Theorem 9. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and suppose $u \in C^\infty(\overline{D})$ is a positive function. For $\epsilon \geq 0$ and a bounded Borel function f the solution φ_ϵ to the Dirichlet problem* (10) *satisfies*

$$\varphi_\epsilon(z) = \varphi_0(z) + \epsilon \int_D u \varphi_0 g_z \, dA + o(\epsilon)$$

as $\epsilon \to 0$, where the error term converges uniformly in z.

Proof. Let g_w^* denote the Green function of the operator $\Delta - \epsilon u$ in the region D. Note that

$$\begin{cases} (\Delta - \epsilon u)(\varphi_\epsilon - \varphi_0) = \epsilon u \varphi_0 & \text{in } D \\ \varphi_\epsilon - \varphi_0 = 0 & \text{on } \partial D, \end{cases}$$

whence an expression for φ_ϵ is given by

$$\varphi_\epsilon(z) = \varphi_0(z) + \epsilon \int_D u \varphi_0 g_z^* \, dA.$$

From the perturbation formula (5) we have

$$\varphi_\epsilon(z) = \varphi_0(z) + \epsilon \int_D u \varphi_0 g_z \, dA + o(\epsilon),$$

as desired. □

Corollary 10. *With the same assumptions as the previous theorem, the linearization of φ_ϵ has the pointwise bound*

$$|\delta \varphi_\epsilon(z)| \leq \|u\|_2 \|g_z\|_2 \|f\|_\infty.$$

Proof. From the maximum modulus principle for harmonic functions, $\sup_D |\varphi_0| \leq \sup_{\partial D} |f|$. The result follows from this and the Cauchy–Schwarz inequality. □

3.2. Elliptic growth and an inverse problem

Consider a bounded domain $D \subset \mathbb{C}$ with smooth, analytic boundary and fix $w \in D$. Given a positive function $u \in C^\infty(\overline{D})$, an elliptic growth process governed by the Schrödinger operator $\Delta - u$ is a nested sequence of domains $\{D(t) : t \geq 0\}$ such that $D(0) = D$ and

$$V_n(\zeta) = \partial_n g_w^*(\zeta),$$

where V_n denotes the outward normal velocity of the boundary $\partial D(t)$ and g_w^* denotes the Green function of the operator $\Delta - u$ for the domain $D(t)$. That is, the Green function "pushes" the boundary of the domain outward, changing the domain – and hence the Green function – in a nonlinear, nonlocal manner. In beginning to understand the nuances of this process, we ask the following question. Which families of nested, growing domains can be produced by an elliptic growth process?

Our first approach to addressing this question is to consider the local problem. Given the initial domain D we seek which velocity fields V can emerge from different choices of the function u. The intuition is as follows. Let V_0 be the velocity field that arises from elliptic growth with $u \equiv 0$ (this is the well-known Laplacian growth process). If V is another velocity field close to V_0, can we choose $u \sim 0$ so that V arises from elliptic growth via $\Delta - u$? In other words, if we view the variation as a linear map $u \mapsto \delta \partial_n g_w^*$ on an appropriate function space, what is the image of this map? We are led to the following theorem.

Theorem 11. *Let $D \subset \mathbb{C}$ be a bounded domain with smooth, analytic boundary and fix $w \in D$. Define*

$$Au(\zeta) = \int_D u(z)g(z, w)P(z, \zeta)\, dA(z).$$

Then A is a linear map $L^2(D) \to L^2(\partial D)$ with dense range.

Proof. To prove A is defined on all of $L^2(D)$ it suffices to show that $g_w P_\zeta \in L^2(D)$ for each $w \in D$ and $\zeta \in \partial D$. Away from w and ζ the function $g_w P_\zeta$ is smooth; furthermore, the singularity at w is logarithmic, which is square-integrable. It remains to understand the singularity at ζ. Having assumed the boundary of D is smooth and analytic, there is a conformal map from a neighborhood of ζ to the unit disk which transforms the Poisson kernel and Green function canonically. Hence it suffices to understand the singularity at ζ in the case when $D = \mathbb{D}$, the unit disk. Without loss of generality, we can assume $w = 0$ temporarily; a Möbius transform can preserve the disk and send w to 0, which does not disturb the behavior of the singularity at ζ. Note that

$$g_0(z)P(z, \zeta) = \frac{\ln|z|}{4\pi^2}\mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right).$$

Taking the limit $z \to \zeta$ gives

$$\lim_{z \to \zeta} g_0(z)P(z, \zeta) = -\frac{1}{2\pi^2},$$

so $g_0 P_\zeta$ remains bounded near ζ. At this point we return to the general domain D; we can write $g_w(z)P(z, \zeta) = h_\zeta(z)$, a bounded function in a neighborhood of $\overline{D}$ except at w, where it has a logarithmic singularity. We conclude $g_w P_\zeta \in L^2(D)$.

Next we show that A maps into $L^2(\partial D)$. Note that

$$\int_{\partial D} |Au|^2\, ds \leq \int_{\partial D} \|u\|_2 \|h_\zeta\|_2\, ds(\zeta) \leq \|u\|_2 \sup_{\zeta \in \partial D} \|h_\zeta\|_2 \cdot \mathrm{length}(\partial D),$$

so it suffices to show that $\sup_{\partial D} \|h_\zeta\|_2 < \infty$. This follows easily from the fact that $\zeta \mapsto \|h_\zeta\|_2$ is a continuous function on the compact set ∂D.

Next we show that the range of A is dense in $L^2(\partial D)$. Suppose $v \in L^2(\partial D)$ is chosen so that

$$\int_{\partial D} v(\zeta)(Au)(\zeta)\, ds(\zeta) = 0$$

for all $u \in L^2(D)$. Using the definition of A and swapping integrals gives

$$\int_D u(z)\left(\int_{\partial D} v(\zeta)h_\zeta(z)\, ds(\zeta)\right) dA(z) = 0.$$

Since u is arbitrary, the inner integral must vanish for almost all $z \in D$. Note that $h_\zeta(z) = g_w(z)P(z, \zeta)$ and $g_w(z) < 0$ for all $z \in D$, so

$$\int_{\partial D} v(\zeta)P(z, \zeta)\, ds(\zeta) = 0$$

for almost all $z \in D$. This integral is a harmonic function in z which is zero almost everywhere in D, hence everywhere by continuity. The domain is regular enough so that, when z approaches ∂D, we recover the function v. Hence $v = 0$ in $L^2(\partial D)$. This shows the orthogonal complement of the range of A is trivial, so the range of A is dense in $L^2(\partial D)$.   □

For physical reasons, we might prefer function spaces different from L^2. One natural candidate for velocity fields is $L^1(\partial D)$, since the L^1 distance between velocity functions measures the difference in areas due to the domain growing in different ways. Furthermore we could consider only smooth perturbation functions u, in the spirit of the original perturbation result. To these ends we have the following corollary.

Corollary 12. *With the same hypotheses as the previous theorem, A maps $C^\infty(\overline{D}) \to L^1(\partial D)$ and has dense range.*

Proof. Since ∂D has finite measure, $L^2(\partial D)$ is dense in $L^1(\partial D)$. Furthermore $C^\infty(\overline{D})$ is dense in $L^2(D)$, and the result follows.   □

The inverse problem is not fully solved with these results. First we should only consider nonnegative functions u and characterize the range of the map A. Second, the restriction of the domain of A to smooth functions begs the following question. Can every smooth function in $L^1(\partial D)$ be obtained from A when restricted to $C^\infty(\overline{D})$? Finally, which of these results can be carried over to elliptic growth governed by a Laplace–Beltrami operator? In that setting the relevant integrals are more delicate and require further analysis. All of these problems provide directions for future work.

Acknowledgment

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